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The Best Constants in the Strong Convexity of $|\xi|^p$ and in the Strong Monotonicity of $|\xi|^{p-2}\xi$

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Dedicated to the memory of Hedy Attouch.

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We prove that the function $\xi \in \mathbb{R}^N \to |\xi|^p$ is strongly convex for every p, $p \in]1, +\infty[$, and we give a characterization, through the resolution of an algebraic equation, of the best constant which appears in the strong convexity inequality. The same proof allows us to prove that the function $\xi \in \mathbb{R}^N \to |\xi|^{p-2}\xi$ is strongly monotone and to identify the explicit value of the best constant.

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1. Introduction and main results

Let $N \in \mathbb{N}$ and $p \in]1, +\infty[$. It is "well known" that the function $\xi \in \mathbb{R}^N \to |\xi|^p \in \mathbb{R}$ is *strongly convex*, i.e., satisfies the strong convexity inequality

$$\exists\,\lambda \in]0, +\infty[\,: |\xi|^p \geq |\eta|^p + p|\eta|^{p-2}\eta(\xi - \eta) + \lambda|\xi - \eta|^2(|\xi| + |\eta|)^{p-2},\ \forall \xi, \eta \in \mathbb{R}^N, \quad (1.1)$$

and that the function $\xi \in \mathbb{R}^N \to |\xi|^{p-2}\xi \in \mathbb{R}^N$ is *strongly monotone*, i.e., satisfies the strong monotonicity inequality

$$\exists\,\mu \in]0, +\infty[\,: \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta) \geq \mu|\xi - \eta|^2(|\xi| + |\eta|)^{p-2},\ \forall \xi, \eta \in \mathbb{R}^N. \quad (1.2)$$

Whenever one knows that there exists some $\lambda \in]0, +\infty[$ such that (1.1) holds true, one can define the *best constant* α_p as the supremum of those λ's such that (1.1) holds true, and try to give a formula for α_p. Similarly, whenever one knows that there exists some $\mu \in]0, +\infty[$ such that (1.2) holds true, one can define the *best*

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constant β_p as the supremum of those μ's such that (1.2) holds true, and try to give a formula for β_p.

A clear and simple proof of the strong monotonicity inequality (1.2) of $|\xi|^{p-2}\xi$ can be found in particular in [3], Lemma 5.1 and Lemma 5.2, but we were not able to locate any precise reference of the proof of the strong convexity inequality (1.1) of $|\xi|^p$. On the other hand, to the best of our knowledge, the values of the best constants α_p in (1.1) and β_p in (1.2) are unknown, even if the papers [1] and [2] provide partial results for the optimal constants which appear in variants of the strong monotonicity inequality (1.2) (see Remark 3.8 in Section 3 below).

In this paper we give a proof of the strong convexity of the function $|\xi|^p$, as well as a characterization of the best constant α_p for which (1.1) holds true; this best constant is computed through the resolution of an algebraic equation (see Theorem 1.1 below). The same proof allows us to prove the strong monotonicity of the function $|\xi|^{p-2}\xi$ and to identify the value of the best constant β_p for which (1.2) holds true; in this case, the best constant is explicit (see Theorem 1.2 below).

The strong convexity of the function $|\xi|^p$ and the strong monotonicity of the function $|\xi|^{p-2}\xi$ are of critical use in the proof of many existence and regularity results of the theory of nonlinear elliptic PDEs (see, for instance, [2] and the references therein). Independently of its own interest, the determination of the best constant is useful for the evaluation of the error estimate in the numerical approximation for such problems.

As far as the strong convexity of the function $|\xi|^p$ is concerned, our result is the following one.

Theorem 1.1. *Let*
$$p \in \,]1, +\infty[\text{ and } N \in \mathbb{N}. \tag{1.3}$$
The function
$$\xi \in \mathbb{R}^N \to |\xi|^p \in \mathbb{R} \tag{1.4}$$
is strongly convex, i.e., [1]
$$\exists\,\lambda \in \,]0, +\infty[\,: |\xi|^p \geq |\eta|^p + p|\eta|^{p-2}\eta(\xi-\eta) + \lambda|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}, \ \forall \xi, \eta \in \mathbb{R}^N. \tag{1.5}$$
The best constant α_p in (1.5) is given by
$$\alpha_p = \begin{cases} \dfrac{m_p^p + pm_p + p - 1}{(m_p+1)^p}, & \text{if } p \in \,]2, +\infty[, \\ 1, & \text{if } p = 2, \\ \dfrac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}}, & \text{if } p \in \,]1, 2\,[, \end{cases} \tag{1.6}$$
where, if $p \neq 2$, the real number m_p is defined in the following way:

- *if $p \in \,]2, +\infty[\,$, m_p is the unique solution to*
$$\sigma_p(m) = 0, \quad m \in \,]1, +\infty[, \tag{1.7}$$
where the function σ_p is defined by
$$\sigma_p : m \in \,]0, +\infty[\to \sigma_p(m) = m^{p-1} - (p-1)m - (p-2); \tag{1.8}$$

[1] When $p \in \,]1, 2\,[$, the expressions $|\eta|^{p-2}\eta$ and $|\xi - \eta|^2(|\xi| + |\eta|)^{p-2}$ are indeterminate when $\eta = 0$ and when $\xi = \eta = 0$, respectively. By convention, the values of these expressions will be zero in both cases, see Remark 3.1.

- *if* $p\in\,]1,2\,[$*, the real number* m_p *is the unique solution to*

$$\sigma_p(m) = 0, \quad m\in\,]0,1[, \tag{1.9}$$

where the function σ_p *is defined by*

$$\sigma_p : m\in\,]0,+\infty[\to \sigma_p(m) = p(p-1)m^2-(4-p)m^p+p(5-2p)m-pm^{p-1}+(p-2)^2. \tag{1.10}$$

Moreover, one has

$$\frac{1}{2^p} < \alpha_p < \frac{p}{2^{p-1}} < 1, \ \textit{if} \ \ p\in\,]2,+\infty[, \tag{1.11}$$

$$\frac{p(p-1)}{2} < \alpha_p < p-1 < 1 < \frac{p}{2^{p-1}}, \ \textit{if} \ \ p\in\,]1,2\,[. \tag{1.12}$$

As far as the strong monotonicity of the function $|\xi|^{p-2}\xi$ is concerned, our result is the following one.

Theorem 1.2. *Let*

$$p\in\,]1,+\infty[\ \textit{and} \ N \in \mathbb{N}. \tag{1.13}$$

The function

$$\xi \in \mathbb{R}^N \to |\xi|^{p-2}\xi \in \mathbb{R}^N \tag{1.14}$$

is strongly monotone, i.e.

$$\exists\, \mu\in\,]0,+\infty[: \left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta) \geq \mu|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}, \ \forall\xi,\eta\in\mathbb{R}^N. \tag{1.15}$$

The best constant β_p *in* (1.15) *is given by*

$$\beta_p = \begin{cases} \dfrac{1}{2^{p-2}}, & \textit{if } p\in\,]2,+\infty[, \\ 1, & \textit{if } p=2, \\ \dfrac{p-1}{2^{p-2}}, & \textit{if } p\in\,]1,2\,[. \end{cases} \tag{1.16}$$

The contents of this paper is as follows.

In Section 2, Proposition 2.2 and Proposition 2.1, we state some qualitative properties of α_p, the best constant in (1.5), and of m_p, the quantity which is used in formula (1.6) to compute the value of α_p. These results are illustrated by Figure 1 which gives the graphs of the functions α_p and m_p as functions of p.

In Section 3, we make some remarks concerning Theorem 1.1 and Theorem 1.2, see in particular Remarks 3.5, 3.6, and 3.7 on variants of the strong convexity inequality of the function $|\xi|^p$ and of the strong monotonicity inequality of the function $|\xi|^{p-2}\xi$ written in terms of $|\xi-\eta|^p$.

Theorem 1.1 and Theorem 1.2 will be proved in section 4 and in section 5, respectively; a sketch of the proof of Theorem 1.1 is given in subsection 4.1.

Proposition 2.1 and Proposition 2.2 will be proved in section 6 and in section 7, respectively.

2. Behaviour of m_p and α_p with respect to p

In this section we state qualitative properties of α_p, the best constant in (1.5) given by formula (1.6). A necessary step in this direction is the study of qualitative properties of m_p, the solution to the algebraic equation

$$\sigma_p(m) = 0,$$

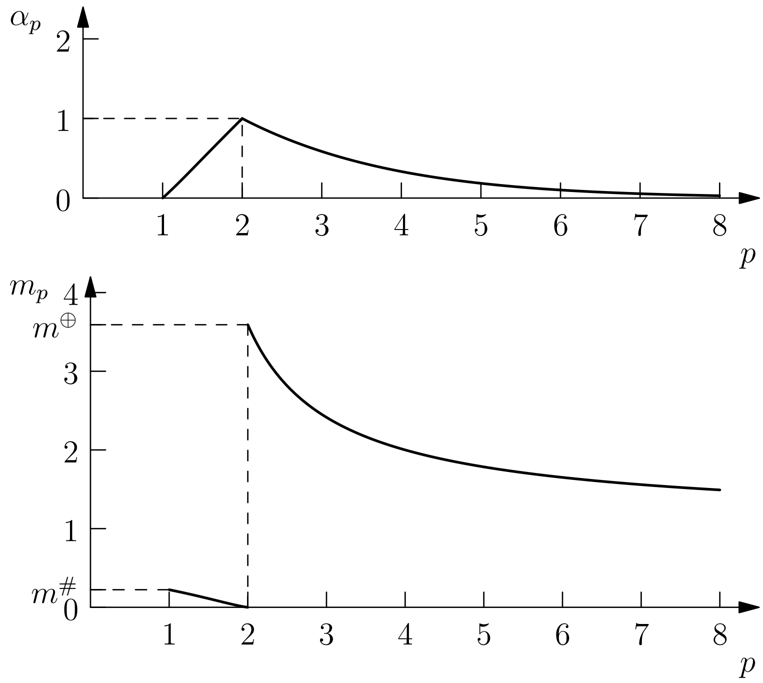

Figure 1: The graphs[2] of the functions α_p and m_p in function of p

or more exactly the unique solution to equation (1.7), with the function σ_p defined by (1.8), if $p\in\,]2,+\infty[$, and to equation (1.9), with the function σ_p defined by (1.10), if $p\in\,]1,2\,[$.

We state the properties of m_p in Proposition 2.1 and the properties of α_p in Proposition 2.2. These properties are illustrated by Figure 1. Proposition 2.1 and Proposition 2.2 will be proved in section 6 and in section 7, respectively.

