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Some Remarks on Quasiconvexity

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Dedicated to the memory of Hedy Attouch.

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This article is dedicated to Hedy Attouch, colleague and friend, with whom I shared many studies on *Gamma convergence* and other types of variational convergence [see the author, *Su una convergenza di funzioni convesse*, Boll. Unione Mat. Italiana 8 (1973) 137–158], as well described in his book [see H. Attouch, *Variational convergence for functions and operators*, Applicable Mathematics Series, Pitman, Boston (1984)]. I have a vivid memory of the discussions we had together on these and other topics in calculus of variations.

This manuscript deals with some properties of *quasiconvex functions* and their applications to problems of the calculus of variations, in particular lower semicontinuity, existence and regularity – partial and everywhere regularity – of minimizers of convex and quasiconvex energy integrals under p, q*-growth conditions*. Motivated by a *model energy integral* relevant in *nonlinear elasticity* and by the *phenomenon of cavitation*, we point out the interest to study *lower semicontinuity* and *relaxation* in the context of *subcritical growth*. Subcritical growth appears to be a context for a real challenge on quasiconvexity.

Keywords: Quasiconvex functions, subcritical growth, cavitation, p,q-growth conditions, calculus of variations, elliptic systems.

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Contents

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1. Prologue

Under some aspects this manuscript is a continuation of the recent paper [85], published in the *Journal of Convex Analysis*, with the title *"Some remarks on convexity"*, devoted to some applications of convexity to the calculus of variations. We consider here *energy integrals* of the form

$$F(u) = \int_{\Omega} f(x, u, Du)\, dx\,, \tag{1.1}$$

with $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *Carathéodory integrand*, *quasiconvex* with respect to the gradient variable and general dimensional parameters $n, m \geq 1$; while in the paper [85] we emphasized the scalar case $n = m = 1$, although with general integrand $f = f(x, u, \xi)$ in (1.1) having *full dependence* on $x \in \Omega$ and $u, \xi \in \mathbb{R}$. Still some interesting open problems remain also when $n = m = 1$; for instance as described by Botteron-Marcellini [10] with some results specific for the *linear growth*, which can be applied to classical examples of the calculus of variations. In many of these applications the "minimizer" may have discontinuities; in particular the solution (minimizer) may "detach" from the given boundary data. For this reason in general it is usual, sometimes necessary, to introduce an extended energy integral $\overline{F}(u)$, similar to (4.3) below. See [6],[10],[16],[73],[85] for physical and geometrical examples.

Well known basic books about these properties, i.e. about convex functions and calculus of variations in the *convex context*, are Rockafellar [96], Ekeland-Temam [48], Buttazzo-Giaquinta-Hildebrandt [12], Thibault [99].

2. A model energy integral in nonlinear elasticity

We describe a mathematical model which does not fit in the classical framework for the existence and the regularity theory and which is motivated by a class of integrals of the *calculus of variations* introduced in *nonlinear elasticity.*

Following the seminal ideas introduced by Morrey [94] in 1952 (see also Morrey's book [95]) and by Ball's fundamental article [5] in 1976, a *model energy-integral* considered in nonlinear elasticity for a generic map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is

$$u \to \int_{\Omega} f\left(Du\left(x\right)\right)\, dx\,, \tag{2.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$, and f is a *quasiconvex* and coercive function, $f : \mathbb{R}^{n\times n} \to \mathbb{R}$, such as for instance,

$$f\left(Du\left(x\right)\right) = \left|Du\left(x\right)\right|^p + g\left(\det Du\left(x\right)\right) \tag{2.2}$$

for some $p > 1$. Here for $n = m \geq 2$, $\det Du\left(x\right)$ represents the Jacobian determinant of the $n \times n$ gradient matrix $\left(Du\right)_{n\times n}$, which for physical reasons in nonlinear elasticity is assumed to be nonnegative; i.e.

$$\det Du\left(x\right) \geq 0\,, \quad \text{a.e. } x \geq \Omega\,. \tag{2.3}$$

Here $g : [0,+\infty) \to [0,+\infty)$ is a real nonnegative function which grows at $+\infty$ at most as a power; i.e. there exist $r, t_0 \geq 1$ and a positive constant c such that

$$g\left(t\right) \leq c\left(1 + t^r\right)\,, \quad \forall\, t \geq t_0\,. \tag{2.4}$$

Note that not necessarily g is increasing in $[0,+\infty)$, not necessarily g is convex in $[0,+\infty)$, see [77]; however in this context always $g\left(t\right)$ is assumed to be increasing for large values of $t \in (0,+\infty)$. Since the Jacobian determinant $\det Du\left(x\right)$ involves products of n partial derivatives of u it grows as an n-power of Du; i.e., $\left|\det Du\left(x\right)\right| \cong \left|Du\left(x\right)\right|^n$; precisely it is well known that the following bound holds (see for instance [61, Chapter 11, (11.76)])

$$\left|\det Du\left(x\right)\right| \leq n^{-\frac{n}{2}} \left|Du\left(x\right)\right|^n.$$

Then, if $x \in \Omega$ is such that $\det Du\left(x\right) \geq t_0$, then $g\left(\det Du\left(x\right)\right) \leq c\left(1 + \det Du\left(x\right)^r\right)$. To simplify this part of exposition, we consider here the case $t_0 = 0$ and in (2.4) $g\left(t\right) \leq c\left(1 + t^r\right)$ for all $t \geq 0$. For the model-energy-integral (2.2) we obtain

$$\left|Du\left(x\right)\right|^p \leq f\left(Du\left(x\right)\right) \leq \text{const}\left(1 + \left|Du\left(x\right)\right|^p + \left|Du\left(x\right)\right|^{rn}\right)\,. \tag{2.5}$$

Therefore the integral (2.1) is coercive in $W^{1,p}\left(\Omega, \mathbb{R}^n\right)$ and, being a model in nonlinear elasticity where *discontinuous* solutions are admitted, it is necessary to exclude the dimensional case $p > n$, when all the functions in the Sobolev space $W^{1,p}\left(\Omega, \mathbb{R}^n\right)$ are continuous. For technical reasons it is convenient to exclude also the limiting case $p = n$, when jump discontinuities are not allowed to functions $u \in W^{1,n}\left(\Omega, \mathbb{R}^n\right)$. Thus, $p < n$.

If we denote by $q := rn$, the described physical reasons in nonlinear elasticity, $p < n \leq q := rn$, imply that the p-growth from below in (2.5) is different from the q-growth from above. We are here in presence of *p, q-growth conditions*, for a constant $M > 0$ and $p < q$,

$$|Du(x)|^p \leq f(Du(x)) \leq M(1 + |Du(x)|^q) . \tag{2.6}$$

3. The p, q-growth conditions in calculus of variations and pde's

3.1. Anisotropic energy integrals

Interior *L^∞-gradient bound*, i.e., the *local Lipschitz continuity*, of minimizers of multiple integrals of the calculus of variations and of weak solutions to nonlinear elliptic equations and systems, under *p, q-growth conditions* and also more general *non standard growth conditions*, have been obtained since 1989 in [78]–[84]. We briefly describe here some aspects, difficulties and recent results, both in the *scalar* and in the *vector-valued* cases. We start from the scalar case, for instance from *anisotropic elliptic equations* with *variable exponents* is a class of differential equations studied by Iwona Chlebicka [18] and her coauthors, mainly in the context of anisotropic Musielak-Orlicz spaces (see also [14]). These elliptic equations are the Euler's first variation of integrals of the calculus of variations of the form

$$F(u) = \int_\Omega \sum_{i=1}^n a_i(x) |u_{x_i}(x)|^{p_i(x)} \, dx \tag{3.1}$$

for some bounded measurable functions $a_i(x)$ and $p_i(x)$, with $0 < \alpha \leq a_i(x) \leq \beta$, $i = 1, 2, \dots, n$, for some positive constants α, β. The exponents $p_i(x)$ usually are assumed to be continuous in Ω for all $i \in \{1, 2, \dots, n\}$. We are here in presence of some *p, q-growth conditions* with

$$p = \inf\{p_i(x) : \ x \in \Omega; \ i = 1, 2, \dots, n\},$$
$$q = \sup\{p_i(x) : \ x \in \Omega; \ i = 1, 2, \dots, n\}.$$

Note that minimizer of the previous anisotropic energy integral may be not smooth – even unbounded – minimizers, also with constant exponents $p_i \geq 2$ and constant coefficients a_i for all $i \in \{1, 2, \dots, n\}$. This happens if the exponents p_i, $i = 1, \dots, n$, are spread out; i.e., if the ratio $\max\{p_i\}/\min\{p_i\} \geq 1$ is not close enough to 1 in dependence on n. For example when $n > 3$ and

$$p_1 = \ldots = p_{n-1} = 2, \qquad p_n = q > \frac{2(n-1)}{n-3} .$$

In a more compact way we can note that discontinuous minimizers may appear if a bound on the ratio $\frac{q}{p}$ of the type $1 \leq \frac{q}{p} < 1 + O\left(\frac{1}{n}\right)$ does not hold. Following an original idea by Boccardo-Marcellini-Sbordone [8], Cupini-Marcellini-Mascolo [26]–[31] considered exponents p_i, $i = 1, \dots, n$, and q, greater than or equal to 1, such that $p_i \leq p_i(x) \leq q$, a.e. $x \in B_r$, $1 \leq i \leq n$, where B_r is a ball of radius $r > 0$ contained in Ω.