Proposition 2.1. *Let μ be the function defined by*

$$\mu : p\in\,]1,+\infty[\setminus\{2\} \to \mu(p) = m_p \in\,]0,+\infty[, \tag{2.1}$$

where m_p is the unique solution to equation (1.7)*, with the function σ_p defined by* (1.8)*, if $p\in\,]2,+\infty[$, and to equation* (1.9)*, with the function σ_p defined by* (1.10)*, if $p\in\,]1,2\,[$. Then*

$$\mu(p) = m_p \to 1, \text{ as } p \to +\infty; \tag{2.2}$$

$$\mu(p) = m_p \to m^{\oplus}, \text{ as } p \to 2^+, \tag{2.3}$$

where $m^{\oplus}$ is the unique solution to

$$\rho^{\oplus}(m) = 0, \quad m\in\,]1,+\infty[, \tag{2.4}$$

with the function $\rho^{\oplus}$ defined by

$$\rho^{\oplus} : m\in\,]0,+\infty[\to m\ln(m) - m - 1; \tag{2.5}$$

[2] Figure 1 suggests that the two functions $p\in\,]1,+\infty[\to \alpha_p$ and $p\in\,]1,+\infty[\to m_p$ seem to be affine in the interval $]1,2\,[$, and even that $\alpha_p = p-1$ and $m_p = m^{\#}(2-p)$. Numerical computations however prove that this is not true and that the maxima of the relative errors are of order 5% and 15%, respectively.
Approximate values of $m^{\oplus}$ and $m^{\#}$ are given by $m^{\oplus} = 3.59112$ and $m^{\#} = 0.22338$.

$$\mu(p) = m_p \to 0, \;\; as\; p \to 2^-; \tag{2.6}$$

$$\mu(p) = m_p \to m^{\#}, \;\; as\; p \to 1^+, \tag{2.7}$$

where $m^{\#}$ is the unique solution to

$$\rho^{\#}(m) = 0, \quad m \in]0,1[, \tag{2.8}$$

with the function $\rho^{\#}$ defined by

$$\rho^{\#} : m \in]0,+\infty[\to -(3m+1)\ln(m) + m^2 + 2m - 3. \tag{2.9}$$

On the other hand, the function μ satisfies

$$\mu \in C^{\infty}(]1,+\infty[\setminus\{2\}), \tag{2.10}$$

with

$$\mu'(p) = \begin{cases} -\dfrac{m_p^{p-1}\ln(m_p) - m_p - 1}{(p-1)(m_p^{p-2}-1)}, \quad if\; p \in]2,+\infty[, \\ -\dfrac{(2p-1)m_p^2 + [1-(4-p)\ln(m_p)]m_p^p + (5-4p)m_p - (1+p\ln(m_p))m_p^{p-1} - 2(2-p)}{2p(p-1)m_p - (4-p)pm_p^{p-1} + p(5-2p) - p(p-1)m_p^{p-2}}, \\ if\; p \in]1,2[. \end{cases} \tag{2.11}$$

Note that the function μ is not defined at $p = 2$, since m_2 is not defined.
Note also that the function μ is discontinuous at $p = 2$, since $\mu(p) \to m^{\oplus}$ as $p \to 2^+$ (see (2.3)), while $\mu(p) \to 0$ as $p \to 2^-$ (see (2.6)). This result is unexpected.

Proposition 2.2. *Let α be the function defined by*

$$\alpha : p \in]1,+\infty[\to \alpha(p) = \alpha_p, \tag{2.12}$$

where α_p is defined by (1.6)*. Then*

$$\alpha(p) = \alpha_p \to 0, \;\; as\; p \to +\infty, \tag{2.13}$$

$$\alpha(p) = \alpha_p \to \alpha_2 = 1, \;\; as\; p \to 2, \tag{2.14}$$

$$\alpha(p) = \alpha_p \to 0, \;\; as\; p \to 1. \tag{2.15}$$

On the other hand, the function α satisfies

$$\alpha(p) \in C^0(]1,+\infty[) \cap C^{\infty}(]1,+\infty[\setminus\{2\}), \tag{2.16}$$

with

$$\alpha'(p) = \begin{cases} \dfrac{m_p^p(\ln(m_p) - \ln(m_p+1)) + m_p(1 - p\ln(m_p+1)) + 1 - (p-1)\ln(m_p+1)}{(m_p+1)^p}, \\ if\; p \in]2,+\infty[, \\ \dfrac{m_p^p(\ln(m_p) - \ln(m_p+1)) - m_p(1 - p\ln(m_p+1)) + 1 - (p-1)\ln(m_p+1)}{(m_p-1)^2(m_p+1)^{p-2}}, \\ if\; p \in]1,2[, \end{cases} \tag{2.17}$$

where m_p is the unique solution to equation (1.7)*, with the function σ_p defined by* (1.8)*, if $p \in]2,+\infty[$, and to equation* (1.9)*, with the function σ_p defined by* (1.10)*, if $p \in]1,2[$. Moreover,*

$$\alpha'(p) < 0, \quad if\; p \in]2,+\infty[. \tag{2.18}$$

In contrast to (2.18), we were not able to prove that the function α is increasing if $p \in]1,2[$, even if Figure 1 strongly suggests it.

To supplement the regularity result (2.16), note that the function α does not belong to $C^1(]1,+\infty[)$. Indeed, if $p \to 2^-$, then $\alpha'(p) \to 1$ in view of (2.17) and (2.6); while, if $p \to 2^+$, then $\alpha'(p) \to 1 - \ln(m^\oplus + 1)$ in view of (2.17), (2.3), (2.4), and (2.5) (hint: use $\ln m^\oplus = \frac{m^\oplus+1}{m^\oplus}$), where $m^\oplus > 1$ in view of (2.4) and (2.5)).

3. Remarks on Theorem 1.1 and Theorem 1.2

Remark 3.1. (Conventions for two indeterminate expressions) When $p \in]1,2[$, the expressions $|\eta|^{p-2}\eta$ and $|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}$ are indeterminate when $\eta = 0$ and when $\xi = \eta = 0$, respectively. In this remark, see also footnote [1], we make conventions to solve this problem.

When $p \in]1,2[$, since $||\eta|^{p-2}\eta| = |\eta|^{p-1}$ if $\eta \neq 0$, we shall make the convention that

$$|\eta|^{p-2}\eta = 0, \text{ if } \eta = 0.$$

Similarly, when $p \in]1,2[$, since

$$0 \le |\xi-\eta|^2\,(|\xi|+|\eta|)^{p-2} = \frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} \le \frac{(|\xi|+|\eta|)^2}{(|\xi|+|\eta|)^{2-p}} = (|\xi|+|\eta|)^p\,, \text{ if } |\xi|+|\eta| > 0,$$

we shall make the convention that

$$|\xi-\eta|^2\,(|\xi|+|\eta|)^{p-2} = \frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} = 0, \text{ if } \xi = \eta = 0. \qquad \square$$

Remark 3.2. (Another characterization of the best constant α_p) The proof of Theorem 1.1 (see section 4, and more specifically subsection 4.2) shows that the best constant α_p in (1.5) satisfies $\alpha_p = A_p$, where the real number A_p is defined by

$$A_p = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \frac{|\xi|^p - |\eta|^p - p|\eta|^{p-2}\eta(\xi-\eta)}{|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}}. \qquad \square \tag{3.1}$$

Remark 3.3. (Explicit values of α_p when $p = 2,3,4$) Formula (1.6) gives the value of the best constant α_p in terms of the solution m_p to equation (1.7) when $p \in]2,+\infty[$, with the function σ_p given by (1.8), and to equation (1.9) when $p \in]1,2[$, with the function σ_p given by (1.10).

This result is explicit when $p = 3$ and $p = 4$; indeed in these cases one has

$$\begin{cases} \sigma_3(m) = m^2 - 2m - 1, \quad m_3 = 1+\sqrt{2}, \quad \alpha_3 = 2-\sqrt{2}, \\ \sigma_4(m) = m^3 - 3m - 2, \quad m_4 = 2, \quad \alpha_4 = \dfrac{1}{3}. \end{cases} \tag{3.2}$$

Remember also that when $p = 2$, $\alpha_2 = 1$ (see (1.6)), and that m_2 is not defined. $\square$

Remark 3.4. (About equations (1.7) and (1.9)) Consider the equation

$$\sigma_p(m) = 0, \quad m \in]0, +\infty[.$$

When $p \in]2, +\infty[$ and the function σ_p is given by (1.8), this equation has a unique solution which actually belongs to $]1, +\infty[$ (see (4.21)); while, when $p \in]1, 2[$ and the function σ_p is given by (1.10), this equation has exactly two solutions, one of them which belongs to $]0, 1[$ and the other one which is equal to 1 (see (4.39), (4.40) and (4.41)). □

Remark 3.5. (Variant of the strong convexity of the function $|\xi|^p$ in terms of $|\xi - \eta|^p$ when $p \in]2, +\infty[$) In this remark we only consider the case where

$$p \in]2, +\infty[.$$

Since in this case

$$|\xi - \eta|^2 \left(|\xi| + |\eta|\right)^{p-2} \geq |\xi - \eta|^2 |\xi - \eta|^{p-2} = |\xi - \eta|^p \quad \forall \xi, \eta \in \mathbb{R}^N, \tag{3.3}$$

we deduce from Theorem 1.1 that

$$|\xi|^p \geq |\eta|^p + p|\eta|^{p-2}\eta(\xi - \eta) + \alpha_p |\xi - \eta|^p, \quad \forall \xi, \eta \in \mathbb{R}^N, \tag{3.4}$$

where α_p is the best constant in (1.5), namely the constant given by (1.6).

Actually, α_p given by (1.6) turns to be also the best constant in (3.4).

Indeed, if one follows along the lines of the proof of Theorem 1.1 to characterize the best constant γ_p in (3.4), one defines successively the functions $\widehat{\gamma}_p(\xi, \eta)$, $\overline{\gamma}_p(m, \tau)$, and finally $\widetilde{\gamma}_p(m)$. This latest function turns to be identical to $\widetilde{\alpha}_p(m)$, which implies that the best constant γ_p in (3.4) is nothing else but the best constant α_p in (1.5). □

Remark 3.6. (No variant of the strong convexity of the function $|\xi|^p$ in terms of $|\xi - \eta|^p$ when $p \in]1, 2[$) Remark 3.5 was concerned with the case $p \in]2, +\infty[$. In this remark we claim that in the case where

$$p \in]1, 2[,$$

it does not exists any $\lambda \in]0, +\infty[$ such that

$$|\xi|^p \geq |\eta|^p + p|\eta|^{p-2}\eta(\xi - \eta) + \lambda |\xi - \eta|^p, \quad \forall \xi, \eta \in \mathbb{R}^N. \tag{3.5}$$

To prove this claim, assume that (3.5) holds true for some $\lambda \in \mathbb{R}$. Take $\eta \in \mathbb{R}^N$ with $|\eta| = 1$ and $\xi = (1 + \varepsilon)\eta$ with $\varepsilon \in]0, 1[$ in (3.5). Then one has

$$(1 + \varepsilon)^p \geq 1 + p\varepsilon + \lambda \varepsilon^p, \quad \forall \varepsilon \in]0, 1[. \tag{3.6}$$

On the other side,

$$(1 + \varepsilon)^p = 1 + \int_1^{1+\varepsilon} p s^{p-1} ds \leq 1 + \varepsilon p (1 + \varepsilon)^{p-1}, \quad \forall \varepsilon \in]0, 1[. \tag{3.7}$$

Combining (3.6) and (3.7) implies that

$$p\varepsilon + \lambda \varepsilon^p \leq \varepsilon p (1 + \varepsilon)^{p-1}, \quad \forall \varepsilon \in]0, 1[,$$

namely,

$$\lambda \leq \varepsilon^{2-p} p \frac{(1 + \varepsilon)^{p-1} - 1}{\varepsilon}, \quad \forall \varepsilon \in]0, 1[. \tag{3.8}$$

Passing to the limit in (3.8) as ε tends to zero and recalling that

$$\frac{(1+\varepsilon)^{p-1}-1}{\varepsilon} \to p-1, \text{ as } \varepsilon \to 0,$$

and that $p\in]1,2[$ implies that $\lambda \leq 0$. This proves the claim. $\square$

Remark 3.7. (Variant of the strong monotonicity of the function $|\xi|^{p-2}\xi$ in terms of $|\xi-\eta|^p$ when $p\in]2,+\infty[$) In this remark, which is similar to Remark 3.5, we only consider the case where

$$p\in]2,+\infty[.$$

Using (3.3) we deduce from Theorem 1.2 that

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta) \geq \beta_p|\xi-\eta|^p, \quad \forall \xi,\eta\in\mathbb{R}^N, \tag{3.9}$$

where β_p is the best constant in (1.15), namely $\beta_p = \frac{1}{2^{p-2}}$ (see (1.16)).

If we denote by δ_p the best constant in (3.9), $\beta_p = \frac{1}{2^{p-2}}$ turns to be also the best constant δ_p in (3.9).

Indeed, following along the lines of the proof of Theorem 1.2 in order to characterize δ_p, we define Δ_p by

$$\Delta_p = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \frac{\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta)}{|\xi-\eta|^p} = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \frac{\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta)}{(|\xi-\eta|^2)^{\frac{p}{2}}}. \tag{3.10}$$

As in (5.2), (5.4), and (5.5) below, we obtain

$$\Delta_p = \min\left\{1,\ \frac{1}{2^{p-2}},\ \inf_{m\in]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]} \overline{\delta}_p(m,\tau)\right\}, \tag{3.11}$$

where the function $\overline{\delta}_p$ is defined by

$$\overline{\delta}_p : (m,\tau)\in (]0,+\infty[\setminus\{1\})\times[-1,1] \to \overline{\delta}_p(m,\tau) = \frac{m^p - m^{p-1}\tau - m\tau + 1}{(m^2-2m\tau+1)^{\frac{p}{2}}}.$$

An easy computation shows that

$$\frac{\partial\overline{\delta}_p}{\partial\tau}(m,\tau) = \frac{m}{(m^2-2m\tau+1)^{\frac{p}{2}+1}}\, h_p(m,\tau),\ \forall (m,\tau)\in([0,+\infty[\setminus\{1\})\times[-1,1], \tag{3.12}$$

where the function h_p is defined by

$$\begin{aligned} h_p &: (m,\tau)\in [0,+\infty[\times[-1,1] \\ &\to h_p(m,\tau) = (2-p)(m^{p-1}+m)\tau + (m^p+1)(p-1) - m^2 - m^{p-2}. \end{aligned}$$

Since $p\in]2,+\infty[$, one has

$$\frac{\partial h_p}{\partial\tau}(m,\tau) = (2-p)(m^{p-1}+m) \leq 0, \quad \forall (m,\tau)\in [0,+\infty[\times[-1,1]. \tag{3.13}$$

On the other hand, one has

$$\begin{cases} h_p(m,1) = (p-1)m^p - (p-2)m^{p-1} - m^{p-2} - m^2 - (p-2)m + p - 1 \\ = (m-1)[m^{p-2} - ((p-1)m+1) - (m+p-1)], \quad \forall m \in [0,+\infty[. \end{cases} \tag{3.14}$$

Since $p \in]2,+\infty[$ and $m \in [0,+\infty[$, one has

$$m \geq 1 \iff m^{p-2} \geq 1, \tag{3.15}$$

which implies that

$$\begin{cases} m^{p-2}((p-1)m+1) - (m+p-1) \geq ((p-1)m+1) - (m+p-1) \\ = (p-2)(m-1) \geq 0, \quad \forall m \in [1,+\infty[, \\ m^{p-2}((p-1)m+1) - (m+p-1) \leq ((p-1)m+1) - (m+p-1) \\ = (p-2)(m-1) \leq 0, \quad \forall m \in [0,1]. \end{cases} \tag{3.16}$$

From (3.13), (3.14), (3.15), and (3.16), one deduces that

$$h_p(m,\tau) \geq 0, \quad \forall (m,\tau) \in [0,+\infty[\times[-1,1]. \tag{3.17}$$

Combining (3.12) and (3.17) implies that $\widetilde{\delta}_p(m) = \inf_{\tau\in[-1,1]} \overline{\delta}_p(m,\tau)$ is reached at $\tau = -1$, i.e.,

$$\widetilde{\delta}_p(m) = \inf_{\tau\in[-1,1]} \overline{\delta}_p(m,\tau) = \overline{\delta}_p(m,-1) = \frac{m^{p-1}+1}{(m+1)^{p-1}}, \quad \forall m \in]0,+\infty[\setminus\{1\}. \tag{3.18}$$

In view of the definition (5.7) below of $\widetilde{\beta}_p(m)$, the latest expression is nothing else but $\widetilde{\beta}_p(m)$ and therefore

$$\widetilde{\delta}_p(m) = \widetilde{\beta}_p(m), \quad \forall m \in]0,+\infty[\setminus\{1\}. \tag{3.19}$$

Moreover, in view of (5.11), one has

$$\inf_{m\in\,]0,+\infty[\setminus\{1\}} \widetilde{\beta}_p(m) = \frac{1}{2^{p-2}}. \tag{3.20}$$

Combining (3.10), (3.11), (3.18), (3.19), and (3.20) implies that

$$\Delta_p = \min\left\{1, \frac{1}{2^{p-2}}, \frac{1}{2^{p-2}}\right\} = \frac{1}{2^{p-2}},$$

which proves that the best constant δ_p in (3.9) is given by

$$\delta_p = \frac{1}{2^{p-2}} = \beta_p. \tag{3.21}$$

Observe also that a proof similar to the one used in Remark 3.6 shows that in the case $p \in]1,2[$, it does not exists any $\mu \in]0,+\infty[$ such that

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta) \geq \mu|\xi-\eta|^p, \quad \forall \xi,\eta \in \mathbb{R}^N. \qquad \square$$

Remark 3.8. (Results already known on the best constants) In this remark we quote the partial results (for the best constants for variants of the strong monotonicity inequality (1.2)) which appear in the papers [1] and [2].

In Lemma 3.2 of [2], the authors proved that if $p\in]2,+\infty[$, the best constant in

$$\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\geq\nu|\xi-\eta|^p,\quad\forall\xi,\eta\in\mathbb{R}^N,\tag{3.22}$$

is given by $\nu=\dfrac{1}{2^{p-2}}$. We prove this result in Remark 3.7 above (see the third paragraph of Remark 3.7 and (3.21)).

On the other hand, in Lemma A.2 of [1] and in Lemma 3.1 of [2], the authors proved that if $p\in]1,2]$, the best constant in the variant of the strong monotonicity of $|\xi|^{p-2}\xi$ inequality (1.15) given by

$$\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\geq\nu\frac{|\xi-\eta|^2}{|\xi|^{2-p}+|\eta|^{2-p}},\quad\forall\xi,\eta\in\mathbb{R}^N,\tag{3.23}$$

is $\nu=\min\{1,2(p-1)\}$ if $p\in]1,2[$, and $\nu=2$ if $p=2$. We prove a similar result in Theorem 1.2 (see (1.15) and the last two lines of (1.16)). Note indeed that the right-hand sides of (1.15) (which is nothing else but (1.2)) and (3.23) are different, even if the coefficients

$$(|\xi|+|\eta|)^{p-2}=\frac{1}{(|\xi|+|\eta|)^{2-p}}\qquad\text{and}\qquad\frac{1}{|\xi|^{2-p}+|\eta|^{2-p}}$$

which appear in these two strong monotonicity inequalities are equivalent for each fixed $p\in]1,2[$, which implies that the results are similar but different. □

4. Proof of Theorem 1.1 (Strong convexity of $|\xi|^p$)

In this section we prove Theorem 1.1. We begin with a sketch of the proof to present its main steps.

4.1. Sketch of the proof of Theorem 1.1

When $p\in]1,+\infty[$, we define the real number A_p by

$$A_p=\inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta}\frac{|\xi|^p-|\eta|^p-p|\eta|^{p-2}\eta(\xi-\eta)}{|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}};\tag{4.1}$$

then A_p is non negative, since the function $|\xi|^p$ is convex. It is easily proved in subsection 4.2 that this function is strongly convex if and only if $A_p>0$, and that in this case one has $A_p=\alpha_p$, where α_p is the best constant in the strong convexity inequality (1.1). We therefore study the minimization problem (4.1) in order to prove that $A_p>0$.

In subsection 4.3, the minimization problem (4.1) is transformed into a simpler minimization problem. One considers in (4.1) first the case where $\eta=0$, and then the case $\eta\neq0$. When $\eta=0$, the infimum in ξ is equal to 1, while when $\eta\neq0$, the infimum reduces by homogeneity to the case where $|\eta|=1$. Setting $|\xi|=m$ with $m\in]0,+\infty[$ and $\xi\eta=m\tau$, where τ is the cosine of the angle between the vectors ξ

and η, the infimum in (ξ,η) when $\eta \neq 0$ is transformed into the infimum in (m,τ) for $m \in]0,+\infty[$ and $\tau \in [-1,1]$ of a certain function $\overline{\alpha}_p(m,\tau)$. It is easy to minimize this function $\overline{\alpha}_p(m,\tau)$ in τ, and to prove that the infimum is given by $\overline{\alpha}_p(m,-1)$ when $p \in]2,+\infty[$ and by $\overline{\alpha}_p(m,1)$ when $p \in]1,2[$.

The next step consists in minimizing these two values with respect to m. This is not difficult when $p \in]2,+\infty[$ and this is done in subsection 4.4. The proof is more tedious when $p \in]1,2[$ and this is done in subsection 4.5. In both cases, the minimum is reached at a unique point m_p which is the solution of an algebraic equation $\sigma_p(m) = 0$. The number A_p defined by (4.1) then appears as an algebraic expression of m_p, and this expression is proved to be strictly positive. Therefore the function $|\xi|^p$ is strongly convex and $A_p = \alpha_p$, so Theorem 1.1 is proved.

4.2. A minimization problem for computing the best constant α_p

Define the real number A_p by (see (4.1))

$$A_p = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \frac{|\xi|^p - |\eta|^p - p|\eta|^{p-2}\eta(\xi-\eta)}{|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}}. \tag{4.2}$$

Since the function $|\xi|^p$ is convex, one has $A_p \geq 0$.

Note that when $\xi = \eta$, the strong convexity inequality (1.5) becomes an equality satisfied by any $\lambda \in]0,+\infty[$; this holds true even if $\xi = \eta = 0$ in view of the convention made in Remark 3.1. Therefore, the function $|\xi|^p$ is strongly convex if and only if $A_p > 0$. In this case, one has

$$\alpha_p = A_p. \tag{4.3}$$

In the proof below, we will study A_p, prove that A_p is given by formula (1.6), and prove that $A_p > 0$.

When $p = 2$, the proof is straightforward. Indeed, in this case

$$A_2 = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \frac{|\xi|^2 - |\eta|^2 - 2\eta(\xi-\eta)}{|\xi-\eta|^2} = \frac{|\xi-\eta|^2}{|\xi-\eta|^2} = 1, \tag{4.4}$$

which is strictly positive, so that the function $|\xi|^2$ is strongly convex and (1.6) is proved. We shall therefore assume in the rest of this section that

$$p \neq 2. \tag{4.5}$$

4.3. Rewriting formula (4.2) which defines the number A_p

Define the function $\widehat{\alpha}_p$ by

$$\widehat{\alpha}_p : (\xi,\eta) \in \left\{\xi,\eta \in \mathbb{R}^N : \xi \neq \eta\right\} \to \widehat{\alpha}_p(\xi,\eta) = \frac{|\xi|^p - |\eta|^p - p|\eta|^{p-2}\eta(\xi-\eta)}{|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}}. \tag{4.6}$$

Then (4.2) is nothing else but

$$A_p = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \widehat{\alpha}_p(\xi,\eta). \tag{4.7}$$

We first observe that[3]

$$A_p = \inf_{\xi\neq\eta} \widehat{\alpha}_p(\xi,\eta) = \min\left\{ \inf_{\xi\neq\eta,\eta=0} \widehat{\alpha}_p(\xi,\eta),\ \inf_{\xi\neq\eta,\eta\neq0} \widehat{\alpha}_p(\xi,\eta) \right\}.$$

[3] From now on, for making notation lighter, we do not recall that ξ and η belong to $\mathbb{R}^N$.