Moreover, let $\overline{p}$ be the *harmonic average* of the $\{p_i\}$ and let $\overline{p}^*$ be the *Sobolev conjugate exponent* of $\overline{p}$; i.e.

$$\frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i} \qquad \overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$$

if $\overline{p} < n$, while $\overline{p}^*$ is any fixed real number greater than $\overline{p}$ if $\overline{p} \geq n$. Under these notations the following result holds.

Theorem 3.1. ([26]–[31]) *Let u be a local minimizer of the anisotropic integral* (3.1) *and let $q \leq \overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$. Then u is locally bounded in Ω.*

3.2. Model examples under p, q-growth conditions

We consider some other model examples, starting from the well known p-Laplacian equation, for an exponent $p > 1$

$$\operatorname{div}\left(|Du|^{p-2}\,Du\right) = 0, \quad \text{or} \quad \operatorname{div}\left(\left(1+|Du|^2\right)^{\frac{p-2}{2}}\,Du\right) = 0,$$

whose energy integrals, respectively, are

$$F_1\left(u\right) = \int_{\Omega} |Du|^p \; dx\,, \quad \text{or} \quad F_2\left(u\right) = \int_{\Omega} \left(1+|Du|^2\right)^{\frac{p}{2}} \; dx\,.$$

We can also consider elliptic equations with more general growth such as, for instance, $\operatorname{div}\left(|Du|^{p(x)-2}\,Du\right) = 0$, with *variable exponent* $p\left(x\right) \in C^0\left(\Omega\right)$; in this case the energy integral $F\left(u\right)$ is $F\left(u\right) = \int_{\Omega} |Du|^{p(x)}\;dx$ and it satisfies the so-called p, q-*growth conditions*. In fact, for regularity we can fix a radius $r > 0$ and a ball B_r such that $\overline{B_r} \subset \Omega$. Then

$$p = \min\left\{p\left(x\right):\; x \in \overline{B_r} \subset \Omega\right\}, \quad q = \max\left\{p\left(x\right):\; x \in \overline{B_r} \subset \Omega\right\}$$

and, for this case with variable exponents we can fix p, q as close as we like by choosing r small, since $p\left(x\right) \in C^0\left(\Omega\right)$; in particular we can satisfy a bound of the type $1 \leq \frac{q}{p} < 1 + O\left(\frac{1}{n}\right)$. We can also consider *double phase equations* of the type

$$\operatorname{div}\left(|Du|^{p-2}\,Du\right) + \operatorname{div}\left(a\left(x\right)|Du|^{q-2}\,Du\right) = 0, \tag{3.2}$$

with $p < q$ and the coefficient $a\left(x\right) \geq 0$ approaching zero at some $x \in \Omega$. The regularity in this case received large consideration, in particular by Esposito-Leonetti-Mingione [49], M.Colombo-Mingione [22], Baroni-M.Colombo-Mingione [7], Eleuteri-Marcellini-Mascolo [45].

From the point of view of energy integrals with *general growth*, for instance:

$$\textit{anisotropic growth} \qquad \int_{\Omega}\sum_{i=1}^{n} a_i\left(x\right)|u_{x_i}(x)|^{p_i}\,dx\,,$$

$$\textit{variable exponents} \qquad \int_{\Omega} |Du\left(x\right)|^{p(x)}\;dx\,, \qquad \int_{\Omega}\left(1+|Du\left(x\right)|^2\right)^{p(x)/2}\;dx\,,$$

$$\textit{logarithmic growth} \qquad \int_{\Omega} |Du(x)|^p \log(1+|Du(x)|)\, dx\,,$$

$$\textit{double phase} \qquad \int_{\Omega} \{a(x)\,|Du(x)|^p + b(x)\,|Du(x)|^q\}\, dx\,,$$

the last example with nonnegative coefficients $a(x)$ and $b(x)$ not equal to zero at the same $x \in \Omega$. By combining them we can see a large variety of energy integrals to be treated within the theories of Musielak-Orlicz spaces.

3.3. Some references on pde's and energy integrals with p, q-growth conditions

The model examples above are special cases of the so-called *p, q-growth conditions,* with $p \leq q$, first considered by the author in [78]–[81] for pde's and energy integrals of the type

$$F(u) = \int_{\Omega} f(x, Du)\, dx\,, \qquad m\,|\xi|^p \leq f(x,\xi) \leq M\,(1+|\xi|^q)\;,$$

for some positive constants m, M and for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. In particular *double phase*, *variable exponents*, *anisotropic growth*, and more *general growth*, has been studied by several authors, in the context of *homogenization* and the *Lavrentiev's phenomenon,* starting from Zhikov [101], [102]. Then, about *Lavrentiev's phenomenon* and *regularity* for *p, q-growth conditions* and *non-uniform elliptic problems*, see Acerbi-Mingione [2], Esposito-Leonetti-Mingione [49], Mihailescu-Pucci-Rădulescu [92], Diening-Harjulehto-Hästö-Ruzicka [43], Colombo-Mingione [22], Baroni-Colombo-Mingione [7], Eleuteri-Marcellini-Mascolo [45], DiMarco-Marcellini [44], DeFilippis-Mingione [39],[40]. Some related recent results are due to Cianchi-Maz'ya [19], Chlebicka-DeFilippis [17], Byun-Oh [13], Eleuteri-Marcellini-Mascolo-Perrotta [46], Ciani-Skrypnik-Vespri [21], Eleuteri-Passarelli [47], Cianchi-Schäffner [20], Crespo-Blanco-Gasinski-Winkert [23], DeFilippis-Koch-Kristensen [38], Marcellini-Nastasi-Pacchiano Camacho [86].

3.4. Elliptic equations with full dependence on x, u, Du

We consider a class of elliptic equations of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x,u,Du) = b(x,u,Du)\,, \qquad x \in \Omega\,, \tag{3.3}$$

where the vector field $(a^i(x,u,\xi))_{i=1,\dots,n}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and the right hand side $b(x,u,\xi)$ is a *Carathéodory function* in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, for instance locally bounded when u is locally bounded, or with more general growth (stated below). We study the local regularity, in particular local Lipschitz continuity and $C^{1,\alpha}$-regularity, of weak solutions to the elliptic equation (3.3) under some *general growth conditions* on the gradient variable $\xi = Du$, as said before, named *p, q-growth conditions.* In the non-uniformly elliptic case what seems in the literature non often studied is when the differential equation (3.3) is not the Euler's first variation of an energy integral. Not always the vector field $a(x,u,\xi) = (a^i(x,u,\xi))_{i=1,\dots,n}$

is the gradient, with respect to the variable $\xi \in \mathbb{R}^n$, of a function $f(x,u,\xi)$. On the contrary in the literature on this subject often the condition $a^i = \partial f / \partial \xi_i$, which simplifies the framework, is one of the main assumption. If f is of class C^2 in ξ then this variational assumption $a^i = \partial f / \partial \xi_i$ implies that

$$\frac{\partial a^i}{\partial \xi_j} = \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 f}{\partial \xi_j \partial \xi_i} = \frac{\partial a^j}{\partial \xi_i},$$

so that the $n \times n$ matrix $\left(\frac{\partial a^i}{\partial \xi_j}\right)$ is symmetric. On the contrary, *we do not assume that this matrix* $\left(\frac{\partial a^i}{\partial \xi_j}\right)$ *is symmetric.* Instead we assume

$$\left|\frac{\partial a^i}{\partial \xi_j} - \frac{\partial a^j}{\partial \xi_i}\right| \leq M\left(1+|\xi|^2\right)^{\frac{p+q-4}{4}} = M\left(1+|\xi|^2\right)^{\frac{(p-2)+(q-2)}{2}\cdot\frac{1}{2}}, \tag{3.4}$$

which requires an *"intermediate growth"* with respect to $|\xi|$, since the exponent $\frac{p+q-4}{2}$ is the average between $p-2$ and $q-2$, of the antisymmetric terms of the matrix $(\partial a^i / \partial \xi_j)$. This condition is automatically satisfied, for instance, either when the vector field $(a^i(x,u,\xi))_{i=1,\ldots,n}$ has the variational structure $a^i = \partial f / \partial \xi_i$, or if $p = q$. This second possibility explains why in the literature this assumption is not considered under the so called *natural growth conditions* (i.e., when $p = q$); in fact if $p = q$ it is an elementary consequence of the triangular inequality and of the other (natural) growth assumptions.

Let us describe our assumptions for Theorem 3.2 below to hold. For every open set Ω', whose closure is contained in Ω, and for every $L > 0$ there exist constants $m, M > 0$ (depending on Ω' and L) such that, for every $x \in \Omega'$, $\lambda, \xi \in \mathbb{R}^n$ and for $|u| \leq L$, the vector field $(a^i(x,u,\xi))_{i=1,\ldots,n}$ satisfies the *ellipticity* and the *growth conditions*

$$m\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a^i}{\partial \xi_j}\lambda_i\lambda_j\,, \tag{3.5}$$

$$\left|\frac{\partial a^i}{\partial \xi_j}\right| \leq M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}, \quad \left|\frac{\partial a^i}{\partial \xi_j} - \frac{\partial a^j}{\partial \xi_i}\right| \leq M\left(1+|\xi|^2\right)^{\frac{p+q-4}{4}}, \tag{3.6}$$

$$\left|\frac{\partial a^i}{\partial x_s}\right| \leq M\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}}, \quad \left|\frac{\partial a^i}{\partial u}\right| \leq M\left(1+|\xi|^2\right)^{\frac{p+q-4}{4}}, \tag{3.7}$$

for some exponents p, q $(2 \leq p \leq q)$ and for every $\lambda, \xi \in \mathbb{R}^n$, $i, j, s = 1, 2, \ldots, n$. We also assume that $|a^i(x,0,0)| \in L_{\text{loc}}^{\frac{q}{q-1}}$, in particular this is satisfied if $a^i(x,0,0)$ is locally bounded in Ω, for every $i = 1, 2, \ldots, n$.

The Carathéodory function $b(x,u,\xi)$ in the right hand side, $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, has the following growth property: for every open set Ω', whose closure is contained in Ω, and for every $L > 0$ there exists a constant $M > 0$ (M depending on Ω' and L) such that

$$|b(x,u,\xi)| \leq M\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}} + b_0(x)\,, \text{ a.e. } x \in \Omega',\ \forall \xi \in \mathbb{R}^n,\ \forall u \in \mathbb{R} : |u| \leq L\,, \tag{3.8}$$

where $b_0 \in L_{\text{loc}}^s(\Omega)$ for an exponent $s > n$.

Before entering in the regularity results, we first emphasize that the notion of *weak solution* needs some care in the Sobolev classes to be considered. In fact, with explicit u-dependence we need to distinguish from summability (precisely $L^{\infty}_{\mathrm{loc}}(\Omega)$-boundedness) of the solution u (which in some applications comes by different arguments, such as, for instance, the *maximum principle*) and the summability class of the gradient Du; in our context $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$. In fact the definition of weak solution is consistent if a-priori $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ and in general (if $q \neq p$) it is not sufficient $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ even if u is bounded. Under p,q-growth conditions (3.5),(3.6) with $q \geq p$, but also under the weaker growth condition

$$\sum_{i=1}^{n} \left| a^i(x,u,\xi) \right| \leq M \left(1+|\xi|^2\right)^{\frac{q-1}{2}},$$

a *weak solution* to is a function $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ such that, for every $\Omega' \subset\subset \Omega$,

$$\int_{\Omega} \sum_{i=1}^{n} \underbrace{a^i(x,u,Du)}_{\in L^{\frac{q}{q-1}}_{\mathrm{loc}}} \underbrace{\varphi_{x_i}}_{\in L^q} \, dx + \int_{\Omega} b(x,u,Du)\,\varphi\, dx = 0, \ \forall \varphi \in W^{1,q}_0(\Omega'), \tag{3.9}$$

where $\frac{q}{q-1}$ and q are conjugate exponents (i.e., as usual, the sum of the reciprocals is equal to 1). This summability condition guarantee the correct definition of the pairings in (3.9); see more details in [25],[29],[83],[84]. In order to state one of the main recent results in this context, we recall the ellipticity and the p,q-growth assumptions. In Theorem 3.2 we assume the ellipticity and p,q-growth condition above and we require a bound on the ratio $\frac{q}{p}$. We obtain $L^{\infty}_{\mathrm{loc}}$-gradient estimates of weak solutions in terms of their L^p_{loc}-gradient norms.

Theorem 3.2. ([29], [83], [84]) *Under the ellipticity and growth conditions* (3.5)–(3.8): *If the exponents $q \geq p \geq 2$ satisfy the bound*

$$\frac{q}{p} < 1 + \frac{2}{n}, \tag{3.10}$$

then every weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ to the PDE is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and there exist constants $c, \alpha, \beta > 0$ such that, for every ϱ and R with $0 < \rho < R \leq \varrho + 1$,

$$\|Du\|_{L^{\infty}(B_{\varrho})} \leq \left(\tfrac{c}{R-\varrho}\right)^{\frac{\alpha\beta q}{\vartheta p}} \left\| \left(1+|Du|^2\right)^{1/2} \right\|^{\alpha}_{L^p(B_R)}. \tag{3.11}$$

The exponents α,ϑ and β in the gradient bound have an explicit representation. To this aim, as usual, we denote by 2^* the Sobolev conjugate of 2; i.e. $2^* = 2n/(n-2)$ when $n > 2$, while we choose 2^* equal to any fixed real number greater than $2q/p$ if $n = 2$. Thus the exponents α, ϑ and β in the gradient bound have an explicit representation. In fact $\beta = (n/q)\vartheta$ and

$$\begin{aligned} \vartheta &= \frac{2^*-2}{2^*\frac{p}{q}-2} \underset{\text{for } n>2}{=} \frac{2q}{np-(n-2)\,q}, \\ \alpha &:= \frac{\vartheta\frac{p}{q}}{1-\vartheta\left(1-\frac{p}{q}\right)} \underset{\text{for } n>2}{=} \frac{2p}{(n+2)\,p-nq}. \end{aligned} \tag{3.12}$$

Since $\vartheta \to \frac{q}{p}$ as $2^* \to +\infty$, when $n = 2$, by the arbitrary possible choice of 2^* then ϑ can be any number close to (and grater than) $\frac{q}{p}$. In any case $\alpha, \vartheta \geq 1$; i.e. for every $n \geq 2$ and every $q \geq p$. The exponent α is greater than or equal to 1, and it is equal to 1 if and only if $q = p$. For the case $n > 2$ the denominator $(n+2)\,p - nq$ of α in (3.12) is positive, since $\frac{q}{p} < 1 + \frac{2}{n}$. Also the denominator of ϑ is positive when $n > 2$; in fact $np - (n-2)\,q > 0$ is equivalent to

$$\frac{q}{p} < \frac{n}{n-2} = 1 + \frac{2}{n-2}$$

which is implied by (3.10).

3.5. Discussion on the existence of weak solutions

By using the previous a-priori estimates and by applying an approximating procedure, in some cases it is possible to prove effectively that there exist weak solutions in the class $W^{1,p}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ of the associated *Dirichlet problems* or other *boundary value problems*, when we fix boundary data with appropriate summability conditions in dependence of the exponents p, q. A recent result is due to Cupini-Marcellini-Mascolo [30], who extended the classical Leray-Lions existence theorem to the context of p, q-growth. Here we propose an approach in a simpler context, with a simplified proof.