Since
$$\inf_{\xi\neq\eta,\eta=0}\widehat{\alpha}_p(\xi,\eta)=\inf_{\xi\neq 0}\frac{|\xi|^p}{|\xi|^2|\xi|^{p-2}}=1,$$
one has
$$A_p=\min\Big\{1,\inf_{\xi\neq\eta,\eta\neq 0}\widehat{\alpha}_p(\xi,\eta)\Big\}. \tag{4.8}$$

As far as the second term $\inf_{\xi\neq\eta,\eta\neq 0}\widehat{\alpha}_p(\xi,\eta)$ of the right-hand side of (4.8) is concerned, since $\eta\neq 0$, we divide by $|\eta|^p$ the numerator and denominator of
$$\widehat{\alpha}_p(\xi,\eta)=\frac{|\xi|^p-p|\eta|^{p-2}\eta\xi+p|\eta|^p-|\eta|^p}{(|\xi|^2-2\xi\eta+|\eta|^2)\,(|\xi|+|\eta|)^{p-2}}.$$

Denoting $\frac{1}{|\eta|}\xi$ again by ξ and $\frac{1}{|\eta|}\eta$ again by η, and observing that $\left|\frac{1}{|\eta|}\eta\right|=1$, we obtain that
$$\inf_{\xi\neq\eta,\,\eta\neq 0}\widehat{\alpha}_p(\xi,\eta)=\inf_{\xi\neq\eta,\,|\eta|=1}\frac{|\xi|^p-p\eta\xi+p-1}{(|\xi|^2-2\xi\eta+1)\,(|\xi|+1)^{p-2}},$$

that we rewrite as
$$\begin{cases}\displaystyle\inf_{\xi\neq\eta,\,\eta\neq 0}\widehat{\alpha}_p(\xi,\eta)=\min\Big\{\inf_{|\xi|=1=|\eta|,\,\xi\neq\eta}\frac{|\xi|^p-p\eta\xi+p-1}{(|\xi|^2-2\xi\eta+1)\,(|\xi|+1)^{p-2}}\,,\\ \displaystyle\qquad\qquad\inf_{|\xi|\neq 1,\,|\eta|=1}\frac{|\xi|^p-p\eta\xi+p-1}{(|\xi|^2-2\xi\eta+1)\,(|\xi|+1)^{p-2}}\Big\}.\end{cases} \tag{4.9}$$

Denoting by τ the cosine of the angle between the two vectors $\xi\neq 0$ and $\eta\neq 0$, one has
$$\forall\xi,\eta\in\mathbb{R}^N\quad\exists\tau\in[-1,1]:\eta\xi=|\eta||\xi|\tau,$$
and the first term of the right-hand side of (4.9) is nothing else but
$$\inf_{|\xi|=1=|\eta|,\,\xi\neq\eta}\frac{|\xi|^p-p\eta\xi+p-1}{(|\xi|^2-2\xi\eta+1)\,(|\xi|+1)^{p-2}}=\inf_{\tau\in[-1,1[}\frac{1-p\tau+p-1}{(1-2\tau+1)\,(1+1)^{p-2}}=\frac{p}{2^{p-1}}, \tag{4.10}$$

while for the second term of the right-hand side of (4.9), setting $|\xi|=m\in\,]0,+\infty[\setminus\{1\}$ and
$$\overline{\alpha}_p:(m,\tau)\in([0,+\infty[\setminus\{1\})\times[-1,1]\to\overline{\alpha}_p(m,\tau)=\frac{m^p-pm\tau+p-1}{(m^2-2m\tau+1)\,(m+1)^{p-2}}, \tag{4.11}$$

one has
$$\begin{cases}\displaystyle\inf_{|\xi|\neq 1,\,|\eta|=1}\frac{|\xi|^p-p\eta\xi+p-1}{(|\xi|^2-2\xi\eta+1)\,(|\xi|+1)^{p-2}}\\ \displaystyle=\inf_{m\in\,]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]}\frac{m^p-pm\tau+p-1}{(m^2-2m\tau+1)\,(m+1)^{p-2}}\\ \displaystyle=\inf_{m\in\,]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]}\overline{\alpha}_p(m,\tau).\end{cases} \tag{4.12}$$

Combining (4.8), (4.9), (4.10), (4.11), and (4.12), we have proved that, for $p\neq 2$,
$$A_p=\min\left\{1,\ \frac{p}{2^{p-1}},\inf_{m\in\,]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]}\overline{\alpha}_p(m,\tau)\right\}. \tag{4.13}$$

In the two next subsections 4.4 and 4.5, we shall compute the last term of the right-hand side of (4.13).

We notice that, for any fixed $m\in\,]0,+\infty[\setminus\{1\}$, the function $\tau\in[-1,1]\to\overline{\alpha}_p(m,\tau)$ belongs to $C^\infty([-1,1])$ and satisfies

$$\frac{\partial\overline{\alpha}_p}{\partial\tau}(m,\tau)=\frac{m}{(m+1)^{p-2}\,(m^2-2m\tau+1)^2}(2m^p-pm^2+p-2),\quad \forall\tau\in[-1,1],\tag{4.14}$$

and that, denoting by f_p the function

$$f_p: m\in\,]0,+\infty[\to f_p(m)=2m^p-pm^2+p-2,\tag{4.15}$$

the sign of $\dfrac{\partial\overline{\alpha}_p}{\partial\tau}(m,\tau)$ is the sign of $f_p(m)$.

We will prove (see (4.17) and (4.29) below) that the sign of $f_p(m)$ depends only on p, and that for every $m\in\,]0,+\infty[$ one has $f_p(m)\geq 0$ when $p\in\,]2,+\infty[$ and $f_p(m)\leq 0$ when $p\in\,]1,2\,[$, which implies that

$$\inf_{\tau\in[-1,1]}\overline{\alpha}_p(m,\tau)=\begin{cases}\overline{\alpha}_p(m,-1) & \text{when } p\in\,]2,+\infty[,\\ \overline{\alpha}_p(m,1) & \text{when } p\in\,]1,2\,[.\end{cases}$$

For this reason, we define the function $\widetilde{\alpha}$ by

$$\begin{cases}\widetilde{\alpha}:(p,m)\in(]1,+\infty[\setminus\{2\})\times([0,+\infty[\setminus\{1\})\to\widetilde{\alpha}(p,m)=\widetilde{\alpha}_p(m),\\[2mm] \widetilde{\alpha}_p(m)=\begin{cases}\overline{\alpha}_p(m,-1)=\dfrac{m^p+pm+p-1}{(m+1)^p}, & \text{if } p\in\,]2,+\infty[,\\[2mm] \overline{\alpha}_p(m,1)=\dfrac{m^p-pm+p-1}{(m-1)^2(m+1)^{p-2}}, & \text{if } p\in\,]1,2\,[.\end{cases}\end{cases}\tag{4.16}$$

We will now study the case where $p\in\,]2,+\infty[$ in subsection 4.4, and then the case where $p\in\,]1,2\,[$ in subsection 4.5.

4.4. Minimizing $\widetilde{\alpha}_p(m)$ in m when $p\in\,]2,+\infty[$

In the whole of this subsection, we assume that $p\in\,]2,+\infty[$.

In this case, since the function f_p defined by (4.15) satisfies

$$f_p(1)=0,\quad f_p'(m)=2pm(m^{p-2}-1)\geq 0\iff m\in[1,+\infty[,$$

one has
$$2m^p-pm^2+p-2=f_p(m)\geq 0,\quad \forall m\in\,]0,+\infty[.\tag{4.17}$$

Combining (4.14), (4.17), and the definition (4.16) of $\widetilde{\alpha}_p(m)$ implies that for any $m\in\,]0,+\infty[\setminus\{1\}$ one has

$$\inf_{\tau\in[-1,1]}\overline{\alpha}_p(m,\tau)=\overline{\alpha}_p(m,-1)=\widetilde{\alpha}_p(m).$$

We have proved that

$$\inf_{\tau\in[-1,1]}\overline{\alpha}_p(m,\tau)=\overline{\alpha}_p(m,-1)=\widetilde{\alpha}_p(m)=\frac{m^p+pm+p-1}{(m+1)^p},\quad \forall m\in\,]0,+\infty[\setminus\{1\}.$$

Therefore
$$\inf_{m\in]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]} \overline{\alpha}_p(m,\tau)= \inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\alpha}_p(m). \tag{4.18}$$

Let us now compute the right-hand side of (4.18). For any fixed $p\in]2,+\infty[$, the function $\widetilde{\alpha}_p$ is smooth and an easy computation shows that

$$\frac{\partial\widetilde{\alpha}_p}{\partial m}(m) = \frac{p}{(m+1)^{p+1}}\,\sigma_p(m), \quad \forall m\in]0,+\infty[, \tag{4.19}$$

where the function $\sigma_p(m)$ is defined by

$$\sigma_p(m) = m^{p-1} - (p-1)m - (p-2), \quad \forall m\in]0,+\infty[,$$

and is nothing else but the function defined by (1.8). This function satisfies

$$\sigma_p(0) = -(p-2) < 0, \quad \sigma_p(1) = -2(p-2) < 0, \quad \sigma_p(m) \to +\infty, \text{ as } m \to +\infty,$$
$$\sigma_p'(m) = (p-1)(m^{p-2}-1) \geq 0 \iff m\in]1,+\infty[. \tag{4.20}$$

This implies that
$$\exists\,!\, m_p \in [0,+\infty[\;:\; \sigma_p(m_p) = 0\,; \tag{4.21}$$

moreover, the number m_p defined by (4.21) is such that

$$m_p > 1$$
$$\sigma_p(m) > 0 \iff m > m_p.$$

In view of these results and of (4.19), the minimum of the function $\widetilde{\alpha}_p$ is reached at $m = m_p$, a number greater than 1, which is the unique solution to (1.7), i.e.,

$$\inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\alpha}_p(m) = \widetilde{\alpha}_p(m_p) = \frac{m_p^p + pm_p + p - 1}{(m_p+1)^p}. \tag{4.22}$$

Combining (4.18) and (4.22) yields

$$\inf_{m\in]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]} \overline{\alpha}_p(m,\tau) = \widetilde{\alpha}_p(m_p) = \frac{m_p^p + pm_p + p - 1}{(m_p+1)^p}. \tag{4.23}$$

Then (4.13) and (4.23) imply that

$$A_p = \min\Big\{1,\, \frac{p}{2^{p-1}},\, \frac{m_p^p + pm_p + p - 1}{(m_p+1)^p}\Big\}. \tag{4.24}$$

We claim that
$$\frac{1}{2^p} < \frac{m_p^p + pm_p + p - 1}{(m_p+1)^p} < \frac{p}{2^{p-1}} < 1, \tag{4.25}$$

which implies that
$$A_p = \frac{m_p^p + pm_p + p - 1}{(m_p+1)^p} > \frac{1}{2^p} > 0, \tag{4.26}$$

and therefore that one has $\alpha_p = A_p$, i.e., formula (1.6) if $p \in]2,+\infty[$, with the estimates (1.11).

Let us now prove the claim (4.25).

As far as the proof of the first inequality of (4.25) is concerned, the facts that

$$m_p^p + pm_p + p - 1 > m_p^p,$$

that the function $s\in]-1,+\infty[\to \dfrac{s}{s+1}$ is increasing, and that m_p is greater than 1 imply that

$$\frac{m_p^p + pm_p + p - 1}{(m_p+1)^p} > \Big(\frac{m_p}{m_p+1}\Big)^p > \Big(\frac{1}{1+1}\Big)^p = \frac{1}{2^p}. \tag{4.27}$$

As far as the proof of the second inequality of (4.25) is concerned, the fact that the minimum of the function $\widetilde{\alpha}_p$ is reached at $m = m_p$ implies that

$$\frac{m_p^p + pm_p + p - 1}{(m_p+1)^p} = \widetilde{\alpha}_p(m_p) = \inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\alpha}_p(m_p) < \widetilde{\alpha}_p(1) = \frac{p}{2^{p-1}}.$$

As far as the proof of the last inequality of the claim (4.25) is concerned, one has to prove that

$$(p-1)\ln(2) - \ln(p) > 0, \quad \text{if } p\in]2,+\infty[.$$

Defining
$$v : p \in [2,+\infty[\to v(p) = (p-1)\ln(2) - \ln(p), \tag{4.28}$$

the latest inequality is equivalent to $v(p) > 0$, if $p\in]2,+\infty[$.

This inequality holds true since

$$v(2) = 0, \quad v'(p) = \ln(2) - \frac{1}{p} > 0 \iff p\ln(2) > 1,$$

and since the latest inequality is satisfied for any $p\in]2,+\infty[$, since $2\ln(2) > 1$, or equivalently $2^2 > e$.

This completes the proof of the claim (4.25). □

Theorem 1.1 is proved when $p\in]2,+\infty[$.

4.5. Minimizing $\widetilde{\alpha}_p(m)$ in m when $p\in]1,2[$

In the whole of this subsection, we assume that $p\in]1,2[$.

This case is different from the previous case where $p\in]2,+\infty[$. Even if the proof follows along the same lines, there are more technical difficulties.