This simplification is possible for instance if the vector field $\left(a^i(x,\xi)\right)_{i=1,\dots,n}$ is independent of u and the right hand side has the form $b(x)$, independent of u and ξ; see Marcellini (either [79] in 1991 or, in a different context, [83] in 2021) and Cupini-Leonetti-Mascolo [24] in 2015. In fact, given a boundary datum with higher summability, $u_0 \in W^{1,p\frac{q-1}{p-1}} \subset W^{1,q}$, for every $\varepsilon \in (0,1]$ we can consider the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{a^i(x,Du) + \varepsilon\,\left(1+|Du|^2\right)^{\frac{q-2}{2}} u_{x_i}\right\} = b(x)\,, \quad x \in \Omega, \\ u = u_0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{3.13}$$

For every ε the differential operator in the left hand side is monotone and satisfies the so-called natural growth conditions of order q; i.e. it satisfies q, q-*growth conditions* and $a^i(x,Du)$ in this case (for ε fixed) has the role of a lower order term. By the theory of monotone operators, the weak solution $u_\varepsilon \in u_0 + W^{1,q}_0(\Omega)$ to the Dirichlet problem exists and is unique (but, please note that the $W^{1,q}(\Omega)$-norm depends on ε). Of course the differential operator satisfies also the original p, q-growth conditions with constants $m, 2M$ independent of $\varepsilon \in (0,1]$. Thus *for* u_ε the L^∞-gradient bound holds, uniformly with respect to $\varepsilon \in (0,1]$, however depending on the $W^{1,p}(\Omega)$-norm. By assuming a suitable higher summability of the boundary datum u_0, precisely the previously mentioned condition $u_0 \in W^{1,p\frac{q-1}{p-1}}(\Omega)$, it is also possible to see ([24], [79],[83]) that u_ε is bounded in $W^{1,p}(\Omega)$ uniformly with respect to $\varepsilon \in (0,1]$. Thus the right hand side in the estimate

$$\|Du_\varepsilon\|_{L^\infty(B_\varrho)} \leq \left(\tfrac{c}{R-\varrho}\right)^{\frac{\alpha\beta q}{\vartheta p}} \left\|\left(1+|Du_\varepsilon|^2\right)^{1/2}\right\|^{\alpha}_{L^p(B_R)}$$

is bounded with a constant independent of ε and the same holds for the left hand side.

With the same proof of Theorem 3.2 we also have a uniform bound of the $W^{2,2}_{\mathrm{loc}}(\Omega)$-norm of u_ε. Therefore we can go to the limit as $\varepsilon \to 0$: up to a subsequence u_ε converges, in the weak topology of $u_0 + W^{1,p}_0(\Omega)$, in the weak topology of $W^{2,2}_{\mathrm{loc}}(\Omega)$, in the strong topology of $W^{1,2}_{\mathrm{loc}}(\Omega)$, and in the weak* topology of $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, to a function $u \in u_0 + W^{1,p}_0(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$. The bound in $W^{2,2}_{\mathrm{loc}}(\Omega)$, uniform with respect to $\varepsilon \in (0,1]$, also implies the convergence of the gradient $Du_\varepsilon(x)$ to the gradient $Du(x)$ a.e. in Ω. We go to the limit as $\varepsilon \to 0^+$ in the integral form of the equation and comes out that u is a weak solution to the original Dirichlet problem with $\varepsilon = 0$.

4. The energy model and the phenomenon of cavitation

Let $n, m \in \mathbb{N}$ with $n, m \geq 1$ and let Ω be an open set in $\mathbb{R}^n$. In this section we consider energy integrals of the form

$$F(u) = \int_\Omega f(x, u, Du)\, dx\,,$$

where $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a *Carathéodory integrand*, *quasiconvex* with respect to the gradient variable. The map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ belongs to the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$ for some $p > 1$. We consider here functions $f = f(x, u, \xi)$ with *full dependence* on $x \in \Omega$, on $u \in \mathbb{R}^m$ and on the gradient variable ξ, which in the general case is a *matrix* $m \times n$; i.e., $\xi \in \mathbb{R}^{m\times n}$. The definition of *quasiconvexity*, introduced by Morrey [95, Section 4.4] (see also Ball [5] and Dacorogna [32]), under the growth condition $|f(x, u, \xi)| \leq M\ (1 + |\xi|^p)$ requires that

$$\int_\Omega f(x_0, u_0, \xi + Du)\, dx\ \geq |\Omega|\ f(x_0, u_0, \xi)\,, \quad \forall\, u \in W^{1,p}_0(\Omega; \mathbb{R}^m)\,, \tag{4.1}$$

for all $x_0 \in \Omega$ and $u_0 \in \mathbb{R}^m$. This condition (4.1) is equivalent to the usual convexity when either $n = 1, m \geq 1$ or $n \geq 1, m = 1$. That is, if one of the two dimensional parameters n, m is equal to 1 then quasiconvexity reduces to (and is equivalent to) convexity in the gradient variable ξ.

Convexity and quasiconvexity in the calculus of variations play a fundamental role. We can refer to Morrey [95], Ball [5], Dacorogna [32] and also, for instance, to [9],[34],[35],[73],[74],[87],[88] and to the references therein.

The energy integral (2.1),(2.2) has been used in 1982 by Ball [6] as a mathematical model for the *phenomenon of cavitation*. See also Marcellini [75],[76],[77] and Celada-Perrotta [16]. Under the constraint (2.3), i.e. $\det Du(x) \geq 0$, for a.e. $x \geq \Omega$, this energy integral

$$u \to \int_\Omega f(Du(x))\, dx = \int_\Omega \{|Du(x)|^p + g(\det Du(x))\}\, dx$$

is also associated to some other mathematical difficulties which deserve to be pointed out. For instance:

− it is related to *vector valued maps* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. The first variation is a *system of partial differential equations*, and not a single equation;

– the dependence on the *Jacobian determinant* $\det Du\,(x)$, which does not produce a convex function f; but a *polyconvex function* f. We are in the context of *quasiconvex functions* in Morrey's and Ball's sense [5],[95];

– the possible *singularity* of $g\,(\det Du\,(x))$ as $\det Du\,(x) \to 0^+$; that is, we could expect that $g\,(t) \to +\infty$ as $t \to 0^+$;

– it may also happen the *lack of convexity* (for some experimental elastic materials) of $g : (0, +\infty) \to (0, +\infty)$; see details in [77];

– the singularity of $f\,(\xi)$ as $|\xi| \to +\infty$; i.e., the *p-growth from below*, different from the *q-growth from above*, as emphasized above in Sections 2,3, which gives the genuine p, q-growth, with p strictly less than $q := rn$.

Let us denote by F the model energy integral

$$F\,(u) := \int_\Omega \{|Du\,(x)|^p + g\,(\det Du\,(x))\}\,dx\,, \quad u \in W^{1,\infty}\,(\Omega, \mathbb{R}^n) \tag{4.2}$$

and by $\overline{F}$ its associated *relaxed energy functional*

$$\overline{F}\,(u) := \inf \left\{ \liminf_k\, F\,(u_k) : \ u_k \in W^{1,\infty}\,(\Omega, \mathbb{R}^n)\,,\ u_k \rightharpoonup u \text{ in } W^{1,p}\,(\Omega, \mathbb{R}^n) \right\}, \tag{4.3}$$

where "$\rightharpoonup$" denotes the *weak topology* of $W^{1,p}\,(\Omega, \mathbb{R}^n)$. A similar definition was originally used in a different context by Lebesgue in his thesis [70]. Later by Serrin [98], DeGiorgi and others (see for instance Giusti [64]). In this context the scheme and related semicontinuity results have been given by the author in [76]. Recent relevant applications of this approach have been produced by Schmidt [97], DeFilippis [37], DeFilippis-Stroffolini [41] and Gmeineder-Kristensen [65] (see Section 5.4.3 below).