In this case, since the function f_p defined by (4.15) satisfies

$$f_p(1) = 0, \quad f_p'(m) = 2pm(m^{p-2} - 1) \geq 0 \iff m \in [0,1],$$

one has
$$2m^p - pm^2 + p - 2 = f_p(m) \leq 0, \quad \forall m\in]0,+\infty[. \tag{4.29}$$

Combining (4.14), (4.29), and the definition (4.16) of $\widetilde{\alpha}_p(m)$ implies that for any $m\in]0,+\infty[\setminus\{1\}$ one has

$$\inf_{\tau\in[-1,1]} \overline{\alpha}_p(m,\tau) = \overline{\alpha}_p(m,1) = \widetilde{\alpha}_p(m).$$

We have proved that

$$\inf_{\tau\in[-1,1]} \overline{\alpha}_p(m,\tau) = \overline{\alpha}_p(m,1) = \widetilde{\alpha}_p(m) = \frac{m^p - pm + p - 1}{(m-1)^2(m+1)^{p-2}}, \quad \forall m\in]0,+\infty[\setminus\{1\}.$$

Therefore,
$$\inf_{m\in]0,+\infty[\setminus\{1\},\, \tau\in[-1,1]} \overline{\alpha}_p(m,\tau) = \inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\alpha}_p(m). \tag{4.30}$$

Let us now compute the right-hand side of (4.30). For any fixed $p \in]1,2[$, the function $\widetilde{\alpha}_p$ is smooth in $[0,+\infty[\setminus\{1\}$ and an easy (but tedious) computation shows that

$$\frac{\partial\widetilde{\alpha}_p}{\partial m}(m) = \frac{1}{(m-1)^3(m+1)^{p-1}}\sigma_p(m), \quad \forall m\in]0,+\infty[\setminus\{1\}, \tag{4.31}$$

where the function σ_p is defined by

$$\sigma_p(m) = p(p-1)m^2 - (4-p)m^p + p(5-2p)m - pm^{p-1} + (p-2)^2$$

and is nothing else but the function defined by (1.10). Note that $\sigma_p(1) = 0$. It can be proved (see Lemma 4.1 below) that

$$\exists\,!\, m_p \in [0,+\infty[\setminus\{1\} \ : \ \sigma_p(m_p) = 0;$$

moreover, the number m_p is such that

$$0 < m_p < \frac{3-p}{4-p} < 1,$$

$$\sigma_p(m) < 0 \iff m \in]m_p, 1[, \quad \sigma_p(m) > 0 \iff m \in]0, m_p[\cup]1, +\infty[.$$

Note that, in contrast to the case where $p \in]2,+\infty[$, the sign of $\frac{\partial \widetilde{\alpha}_p}{\partial m}(m)$ is no more the sign of $\sigma_p(m)$, but the sign of $(m-1)\sigma_p(m)$. Therefore

$$\frac{\partial \widetilde{\alpha}_p}{\partial m}(m) < 0 \iff m \in]0, m_p[, \quad \frac{\partial \widetilde{\alpha}_p}{\partial m}(m) > 0 \iff m \in]m_p, +\infty[\setminus\{1\}. \tag{4.32}$$

Note also that, using Taylor expansion of the numerator around the point $m = 1$, which is licit since the function s^p is C^∞ at the point $s = 1$, one has

$$\widetilde{\alpha}_p(m) = \frac{m^p - pm + p - 1}{(m-1)^2(m+1)^{p-2}} \to \frac{p(p-1)}{2^{p-1}}, \text{ as } m \to 1, \tag{4.33}$$

which proves that the function $\widetilde{\alpha}_p$ is continuous at $m = 1$. In view of (4.32) and (4.33), the minimum of the function $\widetilde{\alpha}_p$ is reached at $m = m_p$, a positive number smaller than 1, which is the unique solution to (1.9), i.e.,

$$\inf_{m \in]0,+\infty[\setminus\{1\}} \widetilde{\alpha}_p(m) = \widetilde{\alpha}_p(m_p) = \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}}. \tag{4.34}$$

Combining (4.30) and (4.34) yields

$$\inf_{m \in]0,+\infty[\setminus\{1\},\, \tau \in [-1,1]} \overline{\alpha}_p(m,\tau) = \widetilde{\alpha}_p(m_p) = \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}}. \tag{4.35}$$

Then (4.13) and (4.35) imply that

$$A_p = \min\left\{1, \ \frac{p}{2^{p-1}}, \ \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}}\right\}. \tag{4.36}$$

We claim that

$$\frac{p(p-1)}{2} < \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}} < p - 1 < 1 < \frac{p}{2^{p-1}}, \tag{4.37}$$

which implies that

$$A_p = \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}} > \frac{p(p-1)}{2} > 0, \tag{4.38}$$

and therefore that one has $\alpha_p = A_p$, i.e., formula (1.6) if $p \in]1, 2[$, with the estimates (1.12).

Let us now prove the claim (4.37).

As far as the proof of the first inequality of (4.37) is concerned, writing the second order Taylor expansion of the function s^p around the point $s = 1$, which is licit since the function s^p is C^∞ at the point $s = 1$, one has

$$\forall s \in]0,1[, \quad \exists\, c(s) \in]s,1[\ :\ s^p = 1 + p(s-1) + \frac{p(p-1)}{2}(c(s))^{p-2}(s-1)^2.$$

Taking $s = m_p$ yields

$$\exists\, c(m_p) \in]m_p, 1[\ :\ \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}} = \frac{p(p-1)}{2}\left(\frac{c(m_p)}{m_p+1}\right)^{p-2},$$

which implies the first inequality of (4.37) since

$$0 < \frac{c(m_p)}{m_p+1} < 1 \text{ and } p - 2 < 0.$$

As far as the proof of the second and third inequalites of (4.37) is concerned, (4.34) implies that

$$\frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}} = \widetilde{\alpha}_p(m_p) = \inf_{m \in]0,+\infty[\setminus\{1\}} \widetilde{\alpha}_p(m_p) \le \widetilde{\alpha}_p(0) = p - 1 < 1.$$

As far as the proof of the last inequality of (4.37) is concerned, one has to prove that

$$(p-1)\ln(2) - \ln(p) < 0, \quad \text{if } p \in]1,2[.$$

Defining v as in (4.28) by

$$v : p \in [1,2] \to v(p) = (p-1)\ln(2) - \ln(p),$$

the latest inequality is equivalent to $v(p) < 0$, if $p \in]1,2[$.

This inequality holds true since $v(1) = v(2) = 0$ and since v is convex in $]1,2[$ (indeed $v''(p) = \frac{1}{p^2}$).

This completes the proof of the claim (4.37). □

Theorem 1.1 is proved when $p \in]1,2[$.

In this subsection, we have used the following lemma.

Lemma 4.1. *Let $p \in]1,2[$ and σ_p be the function defined by* (1.10)*, i.e.,*

$$\sigma_p : m \in]0,+\infty[\to \sigma_p(m) = p(p-1)m^2 - (4-p)m^p + p(5-2p)m - pm^{p-1} + (p-2)^2.$$

Then

$$\sigma_p(1) = 0, \tag{4.39}$$

$$\exists!\, m_p \in [0,+\infty[\setminus\{1\}\ :\ \sigma_p(m_p) = 0. \tag{4.40}$$

Moreover, the number m_p defined by (4.40) *is such that*

$$0 < m_p < \frac{3-p}{4-p} < 1, \tag{4.41}$$

$$\sigma_p(m) < 0 \iff m \in]m_p, 1[, \quad \sigma_p(m) > 0 \iff m \in]0, m_p[\cup]1, +\infty[. \tag{4.42}$$

Proof. Equality (4.39) is straightforward.

An easy computation shows that, for every $m\in\,]0,+\infty[$,

$$\sigma_p'(m) = 2p(p-1)m - (4-p)pm^{p-1} + p(5-2p) - p(p-1)m^{p-2},$$

$$\sigma_p''(m) = 2p(p-1) - (4-p)p(p-1)m^{p-2} + p(p-1)(2-p)m^{p-3},$$

$$\sigma_p'''(m) = p(p-1)(2-p)m^{p-4}\left((4-p)m-(3-p)\right).$$

Note that
$$\sigma_p'''(m) > 0 \iff m > \frac{3-p}{4-p}. \tag{4.43}$$

Since $\sigma_p''(m) \to +\infty$ as $m \to 0^+$, $\sigma_p''(1) = 0$, and $\dfrac{3-p}{4-p} \in\,]0,1[$, property (4.43) implies that there exists a unique $\overline{\overline{m}}_p \in \left]0,\dfrac{3-p}{4-p}\right[$ such that

$$\begin{cases} \sigma_p''(\overline{\overline{m}}_p) = 0, \quad \sigma_p''(m) < 0 \iff m\in\,]\overline{\overline{m}}_p,1[, \\ \qquad\qquad\qquad\quad \sigma_p''(m) > 0 \iff m\in\,]0,\overline{\overline{m}}_p[\cup]1,+\infty[. \end{cases} \tag{4.44}$$

Since
$$\sigma_p'(m) \to -\infty, \text{ as } m \to 0^+, \text{ and } \sigma_p'(1) = 0, \tag{4.45}$$

property (4.44) implies that there exists a unique $\overline{m}_p \in \left]0,\overline{\overline{m}}_p\right[$ such that

$$\sigma_p'(\overline{m}_p)=0,\ \sigma_p'(m)<0 \iff m\in\,]0,\overline{m}_p[,\ \sigma_p'(m)>0 \iff m\in\,]\overline{m}_p,+\infty[\setminus\{1\}. \tag{4.46}$$

Finally, since
$$\sigma_p(0) = (p-2)^2 > 0 \text{ and } \sigma_p(1) = 0, \tag{4.47}$$

property (4.46) implies that there exists a unique $m_p\in\,]0,\overline{m}_p[$ such that

$$\sigma_p(m_p) = 0,\ \sigma_p(m) < 0 \iff m\in\,]m_p,1[,\ \sigma_p(m) > 0 \iff m\in\,]0,m_p[\cup]1,+\infty[.$$

Moreover,
$$0 < m_p < \overline{m}_p < \overline{\overline{m}}_p < \frac{3-p}{4-p} < 1. \tag{4.48}$$

The proof of the Lemma 4.1 is complete. □

5. Proof of Theorem 1.2 (Strong monotonicity of the function $|\xi|^{p-2}\xi$)

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1 and is even easier. For this reason here we give only a sketch of it.

When $p\in\,]1,+\infty[$, we define the real number B_p by

$$B_p = \inf_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \frac{\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta)}{|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}}, \tag{5.1}$$

and we study B_p as we did in Section 4 for A_p.

Of course, when $p = 2$, the strong monotonicity inequality (1.15) holds true with the best constant $B_p = 1$. We shall therefore assume in what follows that

$$p \neq 2$$

and we shall later distinguish between the cases $p\in\,]2,+\infty[$ and $p\in\,]1,2\,[$.

Define the function $\widehat{\beta}_p$ by

$$\widehat{\beta}_p : (\xi,\eta) \in \left\{\xi,\eta \in \mathbb{R}^N : \xi \neq \eta\right\} \to \widehat{\beta}_p(\xi,\eta) = \frac{\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta)}{|\xi-\eta|^2(|\xi|+|\eta|)^{p-2}}. \tag{5.2}$$

Then (5.1) is nothing else but

$$\begin{cases} B_p = \inf\limits_{\xi,\eta\in\mathbb{R}^N:\,\xi\neq\eta} \widehat{\beta}_p(\xi,\eta) = \min\left\{\inf\limits_{\xi\neq\eta,\eta=0} \widehat{\beta}_p(\xi,\eta),\ \inf\limits_{\xi\neq\eta,\eta\neq0} \widehat{\beta}_p(\xi,\eta)\right\} \\ \qquad = \min\left\{1,\ \inf\limits_{\xi\neq\eta,\eta\neq0} \widehat{\beta}_p(\xi,\eta)\right\}. \end{cases} \tag{5.3}$$

Arguing as in the proof of Theorem 1.1 implies that (compare with (4.13))

$$B_p = \min\left\{1,\ \frac{1}{2^{p-2}},\ \inf_{m\in\,]0,+\infty[\setminus\{1\},\,\tau\in[-1,1]} \overline{\beta}_p(m,\tau)\right\}, \tag{5.4}$$

where the function $\overline{\beta}_p$ is defined by

$$\overline{\beta}_p : (m,\tau) \in (]0,+\infty[\setminus\{1\}) \times [-1,1] \to \overline{\beta}_p(m,\tau) = \frac{m^p - m^{p-1}\tau - m\tau + 1}{(m^2-2m\tau+1)\,(m+1)^{p-2}}. \tag{5.5}$$

As in the proof of Theorem 1.1, we shall compute in the two next subsections 5.1 and 5.2 the last term of the right-hand side of (5.4), and firstly

$$\inf_{\tau\in[-1,1]} \overline{\beta}_p(m,\tau), \quad \forall m\in\,]0,+\infty[\setminus\{1\}.$$

To this aim, an easy computation shows that

$$\frac{\partial\overline{\beta}_p}{\partial\tau}(m,\tau) = \frac{m}{(m+1)^{p-2}\,(m^2-2m\tau+1)^2}\, g_p(m), \tag{5.6}$$

for all $(m,\tau) \in (]0,+\infty[\setminus\{1\}) \times [-1,1]$, where the function g_p is defined by

$$g_p : m \in]0,+\infty[\to g_p(m) = (m^{p-2}-1)(m^2-1).$$

Therefore, the sign of $\dfrac{\partial\overline{\beta}_p}{\partial\tau}(m,\tau)$ is the sign of $g_p(m)$.