In order to formulate a problem, which we state below in this section and in more detail in the next sections too, we consider the exponent $p < n$ and we further limit it to be in the interval $p \in [n-1, n)$, starting from

$$n - 1 + \frac{1}{n+1} < p < n \tag{4.4}$$

or equivalently $n^2/(n+1) < p < n$. We already noticed that the Jacobian determinant $\det Du\,(x)$ involves products of n partial derivatives of u and thus it growth as an n-power of Du; i.e., $|\det Du\,(x)| \cong |Du\,(x)|^n$ for large gradient values. Therefore, since $p < n$, then $\det Du\,(x)$ is not summable; i.e., it is not an L^1-function. Thus the determinant of the $n \times n$ matrix (Du) in general is not defined as a function for all maps $u \in W^{1,p}\,(\Omega, \mathbb{R}^n)$ when $p < n$. However, under the bound (4.4) the Jacobian determinant $\det Du$ is well defined as a *distribution*. Under the constraint $\det Du\,(x) \geq 0$ in (2.3), then this distributional determinant $\det Du$ is a *measure*, precisely a *nonnegative measure*. By using the *Lebesgue decomposition* we can split $\det Du$ in the the *regular part* of the measure, which we denote by $\det_R Du$, and which is defined as the part of the nonnegative measure $\det Du$ absolutely continuos with respect to the Lebesgue measure, and its complement, which is the *singular part* and we denote it by $\det_S Du$. Thus this decomposition gives

$$\det Du = \det{}_R\, Du + \det{}_S\, Du\,, \quad \text{for } u \in W^{1,p}\,(\Omega, \mathbb{R}^n) \ \text{ if } p < n\,,$$

while $\det Du = \det_R Du$ and $\det_S Du = 0$ for $u \in W^{1,n}\,(\Omega, \mathbb{R}^n)$.

Having in mind the *cavitation phenomenon* in *nonlinear elasticity*, the author [77, Section 6] proposed in 1989 the following conjecture, related to the analytic representation of the relaxed energy functional $\overline{F}(u)$ in (4.3):

Under the bound for p (4.4) *and the gradient constraint* (2.3)*, i.e.* $\det Du(x) \geq 0$ *for a.e.* $x \geq \Omega$*, the relaxed energy functional* $\overline{F}(u)$*, defined in* (4.3)*, can be represented in the form*

$$\overline{F}(u) = \int_\Omega \{|Du(x)|^p + g(\det{}_R Du(x))\}\, dx \,+\, \overline{g} \cdot |\det{}_S Du(x)| \,, \tag{4.5}$$

for every $u \in W^{1,p}(\Omega, \mathbb{R}^n)$*, where* $|\det_S Du(x)|$ *is the total variation of the singular part* $\det_S Du(x)$ *the nonnegative measure* $\det Du(x)$ *and* $\overline{g} := \lim_{t \to +\infty} \frac{g(t)}{t}$.

Originally, in the '80, this representation formula was proved in [75], [76] when $u \in W^{1,n}(\Omega, \mathbb{R}^n)$. This equivalently in this case means that the following lower semicontinuity inequality holds for the original energy integral F

$$F(u) \leq \liminf_k F(u_k) \text{ for all } u, u_k \in W^{1,n}(\Omega, \mathbb{R}^n)\,,\ u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^n)\,. \tag{4.6}$$

In other words, $\overline{F}(u) = F(u)$ for all $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $\overline{F}$ is an extension (semicontinuous extension) of F from $W^{1,n}(\Omega, \mathbb{R}^n)$ to $W^{1,p}(\Omega, \mathbb{R}^n)$. Another case also originally obtained was for $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ of the form $u(x) := v(|x|)\frac{x}{|x|}$ with $v(0) > 0$; in this case $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ for all $p < n$, while $u \notin W^{1,n}(\Omega, \mathbb{R}^n)$. More details in Section 5.3.3, see in particular formula (5.16).

In the next section we will describe positive steps and improvements after the first results [75],[76] obtained in 1985-1986.

5. The lower semicontinuity in supercritical, critical and subcritical growths

We describe the developments and the steps to achieve the lower semicontinuity, first in supercritical and critical growth, and later in subcritical growth, both for *quasiconvexity* and *polyconvexity.* We start from the original definition of quasiconvexity by Charles Morrey.

5.1. The origin of quasiconvexity and supercritical growth

5.1.1. Morrey's seminal start [94], [95]

The definition of quasiconvexity and the first quasiconvex lower semicontinuity theorem are due to Morrey [94] in the year 1952; Morrey's book [95] was published in 1966. Morrey's lower semicontinuity result is related to the weak convergence in $W^{1,\infty}(\Omega, \mathbb{R}^m)$. For instance, when applied to the model energy integral (4.2), the lower semicontinuity can be described as

$$F(u) = \int_\Omega f(x, u, Du)\, dx\,, \quad 0 \leq f(x, u, \xi)\,, \quad f \text{ quasiconvex in } \xi \tag{5.1}$$

$$F(u) \leq \liminf_k\ F(u_k) \text{ for all } u, u_k \in W^{1,\infty}(\Omega, \mathbb{R}^m)\,,\ u_k \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega, \mathbb{R}^m)\,.$$

Morrey himself obtained a lower semicontinuity theorem in the weak topology of $W^{1,p}(\Omega,\mathbb{R}^m)$ under a kind of uniform continuity of $f(x,u,\xi)$ with respect to all the variables $(x,u,\xi)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^{m\times n}$. In this case the Morrey's lower semicontinuity result in $W^{1,p}$ can be schematized in this form

$$\begin{aligned}
&F(u)=\int_\Omega f(x,u,Du)\,dx\,,\quad f(x,u,\xi)\ \text{ quasiconvex in } \xi\\
&0\le f(x,u,\xi)\le M\left(1+|u|^p+|\xi|^p\right)\,,\ \text{ and uniform continuous in }\ (x,u) \qquad (5.2)\\
&F(u)\le\liminf_k\ F(u_k)\ \text{ for } u,u_k\in W^{1,p}(\Omega,\mathbb{R}^m)\,,\ u_k\rightharpoonup u\ \text{ in } W^{1,p}(\Omega,\mathbb{R}^m)\,.
\end{aligned}$$

This Morrey's theorem was generalized by Meyers [91] to multiple integrals of any order.

5.1.2. Ball's celebrated work [5] in 1976

John Ball's article [5] in 1976 is fundamental from several points of view. In particular for having pointed out the relevance of Morrey's work in the applications to nonlinear elasticity. Ball also obtained new results that have been a reference point for many mathematicians after his work. John Ball extended Meyers's theorem valid in the *quasiconvex case* [5, Theorem 7.1]. We mention here in particular Ball's lower semicontinuity result in the *polyconvex case* [5, see the proof of Theorem 7.3]. For example, when applied to the model energy integral (4.2), the lower semicontinuity can be described by

$$\begin{aligned}
&F(u)=\int_\Omega f(x,u,Du)\,dx\,,\quad 0\le f(x,u,\xi)\,,\quad f\ \text{ polyconvex}\\
&\textit{for instance}\ \ F(u)=\int_\Omega\left\{|Du(x)|^p+g(\det Du(x))\right\}dx\,,\ \text{ with }\ p\le n \qquad (5.3)\\
&F(u)\le\liminf_k\ F(u_k)\ \text{ with } u_k \text{ satisfying some qualified \textit{equi-summability} on}\\
&\text{Orlicz-Sobolev spaces, in particular }\ \det Du_k\rightharpoonup\det Du\ \text{ in } w-L^1(\Omega)\,.
\end{aligned}$$

It was (and still is) a smart application of the lower semicontinuity result by Ekeland-Temam [48] obtained in the convex case (see [89], too).

5.2. Quasiconvex lower semicontinuity under critical growth

To describe how some research in this field started in Italy in the '80s after Morrey's and Ball's work, it can be useful to mention the description made in [62] by Nicola Fusco and Giuseppe Mingione in the special issue of *Nonlinear Analysis* dedicated in 2018 to Carlo Sbordone:

"Next to integral representation theorems, Carlo was interested in lower semicontinuity problems as the two things are obviously related. The arrival of his longterm collaborator and friend Paolo Marcellini at Napoli gave strong impulse to the study of such a concept. A complete answer to the lower semicontinuity problem was given in the convex case by Marcellini & Sbordone [89] while a first result in the quasiconvex case was obtained by Fusco under an additional equiintegrability assumption

on the converging sequence [59]. This generated a first sharp existence result, contained in the pioneering paper of Marcellini & Sbordone [90], based on a clever use of Gehring's lemma in combination with Fusco's previous result. The whole line of research then culminated with the paper by Acerbi-Fusco [1], where finally lower semicontinuity with respect to the natural topology was proved, bringing such a longstanding and important issue to a conclusion. Eventually, again Marcellini was able to find alternative proofs [75], and to extend the lower semicontinuity results to certain non-standard situations [76]."

In the next Sections (from 5.2.1 to 5.2.3) we briefly describe these quoted results in the Fusco-Mingione's article [62], starting from Fusco [59].