We will prove (see (5.8) and (5.12) below) that the sign of $g_p(m)$ depends only on p, and that for every $m \in]0,+\infty[$ one has

$$g_p(m) \geq 0 \text{ when } p\in\,]2,+\infty[, \quad g_p(m) \leq 0 \text{ when } p\in\,]1,2\,[,$$

which implies that

$$\inf_{\tau\in[-1,1]} \overline{\beta}_p(m,\tau) = \begin{cases} \overline{\beta}_p(m,-1) & \text{when } p\in\,]2,+\infty[, \\ \overline{\beta}_p(m,1) & \text{when } p\in\,]1,2\,[. \end{cases}$$

For this reason, we define the function $\widetilde{\beta}$ by

$$\begin{cases} \widetilde{\beta} : (p,m) \in (]1,+\infty[\setminus\{2\}) \times (]0,+\infty[\setminus\{1\}) \to \widetilde{\beta}(p,m) = \widetilde{\beta}_p(m), \\ \widetilde{\beta}_p(m) = \begin{cases} \overline{\beta}_p(m,-1) = \dfrac{m^{p-1}+1}{(m+1)^{p-1}}, & \text{if } p\in\,]2,+\infty[, \\ \overline{\beta}_p(m,1) = \dfrac{m^{p-1}-1}{(m-1)(m+1)^{p-2}}, & \text{if } p\in\,]1,2\,[. \end{cases} \end{cases} \tag{5.7}$$

We will now study the case where $p\in\,]2,+\infty[$ in subsection 5.1, and then the case where $p\in\,]1,2\,[$ in subsection 5.2.

5.1. Minimizing $\widetilde{\beta}_p(m)$ in m when $p \in]2,+\infty[$

In the whole of this subsection we assume that $p \in]2,+\infty[$.

In this case, since

$$g_p(m) = (m^{p-2}-1)(m^2-1) \geq 0, \quad \forall m \in]0,+\infty[, \tag{5.8}$$

formula (5.6) and definition (5.7) imply that

$$\inf_{\tau\in[-1,1]} \overline{\beta}_p(m,\tau) = \overline{\beta}_p(m,-1) = \widetilde{\beta}_p(m) = \frac{m^{p-1}+1}{(m+1)^{p-1}}, \quad \forall m \in]0,+\infty[\setminus\{1\}.$$

Therefore,
$$\inf_{m\in]0,+\infty[\setminus\{1\},\, \tau\in[-1,1]} \overline{\beta}_p(m,\tau) = \inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\beta}_p(m). \tag{5.9}$$

Let us now compute the right-hand side of (5.9). For any fixed $p \in]2,+\infty[$ the function $\widetilde{\beta}_p$ is smooth and an easy computation shows that

$$\frac{\partial\widetilde{\beta}_p}{\partial m}(m) = \frac{p-1}{(m+1)^p}(m^{p-2}-1), \quad \forall m \in]0,+\infty[, \tag{5.10}$$

which implies that
$$\inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\beta}_p(m) = \widetilde{\beta}_p(1) = \frac{1}{2^{p-2}}. \tag{5.11}$$

Finally, combining (5.4), (5.9), and (5.11) gives

$$B_p = \min\left\{1, \frac{1}{2^{p-2}}, \frac{1}{2^{p-2}}\right\} = \frac{1}{2^{p-2}} > 0,$$

so that $\beta_p = B_p = \dfrac{1}{2^{p-2}}$, i.e., (1.16) in the case $p \in]2,+\infty[$.

5.2. Minimizing $\widetilde{\beta}_p(m)$ in m when $p \in]1,2[$

In the whole of this subsection we of course assume that $p \in]1,2[$.

In this case, since

$$g_p(m) = (m^{p-2}-1)(m^2-1) \leq 0, \quad \forall m \in]0,+\infty[, \tag{5.12}$$

formula (5.6) and definition (5.7) imply that

$$\inf_{\tau\in[-1,1]} \overline{\beta}_p(m,\tau) = \overline{\beta}_p(m,1) = \widetilde{\beta}_p(m) = \frac{m^{p-1}-1}{(m-1)(m+1)^{p-2}}, \quad \forall m \in]0,+\infty[\setminus\{1\}.$$

Therefore
$$\inf_{m\in]0,+\infty[\setminus\{1\},\, \tau\in[-1,1]} \overline{\beta}_p(m,\tau) = \inf_{m\in]0,+\infty[\setminus\{1\}} \widetilde{\beta}_p(m). \tag{5.13}$$

Let us now compute the right-hand side of (5.14). For any fixed $p \in]1,2[$ the function $\widetilde{\beta}_p$ is smooth in $[0,+\infty[\setminus\{1\}$ and an easy computation shows that

$$\frac{\partial\widetilde{\beta}_p}{\partial m}(m) = \frac{1}{(m-1)^2(m+1)^{p+1}} k_p(m), \quad \forall m \in]0,+\infty[\setminus\{1\}, \tag{5.14}$$

where the function k_p is defined by

$$k_p(m) = (p-1)m - (3-p)m^{p-1} + 3 - p - (p-1)m^{p-2}, \quad \forall m \in]0,+\infty[.$$

Studying k_p' and k_p'' it is easily seen that

$$k_p(m) \geq 0 \iff m \in [1, +\infty[,$$

which implies, thanks to (5.14), that $\widetilde{\beta}_p$ is an increasing function on $]1, +\infty[$ and a decreasing function on $]0, 1[$. Moreover, L'Hôpital's rule implies that

$$\widetilde{\beta}_p(m) \to \frac{p-1}{2^{p-2}}, \text{ as } m \to 1.$$

Therefore one has
$$\inf_{m \in]0,+\infty[\setminus\{1\}} \widetilde{\beta}_p(m) = \frac{p-1}{2^{p-2}}, \tag{5.15}$$

and combining (5.4), (5.13), and (5.15) gives

$$B_p = \min\left\{1, \frac{1}{2^{p-2}}, \frac{p-1}{2^{p-2}}\right\}. \tag{5.16}$$

Since when $p \in]1, 2[$ one has $0 < \frac{p-1}{2^{p-2}} < 1 < \frac{1}{2^{p-2}}$, formula (5.16) gives $B_p = \frac{p-1}{2^{p-2}}$, so that $\beta_p = B_p = \frac{p-1}{2^{p-2}}$, i.e., (1.16) in the case $p \in]1, 2[$.

The proof of Theorem 1.2 is complete. □

6. Proof of Proposition 2.1 (Properties of m_p)

The proof will be split in five steps.

Step 1: Proof of (2.2) ($\mu(p) = m_p \to 1$, as $p \to +\infty$)

If $p \in]2, +\infty[$, m_p is the unique solution to equation (1.7), with the function σ_p defined by (1.8), i.e.,

$$m_p^{p-1} - (p-1)m_p - (p-2) = 0, \quad m_p \in]1, +\infty[, \quad \text{if } p \in]2, +\infty[,$$

that we rewrite as
$$\frac{m_p^{p-2}}{p-2} = \frac{p-1}{p-2} + \frac{1}{m_p}, \quad m_p \in]1, +\infty[, \quad \text{if } p \in]2, +\infty[. \tag{6.1}$$

We claim that (2.2) holds true. Indeed, if (2.2) does not hold, for any $M \in]1, +\infty[$, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \in]2, +\infty[$ such that

$$p_n \to +\infty, \quad m_{p_n} > M, \quad \forall n \in \mathbb{N}. \tag{6.2}$$

Then one has
$$m_{p_n}^{p_n-2} > M^{p_n-2}, \quad \forall n \in \mathbb{N}, \tag{6.3}$$

while, thanks to $\frac{p-1}{p-2} \leq 2 \iff p \in [3, +\infty[$ and to (6.2), one has

$$\frac{p_n-1}{p_n-2} + \frac{1}{m_{p_n}} \leq 2 + \frac{1}{M}, \quad \text{for } n \text{ large enough}. \tag{6.4}$$

Finally, combining (6.3), (6.1), and (6.4) gives

$$\frac{e^{(p_n-2)\ln(M)}}{p_n-2} = \frac{M^{p_n-2}}{p_n-2} \leq \frac{m_{p_n}^{p_n-2}}{p_n-2} = \frac{p_n-1}{p_n-2} + \frac{1}{m_{p_n}} \leq 2 + \frac{1}{M}, \text{ for } n \text{ large enough},$$

which can not hold true since $p_n \to +\infty$. This proves (2.2).

Step 2: Proof of (2.3) ($\mu(p) = m_p \to m^{\oplus}$, as $p \to 2^+$)

Before proving (2.3), let us observe that equation (2.4) has a unique solution $m_{\oplus}$. This follows from $\rho^{\oplus}(1) = -2$, $\rho^{\oplus}(m) \to +\infty$, as $m \to +\infty$, and from

$$\rho^{\oplus\prime}(m) = \ln m > 0 \iff m \in]1, +\infty[. \tag{6.5}$$

Let us now prove (2.3).

For this purpose, let us set $p = 2+\varepsilon$, where $\varepsilon \in]0,1[$, and recall (see (1.7) and (1.8)) that $m_{2+\varepsilon}$ is the unique solution to

$$\sigma_{2+\varepsilon}(m) = 0, \quad m \in]1, +\infty[, \tag{6.6}$$

where the function $\sigma_{2+\varepsilon}$ is defined by

$$\sigma_{2+\varepsilon} : m \in]0, +\infty[\to \sigma_{2+\varepsilon}(m) = m^{1+\varepsilon} - (1+\varepsilon)m - \varepsilon = me^{\varepsilon \ln(m)} - (1+\varepsilon)m - \varepsilon,$$

so that
$$e^{\varepsilon \ln(m_{2+\varepsilon})} - 1 = \varepsilon \, \frac{1 + m_{2+\varepsilon}}{m_{2+\varepsilon}}, \quad \forall \varepsilon \in]0,1[. \tag{6.7}$$

Since $m_{2+\varepsilon} \in]1, +\infty[$, equation (6.7) implies that

$$e^{\varepsilon \ln(m_{2+\varepsilon})} \to 1, \text{ as } \varepsilon \to 0,$$

namely,
$$\varepsilon \ln(m_{2+\varepsilon}) \to 0, \text{ as } \varepsilon \to 0. \tag{6.8}$$

Combining (6.7) with (6.8) implies that

$$\frac{1 + m_{2+\varepsilon}}{m_{2+\varepsilon}} \, \frac{1}{\ln(m_{2+\varepsilon})} = \frac{e^{\varepsilon \ln(m_{2+\varepsilon})} - 1}{\varepsilon \ln(m_{2+\varepsilon})} \to 1, \text{ as } \varepsilon \to 0, \tag{6.9}$$

which implies that $m_{2+\varepsilon}$ tends to a limit $m^{\oplus} \in]1, +\infty[$, as ε tends to zero, where $m^{\oplus}$ satisfies

$$\frac{1 + m^{\oplus}}{m^{\oplus}} \, \frac{1}{\ln(m^{\oplus})} = 1, \quad m^{\oplus} \in]1, +\infty[,$$

or in other terms that $m^{\oplus}$ is the unique solution of (2.4). This proves (2.3).

Step 3: Proof of (2.6) ($\mu(p) = m_p \to 0$, as $p \to 2^-$)

Before proving (2.6), let us observe that the equation

$$\rho^{\ominus}(m) = 0, \quad m \in]0, +\infty[, \tag{6.10}$$

with the function $\rho^{\ominus}$ defined by[4]

$$\rho^{\ominus} : m \in]0, +\infty[\to \rho^{\ominus}(m) = 4m - 4m^2 + (2m^2 + 2m) \ln(m) \tag{6.11}$$

has exactly two solutions, namely $m = 0$ and $m = 1$. This follows from the fact that

$$\rho^{\ominus}(0) = 0, \quad \rho^{\ominus}(1) = 0, \text{ and } \rho^{\ominus}(m) < 0 \iff m \in]0,1[,$$

which in turn follows from a detailed study of $\rho^{\ominus\prime\prime\prime}$, $\rho^{\ominus\prime\prime}$, $\rho^{\ominus\prime}$, and $\rho^{\ominus}$ similar to the study we have made for σ_p in the proof of Lemma 4.1.

[4] By convention we set $m \ln(m) = 0$ when $m = 0$.

Let us now prove (2.6).

For this purpose, let us set $p = 2-\varepsilon$, where $\varepsilon \in]0,1[$, and recall (see (1.9) and (1.10)) that $m_{2-\varepsilon}$ is the unique solution to

$$\sigma_{2-\varepsilon}(m) = 0, \quad m \in]0,1[, \tag{6.12}$$

with $\sigma_{2-\varepsilon}$ defined by

$$\begin{cases} \sigma_{2-\varepsilon} : m \in]0,+\infty[\to \sigma_{2-\varepsilon}(m), \\ \sigma_{2-\varepsilon}(m) = (2-\varepsilon)(1-\varepsilon)m^2 - (2+\varepsilon)m^{2-\varepsilon} + (2-\varepsilon)(1+2\varepsilon)m - (2-\varepsilon)m^{1-\varepsilon} + \varepsilon^2 \\ = (2-\varepsilon)(1-\varepsilon)m^2 + (2-\varepsilon)(1+2\varepsilon)m + \varepsilon^2 - ((2+\varepsilon)m^2 + (2-\varepsilon)m)\, e^{-\varepsilon \ln(m)}, \end{cases}$$

and that (see (4.41))

$$0 < m_{2-\varepsilon} \le \frac{1+\varepsilon}{2+\varepsilon} \le \frac{2}{3}, \quad \forall \varepsilon \in]0,1[. \tag{6.13}$$

The Maclaurin expansion of e^t around $t = 0$ asserts that

$$\forall t \in \mathbb{R}, \quad \exists\, \theta(t) \in]\min\{0,t\}, \max\{0,t\}[\,:\; e^t = 1 + t + \frac{1}{2} e^{\theta(t)} t^2. \tag{6.14}$$

Therefore one has

$$\begin{cases} \sigma_{2-\varepsilon}(m) = (2-\varepsilon)(1-\varepsilon)m^2 + (2-\varepsilon)(1+2\varepsilon)m + \varepsilon^2 \\ - \big((2+\varepsilon)m^2 + (2-\varepsilon)m\big) \Big(1 - \varepsilon \ln(m) + \frac{1}{2} e^{\theta(-\varepsilon \ln(m))} \varepsilon^2 \ln^2(m)\Big) \\ = \varepsilon \rho^{\ominus}(m) + \varepsilon^2 r^{\ominus}(m,\varepsilon), \quad \forall m \in]0,1[, \quad \forall \varepsilon \in]0,1[, \end{cases} \tag{6.15}$$

where the function $\rho^{\ominus}$ is defined by (6.11) and where the function $r^{\ominus}(m,\varepsilon)$ is defined by

$$\begin{cases} r^{\ominus}(m,\varepsilon) = (m-1)^2 + m(m-1)\ln(m) \\ + \Big(-m - 1 + \frac{(1-m)}{2}\varepsilon\Big)\, m \ln^2(m)\, e^{\theta(-\varepsilon \ln(m))}, \quad \forall m \in]0,1[, \quad \forall \varepsilon \in]0,1[, \end{cases} \tag{6.16}$$

where $\theta(-\varepsilon \ln(m))$ satisfies

$$0 < \theta(-\varepsilon \ln(m)) < -\varepsilon \ln(m), \quad \forall m \in]0,1[, \quad \forall \varepsilon \in]0,1[. \tag{6.17}$$

We claim that (2.6) holds true. Indeed, if (2.6) does not hold true, for any $M \in]0, \frac{2}{3}[$ (see (6.13)) there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \in]0,1[$ such that

$$\varepsilon_n \to 0, \quad m_{2-\varepsilon_n} \ge M, \quad \forall n \in \mathbb{N}. \tag{6.18}$$

Since $\sigma_{2-\varepsilon_n}(m_{2-\varepsilon_n}) = 0$ for all $n \in \mathbb{N}$, we rewrite (6.15) as

$$\rho^{\ominus}(m_{2-\varepsilon_n}) = -\varepsilon_n r^{\ominus}(m_{2-\varepsilon_n}, \varepsilon_n), \quad \forall n \in \mathbb{N}. \tag{6.19}$$

In view of (6.13) and (6.18), we can assume that there exist a subsequence ε_n, still indexed by n, and a constant $m^\star$ such that

$$m_{2-\varepsilon_n} \to m^\star, \quad m^\star \in \Big[M, \frac{2}{3}\Big], \quad \text{as } \varepsilon_n \to 0. \tag{6.20}$$

In view of (6.17) we have

$$0 < \theta(-\varepsilon_n \ln(m_{2-\varepsilon_n})) < -\varepsilon_n \ln(m_{2-\varepsilon_n}),$$

which implies that $-\varepsilon_n \ln(m_{2-\varepsilon_n}) \to 0 \ln(m^\star) = 0$ as $\varepsilon_n \to 0$, and therefore

$$\theta(-\varepsilon_n \ln(m_{2-\varepsilon_n})) \to 0, \text{ as } \varepsilon_n \to 0. \tag{6.21}$$

Passing to the limit in (6.19) and taking into account (6.16), (6.20), and (6.21), one obtains

$$\rho^{\ominus}(m^\star) = 0,$$

a contradiction since $m^\star \in [M, \frac{2}{3}]$ and since equation (6.10) has only two solutions $m = 0$ and $m = 1$, as noticed at the beginning of this proof.

This proves (2.6).

Step 4: Proof of (2.7) ($\mu(p) = m_p \to m^\#$, as $p \to 1^+$)

Before proving (2.7), we observe that equation (2.8) has a unique solution $m = m^\#$.

More precisely, equation $\rho^\#(m) = 0$, $m \in]0, +\infty[$, has exactly two solutions, namely $m = 1$ and $m = m^\# \in]0, \frac{1}{2}[$. This follows from the facts that

$$\rho^\#(m) \to +\infty, \text{ as } m \to 0, \quad \rho^\#(1) = 0, \text{ and } {\rho^\#}'(m) < 0 \iff m \in]0, c[,$$

for some $c \in]0, \frac{1}{2}[$, a fact which in turn follows from a detailed study of ${\rho^\#}''$, ${\rho^\#}'$, and $\rho^\#$ similar to the study we have made for σ_p in the proof of Lemma 4.1.

Let us now prove (2.7).

For this purpose, let us set $p = 1+\varepsilon$, where $\varepsilon \in]0, 1[$, and recall (see (1.9) and (1.10)) that $m_{1+\varepsilon}$ is the unique solution to $\sigma_{1+\varepsilon}(m) = 0$, $m \in]0, 1[$, with $\sigma_{1+\varepsilon}$ defined by

$$\begin{cases} \sigma_{1+\varepsilon} : m \in]0, +\infty[\to \sigma_{1+\varepsilon}(m), \\ \sigma_{1+\varepsilon}(m) = \varepsilon(1+\varepsilon)m^2 - (3-\varepsilon)m^{1+\varepsilon} + (1+\varepsilon)(3-2\varepsilon)m - (1+\varepsilon)m^\varepsilon + (\varepsilon-1)^2 \\ = (1-m^\varepsilon)(3m+1) + \varepsilon\left((1+\varepsilon)m^2 + (1-2\varepsilon)m + \varepsilon - 2 + (m-1)m^\varepsilon\right). \end{cases}$$

Defining the function $r^\#$ by

$$r^\#(m, \varepsilon) = (1+\varepsilon)m^2 + (1-2\varepsilon)m + \varepsilon - 2 + (m-1)m^\varepsilon, \quad \forall m \in]0, +\infty[, \quad \forall \varepsilon \in]0, 1[,$$

the definition of $m_{1+\varepsilon}$ reads as

$$e^{\varepsilon \ln(m_{1+\varepsilon})} - 1 = m_{1+\varepsilon}^\varepsilon - 1 = \varepsilon \frac{r^\#(m_{1+\varepsilon}, \varepsilon)}{3m_{1+\varepsilon} + 1}, \quad \forall \varepsilon \in]0, 1[. \tag{6.22}$$

Since (see (4.41)), $$0 < m_{1+\varepsilon} < \frac{2-\varepsilon}{3-\varepsilon} < \frac{2}{3}, \quad \forall \varepsilon \in]0, 1[, \tag{6.23}$$

one has $$\exists\, c \in]0, +\infty[\,: \left|\frac{r^\#(m_{1+\varepsilon}, \varepsilon)}{3m_{1+\varepsilon} + 1}\right| < c, \quad \forall \varepsilon \in]0, 1[\,. \tag{6.24}$$

Then (6.22) implies
$$e^{\varepsilon \ln(m_{1+\varepsilon})} - 1 \to 0, \quad \text{as} \quad \varepsilon \to 0, \tag{6.25}$$
i.e., $\varepsilon \ln(m_{1+\varepsilon}) \to 0$ as $\varepsilon \to 0$, which in turn implies that
$$\frac{e^{\varepsilon \ln(m_{1+\varepsilon})} - 1}{\varepsilon \ln(m_{1+\varepsilon})} = \frac{e^t - 1}{t} \to 1, \text{ as } \varepsilon \ln(m_{1+\varepsilon}) = t \to 0. \tag{6.26}$$

Let now $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ converging to 0. By virtue of (6.23), we can assume that, for a subsequence still indexed by n,
$$m_{1+\varepsilon_n} \to m^\star, \text{ with } m^\star \in \left[0, \frac{2}{3}\right].$$

Rewriting (6.22) for such a subsequence, we obtain that
$$\frac{e^{\varepsilon_n \ln(m_{1+\varepsilon_n})} - 1}{\varepsilon_n \ln(m_{1+\varepsilon_n})} = \frac{1}{\ln(m_{1+\varepsilon_n})} \frac{r^\#(m_{\varepsilon_n+1}, \varepsilon_n)}{(3m_{1+\varepsilon_n} + 1)}, \quad \forall n \in \mathbb{N}. \tag{6.27}$$

Let us prove by contradiction that $m^\star > 0$. Indeed if we assume that $m^\star = 0$, passing to the limit in (6.27) and taking into account (6.25) and (6.23), one obtains that $1 = 0$. Therefore one has $m^\star > 0$.

Passing to the limit in (6.27) we obtain that
$$1 = \frac{1}{\ln(m^\star)} \frac{{m^\star}^2 + 2m^\star - 3}{3m^\star + 1},$$
namely that $m^\star$ is a solution to equation (2.8). Since this equation has a unique solution $m^\# \in]0,1[$ and since $m^\star \in]0,1[$, we have proved that $m^\star = m^\#$.

This proves (2.7).