5.2.1. Fusco's [59] lower semicontinuity: from $w^* - W^{1,\infty}$ to $w - W^{1,p+\varepsilon}$

Nicola Fusco [59] extended to the *general quasiconvex case* the lower semicontinuity theorem obtained by Ball in the *polyconvex context*. At the same time Fusco extended Morrey's lower semicontinuity theorem to *Carathéodory integrands*. However, this first Fusco's lower semicontinuity result needs a*bsolutely continuous norms* of the minimizing sequences in $W^{1,p}(\Omega,\mathbb{R}^m)$, being a-priori bounded in $W^{1,p+\varepsilon}(\Omega,\mathbb{R}^m)$. Fusco's lower semicontinuity result is valid in the *quasiconvex case*; it can be described, for every $\varepsilon > 0$, by

$$\begin{aligned}
&F(u) = \int_\Omega f(x,u,Du)\,dx\,,\\
&0 \le f(x,u,\xi) \le M\left(1+|u|^p+|\xi|^p\right)\,,\quad f \text{ Carathéodory and quasiconvex} \qquad (5.4)\\
&F(u) \le \liminf_k\; F(u_k)\ \text{ for } u,u_k \in W^{1,p+\varepsilon}(\Omega,\mathbb{R}^m)\,,\ u_k \rightharpoonup u \text{ in } W^{1,p+\varepsilon}(\Omega,\mathbb{R}^m)\,.
\end{aligned}$$

5.2.2. Marcellini-Sbordone's [90] first existence result in the quasiconvex case

The proof of the existence result by Marcellini-Sbordone, in the *general quasiconvex context*, is based on the existence of a minimizing sequence which is bounded in $W^{1,p+\varepsilon}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$. Precisely, under the coercivity and growth assumption

$$m\,|\xi|^p \le f(x,u,\xi) \le M\,\left(1+|u|^p+|\xi|^p\right)\,,$$

valid for some positive constants m, M and for almost all $x \in \Omega$, all $u \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^{m\times n}$, a delicate use of Gehring's lemma in this vector-valued context allows us to obtain the existence of a minimizing sequence $u_k \in W^{1,p+\varepsilon_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$, for a specific $\varepsilon_0 > 0$, which is bounded in $W^{1,p+\varepsilon_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$. So, by fixing a boundary condition $u_0 \in W^{1,p}(\Omega,\mathbb{R}^m)$, the application of the direct methods of the calculus of variations, together with the Fusco's lower semicontinuity theorem described in the previous Section 5.2.1, gives the existence of a minimizer $u \in u_0 + W^{1,p}_0(\Omega,\mathbb{R}^m)$ and, at the same time, the higher summability of this minimizer $u \in W^{1,p+\varepsilon_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$.

This existence (and regularity) result can be schematized in the following simple form

$$\begin{aligned} &\textit{assumptions}: \; F(u) = \int_\Omega f(x,u,Du)\,dx\,, \quad u_0 \in W^{1,p}(\Omega,\mathbb{R}^m)\,, \;\; p>1\,, \\ &\qquad m\,|\xi|^p \le f(x,u,\xi) \le M\,(1+|u|^p+|\xi|^p)\,, \\ &\qquad f \text{ Carathéodory and quasiconvex} \\ &\textit{conclusion}: \; \exists\,\varepsilon_0>0,\ \exists\,u \in u_0 + W_0^{1,p}(\Omega,\mathbb{R}^m)\cap W_{\mathrm{loc}}^{1,p+\varepsilon_0}(\Omega,\mathbb{R}^m)\;: \\ &\qquad F(u) \le F(v) \text{ for all } v \in u_0 + W_0^{1,p}(\Omega,\mathbb{R}^m)\,. \end{aligned} \tag{5.5}$$

5.2.3. Acerbi-Fusco [1] lower semicontinuity in $w - W^{1,p}$ and the version in [75]

The scheme of the Acerbi-Fusco [1] lower semicontinuity theorem in the weak topology of $W^{1,p}(\Omega,\mathbb{R}^n)$, valid for general *Carathéodory integrands with* p-growth, $p\ge 1$, *quasiconvex* with respect to the gradient variable, is the following

$$\begin{aligned} &F(u) = \int_\Omega f(x,u,Du)\,dx\,, \\ &0 \le f(x,u,\xi) \le M\,(1+|u|^p+|\xi|^p)\,, \quad f \text{ Carathéodory and quasiconvex} \\ &F(u) \le \liminf_k \; F(u_k) \text{ for } u,u_k \in W^{1,p}(\Omega,\mathbb{R}^m)\,,\; u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega,\mathbb{R}^m)\,. \end{aligned} \tag{5.6}$$

The Acerbi-Fusco's theorem is a relevant and not simple result. Its delicate proof does not simplify in the particular model case with $f=f(\xi)$ independent of x,u. Differently from Marcellini [75], who gave an alternative proof of this lower semicontinuity result under slightly more general assumptions, with a simpler argument for the case $f=f(\xi)$. This is the statement in [75] for the general case

$$\begin{aligned} &F(u) = \int_\Omega f(x,u,Du)\,dx\,, \qquad f \;\text{ Carathéodory and quasiconvex} \\ &-m\,(1+|u|^r+|\xi|^r) \le f(x,u,\xi) \le g(x,u)\,(1+|\xi|^p)\,, \quad 1\le r<p\,; \\ &\textit{growth conditions not required for the Carathéodory function } g(x,u)\ge 0 \\ &\text{Conclusion: } \textit{weak lower semicontinuity in } W^{1,p}: \\ &F(u) \le \liminf_k \; F(u_k) \text{ for } u,u_k \in W^{1,p}(\Omega,\mathbb{R}^m)\,,\; u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega,\mathbb{R}^m)\,. \end{aligned} \tag{5.7}$$

We emphasize that in general $1\le r<p$, while $r=1$ for the particular case $p=1$. The lower semicontinuity result in general *is false* if $1<r=p$ (see the details in [75, Example 6.4]). For more recent related results see [52], [100].

5.3. Quasiconvex lower semicontinuity in subcritical growth

We continue the discussion started in Section 2 about the *model energy integral* in nonlinear elasticity (2.1), (2.2)

$$u \to \int_\Omega |Du|^p + g(\det Du)\,dx\,, \tag{5.8}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n\ge 2$, $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$, $p>1$, and $\det Du(x)$ represents the *Jacobian determinant* of the $n\times n$ gradient matrix $(Du)_{n\times n}$.

The physical motivations and the mathematical discussion of Section 2 show that (5.8) is an integral with p-growth from below and q-growth from above, with q different from p. More precisely, as in (2.6), we are here in presence of an energy integral of the type $\int_\Omega f(Du(x))\,dx$, with *p,q-growth conditions* and $p<q$; i.e.,

$$|Du(x)|^p \le f(Du(x)) \le \text{const}(1+|Du(x)|^q),$$

where by $q:=rn$ and $r\ge 1$ is the growth exponent of the nonnegative function $g:(0,+\infty)\to[0,+\infty)$; i.e., $g(t)\le c(1+t^r)$, $t\ge 0$. In the realistic case for applications to nonlinear elasticity $r=1$, $q:=n$, $p<q$; then

$$u\to\int_\Omega f(Du(x))\,dx\,,\quad |Du(x)|^p\le f(Du(x))\le \text{const}\,(1+|Du(x)|^n)\,. \tag{5.9}$$

Since the energy integral in (5.9) is coercive in $W^{1,p}(\Omega,\mathbb{R}^n)$, it is natural to look for lower semicontinuity in the weak topology of $W^{1,p}(\Omega,\mathbb{R}^n)$. The growth-exponent in (5.9) is $q=n$. Therefore the natural problem for this relevant model case is the weak$-W^{1,p}$ lower semicontinuity under the *subcritical growth exponent* $p<n$.

We emphasize a sharp difference with the convex case, when lower semicontinuity holds in the weak-$W^{1,p}$ topology for every $p\ge 1$ [48], [89] and even in the strong L^1-norm topology (see [66], [67]). This is not true in general for quasiconvex integrals; in the following sections we describe positive cases of quasiconvex and polyconvex integrals where weak-$W^{1,p}$ lower semicontinuity holds for some $p<n$.

5.3.1. First approaches [75], [76] to subcritical growth

In the two papers [75], [76] it was first discovered to possibility to obtain lower semicontinuity under subcritical growth; precisely [75] deals with *polyconvex integrals*, while in [76] this fact was extended to the more general *quasiconvex context*. This is the scheme of one of the results in [75, Corollaries 5.6 and 5.7] for the *polyconvex case* (for simplicity we describe here the special case $n=m$; however the result is valid more generally for $n,m\ge 2$)

$$\begin{aligned}
&F(u)=\int_\Omega f(x,Du)\,dx\,,\quad 0\le f(x,\xi)\,,\quad f\ \text{ polyconvex}\\
&i.e.\quad F(u)=\int_\Omega g(x,Du,\text{all minors},\det Du(x))\,dx\,,\\
&g(x,\eta)\ \text{ continuous in }\ (x,\eta)\ \text{ and convex in }\eta\\
&F(u)\le\liminf_k\ F(u_k)\ \text{ for } u,u_k\in W^{1,n}_{\text{loc}}(\Omega,\mathbb{R}^n)\,,\ u_k\rightharpoonup u\ \text{ in } W^{1,p}_{\text{loc}}(\Omega,\mathbb{R}^n)\\
&\text{bound on the topology exponent } p:\ \ \frac{n^2}{n+1}<p\ (\le n)\,.
\end{aligned} \tag{5.10}$$

Please note that the condition $\frac{n^2}{n+1}<p\le n$ forces p to belong to the interval $[n-1,n]$:

$$n-1<\frac{n^2}{n+1}<p\le n\,.$$

Later was proved by Malý [71] (see also Gangbo [63]) that the weak lower semicontinuity in $W^{1,p}_{\text{loc}}$ does not hold in general for $p<n-1$. While some positive

results have been obtained in the full interval $[n-1, n]$, in particular for $f = f(\xi)$, independent of x and u. Details below in the next sections; see for instance (5.20), (5.21) below. With the aim to go in temporal order, we quote here the extension of (5.10) to *general quasiconvex energy integrals*, obtained in [76, Theorem 2.1]:

$$\begin{cases} F(u) = \int_\Omega f(x, Du)\, dx\,, \quad 0 \le f(x,\xi) \le c\,(1 + |\xi|^q)\,, \quad f \text{ quasiconvex in } \xi \\ f(x,\xi) \text{ uniformly continuous in } x, \quad f(x, t\xi) \le c\,(1 + f(x,\xi)) \;\; \forall\, t \in [0,1] \\ F(u) \le \liminf_k F(u_k) \text{ for } u, u_k \in W^{1,q}(\Omega, \mathbb{R}^n)\,,\; u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \\ \text{bound on the topology exponents } p, q: \;\; q\dfrac{n}{n+1} < p \le q\,. \end{cases} \tag{5.11}$$

The condition $q(n/n+1) < p$ corresponds to the previous bound $(n^2/n+1) < p$ for the polyconvex case (5.10), which was obtained with $q = n$. The structure assumption $f(x, t\xi) \le c\,(1 + f(x,\xi))$ for all $t \in [0,1]$ comes out to be satisfied in the polyconvex case (see [76, Lemma 3.1]) and it has been also assumed in some later extensions of this lower semicontinuity theorem for quasiconvex integrands.

5.3.2. Dacorogna-Marcellini [33]: exponent's bound extension in the polyconvex case

After the discovery in (5.10) of the lower semicontinuity under subcritical growth, the first extension of the bound for the exponent p in (5.10) was given by Dacorogna-Marcellini [33] in the *polyconvex case*, arriving to the bound $p > n-1$ in the generic context $n \ge 2$, with also the equality sign $p \ge n-1$ limited to the bidimensional case $n = 2$. This is the result in [33]:

$$\begin{cases} F(u) = \int_\Omega f(Du)\, dx\,, \quad 0 \le f(\xi)\,, \quad f \text{ polyconvex} \\ i.e. \;\; F(u) = \int_\Omega g(Du, \text{all minors}, \det Du(x))\, dx\,, \\ g = g(\eta) \text{ convex in } \eta \\ F(u) \le \liminf_k F(u_k) \text{ for } u, u_k \in W^{1,n}(\Omega, \mathbb{R}^n)\,,\; u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \\ p: \;\; n-1 < p \;\; (\le n), \qquad \text{with } p = 1 \text{ too when } n = 2\,. \end{cases} \tag{5.12}$$

In 1994 Gangbo [63] incorporated in this setting a dependence on x and u as well. Later, DalMaso-Sbordone [36], Fusco-Hutchinson [60] and Celada-DalMaso [15] extended – also when $n > 2$ – the lower semicontinuity to the limiting case $p = n-1$. A more recent related result is in [51]. Counterexamples to lower semicontinuity in the subcritical growth have been provided by Malý [71] for $p < n-1$ and by Gangbo [63].

5.3.3. The Fonseca-Marcellini [57] relaxation in subcritical Sobolev spaces

The manuscript by Fonseca-Marcellini [57] was published in 1997 in the *Journal of Geometric Analysis* although submitted to this Journal in 1994: it is an example of how long sometimes it is necessary to wait for publication! It deals with relaxation of integrals of the calculus of variations of which a prototype example is given by the model (5.8).

We restrict ourselves here to describe the result in [57] for the model case (5.8), (5.9), with relaxed energy as in (4.3); i.e., we denote by F the model energy integral

$$F(u) := \int_{\Omega} \{|Du(x)|^p + g(\det Du(x))\}\, dx$$

and by $\overline{F}$ its associated *relaxed energy functional*

$$\overline{F}(u) := \inf \left\{ \liminf_k \ F(u_k) : u_k \in W^{1,n}(\Omega, \mathbb{R}^n),\ u_k \rightharpoonup u \ \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \right\}, \quad (5.13)$$

where, as before, "$\rightharpoonup$" denotes the *weak topology* of $W^{1,p}(\Omega, \mathbb{R}^n)$, $n \geq 2$. Then, in this special context, the *relaxation theorem* in [57, Theorem 3.1] can be schematized in this way

$$\begin{aligned}
&F(u) = \int_{\Omega} h(Du(x))\, dx, \qquad h(\xi) = f(\xi) + g(\det \xi) \\
&f : \mathbb{R}^{n\times n} \to [0, +\infty), \quad g : \mathbb{R} \to [0, +\infty), \quad Qh(\xi) \ \text{quasiconvex envelope of } W \\
&m|\xi|^p \leq f(\xi) \leq M(1 + |\xi|^p), \quad \xi \in \mathbb{R}^{n\times n}, \quad m|t| \leq g(t) \leq M(1 + |t|), \quad t \in \mathbb{R} \\
&f, g \ \text{ Borel measurable functions}, \quad n - 1 < p < n. \qquad (5.14) \\
&\text{Then} \quad \overline{F}(u) \geq \int_{\Omega} Qh(Du(x))\, dx \qquad \forall\, u \in W^{1,p}(\Omega, \mathbb{R}^n), \\
&\text{and also} \quad \overline{F}(u) = \int_{\Omega} Qh(Du(x))\, dx \qquad \forall\, u \in W^{1,n}(\Omega, \mathbb{R}^n).
\end{aligned}$$

Recall that the *quasiconvex envelope* Qh of a function $h : \mathbb{R}^{n\times n} \to [0, +\infty)$, is defined by

$$Qh(\xi) := \inf \left\{ \frac{1}{|C|} \int_C h(\xi + D\varphi(x))\, dx \ : \varphi \in W_0^{1,\infty}(C; \mathbb{R}^n) \right\}, \qquad (5.15)$$

where C is the n-dimensional cube $\left(-\frac{1}{2}, \frac{1}{2}\right)^n$ (see [32]).