Step 5: Proof of (2.10) and (2.11) (regularity of μ and formula for μ')

Statements (2.10) and (2.11) follow from the definition of m_p and from the implicit function theorem applied respectively to the function (see (1.8))
$$(p,m) \in]2,+\infty[\times]1,+\infty[\to \sigma(p,m) = \sigma_p(m) = m^{p-1} - (p-1)m - (p-2), \tag{6.28}$$
when $p \in]2,+\infty[$, and to the function (see (1.10))
$$\begin{cases} (p,m) \in]1,2\,[\times]0,1[\to \sigma(p,m) = \sigma_p(m) \\ = p(p-1)m^2 - (4-p)m^p + p(5-2p) - pm^{p-1} + (p-2)^2, \end{cases} \tag{6.29}$$
when $p \in]1,2\,[$.

Indeed the two functions $\sigma(p,m)$ defined by (6.28) and (6.29) are C^∞ on their domains of definition; on the other side, their partial derivatives $\frac{\partial \sigma}{\partial m}(p,m) = \sigma_p'(m)$ do not vanish at the point (p, m_p) when $p \in]1,+\infty[\setminus\{2\}$: when $p \in]2,+\infty[$, this results from $\frac{\partial \sigma}{\partial m}(p,m) = \sigma_p'(m) = (p-1)(m^{p-2} - 1)$ and the fact that $m_p \in]1,+\infty[$ (see (1.7)); when $p \in]1,2[$, this results from (4.46), which implies that if $m \in]1,+\infty[$ is a solution to $\sigma_p'(m) = 0$, then either $m = \overline{m}_p$ or $m = 1$, and from the inequality $0 < m_p < \overline{m}_p < 1$ (see (4.48)). Therefore the implicit function theorem applies, which proves (2.10) and (2.11).

The proof of Proposition 2.1 is complete. □

7. Proof of Proposition 2.2 (Properties of α_p)

The proof will be split into four steps.

Step 1: Proof of (2.13) **and** (2.15)
($\alpha(p) = \alpha_p \to 0$, as $p \to +\infty$, and $\alpha(p) = \alpha_p \to 0$, as $p \to 1$)

Convergence (2.13) follows from inequalities (1.11), while convergence (2.15) follows from inequalities (1.12).

Step 2: Proof of (2.14) ($\alpha(p) = \alpha_p \to \alpha_2 = 1$, as $p \to 2$)
Inequalities (1.12) imply that

$$\alpha_p \to 1, \text{ as } p \to 2^-. \tag{7.1}$$

It remains to prove that

$$\alpha_p \to 1, \text{ as } p \to 2^+. \tag{7.2}$$

To this aim, let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence such that

$$p_n \in]2, +\infty[, \quad \forall n \in \mathbb{N}, \quad p_n \to 2, \tag{7.3}$$

and let $m_{p_n} \in]1, +\infty[$ be the unique solution to equation (1.7) with $p = p_n$. Then there exist $m^\star \in [1, +\infty]$ and a subsequence, still indexed by n, such that

$$m_{p_n} \to m^\star.$$

In view of formula (1.6), if $m^\star \in [1, +\infty[$, one has

$$\alpha_{p_n} = \frac{m_{p_n}^{p_n} + p_n m_{p_n} + p_n - 1}{(m_{p_n}+1)^{p_n}} = \frac{e^{p_n \ln m_{p_n}} + p_n m_{p_n} + p_n - 1}{e^{p_n \ln(m_{p_n}+1)}} \to \frac{m^{\star 2} + 2m^\star + 1}{(m^\star+1)^2} = 1,$$

as $n \to +\infty$, while, if $m^\star = +\infty$, one has

$$\alpha_{p_n} = \frac{m_{p_n}^{p_n} + p_n m_{p_n} + p_n - 1}{(m_{p_n}+1)^{p_n}} = \left[e^{p_n \ln\left(\frac{m_{p_n}}{m_{p_n}+1}\right)} \left(1 + \frac{p_n}{e^{(p_n-1)\ln(m_{p_n})}} + \frac{p_n - 1}{e^{p_n \ln(m_{p_n})}} \right) \right] \to 1,$$

as $n \to +\infty$, which prove (7.2) and (2.14).

Step 3: Proof of (2.16) **and** (2.17) (regularity of α and formula for α')

Proposition 2.1 states that the function μ defined by (2.1), namely

$$\mu : p \in]1, +\infty[\setminus \{2\} \to \mu(p) = m_p \in]0, +\infty[,$$

where m_p is the unique solution to equation (1.7), with the function σ_p defined by (1.8), if $p \in]2, +\infty[$, and to equation (1.9), with the function σ_p defined by (1.10), if $p \in]1, 2[$, belongs to $C^\infty(]1, +\infty[\setminus\{2\})$ (see (2.10)). Observe moreover that $\mu(p) = m_p \in]1, +\infty[$ if $p \in]2, +\infty|$ (see (1.7)), and $\mu(p) = m_p \in]0, 1[$ if $p \in]1, 2[$ (see (1.9)), so that one has $\mu(p) = m_p \in]0, +\infty[\setminus \{1\}$ for every $p \in]1, +\infty[\setminus \{2\}$.

On the other hand, the function α defined in Proposition 2.2 by (2.12), namely

$$\alpha : p \in]1, +\infty[\to \alpha(p) = \alpha_p,$$

where α_p is defined by formula (1.6), can be written as

$$\alpha(p) = \alpha_p = \widetilde{\alpha}(p, m_p) = \widetilde{\alpha}_p(m_p),$$

where the function $\widetilde{\alpha}$

$$\widetilde{\alpha} : (p,m) \in (]1,+\infty| \setminus \{2\}) \times (]0,\infty[\setminus\{1\}) \to \widetilde{\alpha}(p,m) = \widetilde{\alpha}_p(m),$$

is given by (4.16) and clearly belongs to $C^\infty((]1,+\infty| \setminus \{2\}) \times (]0,\infty[\setminus\{1\}))$.

These two results immediately imply that the function $\alpha(p) = \widetilde{\alpha}(p,\mu(p))$ belongs to $C^\infty(]1,+\infty| \setminus \{2\})$.

In order to prove that the function α satisfies the regularity $C^0(\,]1,+\infty[\,)$ it remains therefore to prove that $\alpha(p) = \alpha_p$ is continuous at $p = 2$. Recall that the function $\mu(p) = m_p$ is not defined at $p = 2$, and is discontinuous at $p = 2$ since (see (2.3) and (2.6)) $\mu(p) = m_p \to m^\oplus$ as $p \to 2^+$, while $\mu(p) = m_p \to 0$ as $p \to 2^-$. But the function $\alpha(p) = \widetilde{\alpha}(p,m_p) = \widetilde{\alpha}(p,\mu(p))$ is nevertheless continuous at $p = 2$, since

$$\alpha(p) = \alpha_p = \frac{m_p^p + pm_p + p - 1}{(m_p+1)^p} \to \frac{(m^\oplus)^2 + 2m^\oplus + 1}{(m^\oplus+1)^2} = 1, \text{ as } p \to 2^+,$$

$$\alpha(p) = \alpha_p = \frac{m_p^p - pm_p + p - 1}{(m_p-1)^2(m_p+1)^{p-2}} \to \frac{0^2 - 2\times 0 + 1}{(-1)^2\,1^0} = 1, \text{ as } p \to 2^-.$$

This completes the proof of (2.16).

Let us finally compute at every point $p \in]1,+\infty[\setminus\{2\}$ the derivative $\alpha'(p)$ of the function

$$\alpha(p) = \alpha_p = \widetilde{\alpha}_p(m_p) = \widetilde{\alpha}(p,m_p) = \widetilde{\alpha}(p,\mu(p)). \tag{7.4}$$

One has

$$\alpha'(p) = \frac{\partial\widetilde{\alpha}(p,\mu(p))}{\partial p} = \frac{\partial\widetilde{\alpha}}{\partial p}(p,\mu(p)) + \frac{\partial\widetilde{\alpha}}{\partial m}(p,\mu(p))\frac{\partial\mu}{\partial p}(p), \quad \text{if } p\in]1,+\infty[\setminus\{2\}. \tag{7.5}$$

In view of (4.19), (4.31), and of the definition of $\mu(p) = m_p$, one has

$$\frac{\partial\widetilde{\alpha}}{\partial m}(p,m_p) = \begin{cases} \dfrac{p}{(m_p+1)^{p+1}}\sigma_p(m_p) = 0, & \text{if } p\in]2,+\infty[, \\ \dfrac{1}{(m_p-1)^3(m_p+1)^{p-1}}\sigma_p(m_p) = 0, & \text{if } p\in]1,2[. \end{cases} \tag{7.6}$$

Combining (7.5) and (7.6) yields

$$\alpha'(p) = \frac{\partial\widetilde{\alpha}}{\partial p}(p,m_p), \quad \text{if } p\in]1,+\infty[\setminus\{2\}. \tag{7.7}$$

Let us now compute the right-hand side of (7.7).

If $p\in]2,+\infty[$ one has

$$\begin{cases} \alpha'(p) = \dfrac{\partial\widetilde{\alpha}}{\partial p}(p,m_p) \\ = \dfrac{\left(m_p^p\ln(m_p) + m_p + 1\right)(m_p+1)^p - \left(m_p^p + pm_p + p - 1\right)(m_p+1)^p\ln(m_p+1)}{(m_p+1)^{2p}} \\ = \dfrac{m_p^p\left(\ln(m_p) - \ln(m_p+1)\right) + m_p\left(1 - p\ln(m_p+1)\right) + 1 - (p-1)\ln(m_p+1)}{(m_p+1)^p}, \end{cases} \tag{7.8}$$

whereas if $p\in\,]1,2\,[$, one has

$$\begin{cases} \alpha'(p) = \dfrac{\partial\widetilde{\alpha}}{\partial p}(p,m_p) \\ = \dfrac{\left(m_p^p\ln(m_p) - m_p + 1\right)(m_p+1)^{p-2} - \left(m_p^p - pm_p + p - 1\right)(m_p+1)^{p-2}\ln(m_p+1)}{(m_p-1)^2\,(m_p+1)^{2p-4}} \\ = \dfrac{m_p^p\left(\ln(m_p) - \ln(m_p+1)\right) - m_p\left(1 - p\ln(m_p+1)\right) + 1 - (p-1)\ln(m_p+1)}{(m_p-1)^2\,(m_p+1)^{p-2}}. \end{cases}$$

This proves (2.17).

Step 4: Proof of (2.18) $(\alpha'(p) < 0$ if $p\in\,]2,+\infty[)$

To prove that $\alpha'(p) < 0$ if $p\in\,]2,+\infty[$, let us estimate the three terms which appear in the numberator of the last line of (7.8), namely:

$$m_p^p\left(\ln(m_p) - \ln(m_p+1)\right),\ m_p\left(1 - p\ln(m_p+1)\right),\ 1 - (p-1)\ln(m_p+1). \tag{7.9}$$

As far as the first term of (7.9) is concerned, of course one has

$$m_p^p\left(\ln(m_p) - \ln(m_p+1)\right) < 0. \tag{7.10}$$

As far as the second term of (7.9) is concerned, since $p\in\,]2,+\infty[$ and $m_p\in\,]1,+\infty[$, one has

$$1 - p\ln(m_p+1) < 1 - 2\ln 2 = \ln\frac{e}{4} < 0, \quad \text{if } p\in\,]2,+\infty[.$$

Therefore, since $m_p\in\,]1,+\infty[$, one has

$$m_p\left(1 - p\ln(m_p+1)\right) < 1 - 2\ln 2 < 0, \quad \text{if } p\in\,]2,+\infty[. \tag{7.11}$$

As far as the third term of (7.9) is concerned, since $p\in\,]2,+\infty[$ and $m_p\in\,]1,+\infty[$, one has

$$1 - (p-1)\ln(m_p+1) < 1 - \ln 2, \quad \text{if } p\in\,]2,+\infty[. \tag{7.12}$$

Unfortunately, $1 - \ln 2 = \ln\frac{e}{2}$ is a positive number, but adding (7.11) and (7.12) yields, if $p\in\,]2,+\infty[$,

$$m_p\left(1 - p\ln(m_p+1)\right) + 1 - (p-1)\ln(m_p+1) < 2 - 3\ln 2 = \ln\frac{e^2}{8} < 0. \tag{7.13}$$

Then (2.18) follows from (7.8), (7.10), and (7.13).

The proof of Proposition 2.2 is complete. □

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