Moreover [57, Theorem 4.1], if the function $f : \mathbb{R}^{n\times n} \to [0, +\infty)$ is quasiconvex and if $g : \mathbb{R} \to [0, +\infty)$ is convex, then the *relaxed energy* $\overline{F}(u)$ can be also explicitly represented as a measure for $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ of the form $u(x) := v(|x|) \frac{x}{|x|}$. In this case $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ for all $p < n$, and $u \notin W^{1,p}(\Omega, \mathbb{R}^n)$ when $p \geq n$, if $v(0) > 0$. The explicit expression of $\overline{F}(u)$ is

$$\overline{F}(u) = \int_{\Omega} f(Du) + g(\det Du)\, dx \ + |\Omega|\, g_{\infty} \cdot v(0)^n, \qquad (5.16)$$

where the *recession function* $g_{\infty} : \mathbb{R} \to [0, +\infty)$ of g is defined by

$$g_{\infty}(s) := \lim_{t\to+\infty} \frac{g((ts))}{t}.$$

5.3.4. Fonseca-Malý [56]: exponent's bound extension in the quasiconvex case

After the discovery in (5.10),(5.11) of the lower semicontinuity under subcritical growth, following the extension of the bound for the exponent p arriving to $p > n-1$ given by Dacorogna-Marcellini [33] in the *polyconvex case* as described in Section 5.3.2, a similarly extension to the quasiconvex case was obtained by Fonseca-Malý [56, Theorem 4.1]. We give here their interesting result, which generalized (5.11),

$$\begin{aligned} &F(u) = \int_\Omega f(Du)\,dx\,, \quad 0 \le f(\xi) \le M(1+|\xi|^q)\,, \quad f \text{ quasiconvex} \\ &F(u) \le \liminf_k F(u_k) \text{ for } u, u_k \in W^{1,q}(\Omega,\mathbb{R}^m)\,,\ u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega,\mathbb{R}^m) \qquad (5.17) \\ &\text{bound on the topology exponents } p,q: \quad q\frac{n-1}{n} < p \le q\,. \end{aligned}$$

Note that, contrary to (5.11) where an integrand $f = f(x,\xi)$ was considered, here it is studied the case $f = f(\xi)$ independent of the x-variable. However it is not assumed the uniform continuity stated in $(5.11)_2$ and a *better bound* for the topology exponents p, q appears in $(5.17)_3$. Also note that, when $q = n$, then the bound $q\frac{n-1}{n} < p \le q$ reduces to $n-1 < p \le n$.

5.3.5. Kristensen [68] lower semicontinuity of quasiconvex integrals in $BV(\Omega;\mathbb{R}^m)$

In [68] Kristensen considers the energy integral $F(u) = \int_\Omega f(Du)\,dx$, where Ω is a bounded open subset of $\mathbb{R}^n$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n, m \ge 2$, Du denotes the *Jacobian gradient matrix* of u, f is a *non-negative quasiconvex function* defined in the space $\mathbb{R}^{m\times n}$ of $m \times n$ matrices, satisfying the growth conditions

$$0 \le f(\xi) \le M(1+|\xi|^q)\,, \qquad \forall\, \xi \in \mathbb{R}^{m\times n}\,, \tag{5.18}$$

with $q \ge 1$ and $M > 0$. The aim of Kristensen's work is to obtain the lower semicontinuity of this quasiconvex integral in BV, the space of maps u of *bounded variation*.

If $u : \Omega \to \mathbb{R}^m$, $u \in BV(\Omega;\mathbb{R}^m)$, then Du denotes the *distributional gradient* of u and ∇u denotes the *approximate gradient* of u, which is the Radon-Nikodym derivative of the *measure* Du with respect to Lebesgue measure, and which exists almost everywhere for any map u of *bounded variation*. The Kristensen's lower semicontinuity result in [68, Theorem 1.1] is

$$\begin{aligned} &F(u) = \int_\Omega f(Du)\,dx\,, \quad 0 \le f(\xi) \le M(1+|\xi|^q)\,, \quad f \text{ quasiconvex} \\ &\text{bound on the topology exponents } p,q, \text{ with } p=1: \quad q\frac{n-1}{n} < 1 \le q \\ &u \in BV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)\,, \quad u_k \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)\,, \qquad (5.19) \\ &u_k \to u \text{ in } L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^m)\,, \quad \forall\, \omega \subset\subset \Omega\ \exists\, c = c(\omega): \int_\omega |Du_k|\,dx \le c(\omega) \\ &\text{Conclusion:} \quad \int_\Omega f(\nabla u)\,dx \le \liminf_k \int_\Omega f(Du_k)\,dx\,. \end{aligned}$$

A further lower semicontinuity result in [68, Proposition 1.2], with two exponents p, q greater than 1, is

$$\begin{aligned}
&F(u) = \int_\Omega f(Du)\,dx\,, \quad 0 \le f(\xi) \le M(1+|\xi|^q)\,, \quad f \ \text{quasiconvex} \\
&\text{bound on the topology exponents } p,q>1: \quad 1 < q\frac{n-1}{n} \le p \le q \\
&u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)\,, \quad u_k \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)\,, \qquad (5.20)\\
&u_k \rightharpoonup u \ \text{ in } W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)\,, \quad \forall\,\omega \subset\subset \Omega\ \exists\, c = c(\omega): \int_\omega |Du_k|^{q\frac{n-1}{n}}\,dx \le c(\omega) \\
&\text{Conclusion:} \quad \int_\Omega f(Du)\,dx \le \liminf_k \int_\Omega f(Du_k)\,dx\,.
\end{aligned}$$

We emphasize the possibility of the *equality sign* in the bound $(5.20)_2$:

$$\frac{q}{p} \le \frac{n}{n-1} = 1 + \frac{1}{n-1}.$$

In particular, if $q = n$, then the bounds on p become

$$n - 1 \le p \le n\,, \tag{5.21}$$

with also $1 < q(n-1)/n$; i.e., $n \ge 3$. See also [69] for related Kristensen's results.

5.4. Further on lower semicontinuity and relaxation

In this section we mainly consider relaxation of *quasiconvex integrals* outside the traditional regularity space. Problems of relaxation and integral representation for *convex integrals* is widely studied in the book by Buttazzo [11].

5.4.1. Ambrosio-DalMaso [3] and Fonseca-Müller [58] relaxation in $BV(\Omega;\mathbb{R}^m)$

The following relaxation theorem in $BV(\Omega;\mathbb{R}^m)$, in the spirit of formula (4.3) above, was obtained by Ambrosio-DalMaso [3, Theorem 4.1] for quasiconvex integrals of the form $F(u) = \int_\Omega f(Du)\,dx$ with integrand $f = f(\xi)$ of linear growth. A similar generalization to integrals $F(u) = \int_\Omega f(x,u,Du)\,dx$ is due to Fonseca-Müller [58]. The *relaxed energy functional* $\overline{F}$ associated to F is defined by

$$\overline{F}(u) := \inf\left\{\liminf_k\ F(u_k): \ u_k \in C^1(\Omega,\mathbb{R}^m)\,,\ u_k \to u \ \text{ in } L^1(\Omega,\mathbb{R}^m)\right\}. \tag{5.22}$$

The *recession function* $f_\infty : \Omega \subset \mathbb{R}^{m\times n} \to \mathbb{R}$ of f is defined by

$$f_\infty(\xi) := \limsup_{t\to+\infty} \frac{f((t\xi))}{t}\,. \tag{5.23}$$

We recall that if $u \in BV(\Omega;\mathbb{R}^m)$ is a *map of bounded variation*, then Du is a *measure*. We denote by ∇u the density of the absolutely continuous part of the measure Du with respect to Lebesgue measure; we also denote by $D^s u$ is the *singular part* of the measure Du and by $\frac{D^s u}{|D^s u|}$ the Radon-Nikodym derivative of $D^s u$ with respect to its variation $|D^s u|$. Then the Ambrosio-DalMaso [3] result is a representation formula for $\overline{F}$ in $BV(\Omega;\mathbb{R}^m)$ and it can be schematized in this way

$$\begin{aligned} & F(u)=\int_\Omega f(Du)\,dx \quad \forall\, u\in C^1(\Omega,\mathbb{R}^m)\,, \\ & 0\le f(\xi)\le c(1+|\xi|) \quad \forall\,\xi\in\mathbb{R}^{m\times n}\,,\quad f \text{ quasiconvex} \\ & \overline{F}(u)=\int_\Omega f(\nabla u)\,dx+\int_\Omega f_\infty\Big(\frac{D^s u}{|D^s u|}\Big)\,|D^s u|\,,\quad \text{for all } u\in BV(\Omega;\mathbb{R}^m)\,. \end{aligned} \tag{5.24}$$

5.4.2. Fonseca-Fusco-Marcellini's [53] total variation of the Jacobian

Given a map $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ in an open bounded set $\Omega\subset\mathbb{R}^n$, as before, $\det Du(x)$ represents the *Jacobian determinant* of the $n\times n$ *gradient matrix* $(Du)_{n\times n}$. The *total variation* $TV(\Omega,u)$ of the Jacobian $\det Du$ is defined for smooth maps u, say $u\in W^{1,n}(\Omega,\mathbb{R}^n)$, as usual as the integral

$$TV(\Omega,u):=\int_\Omega |\det Du(x)|\,dx\,,\qquad u\in W^{1,n}(\Omega,\mathbb{R}^n)\,. \tag{5.25}$$

This definition is consistent, since $\det Du(x)\in L^1(\Omega)$ when $u\in W^{1,n}(\Omega,\mathbb{R}^n)$. In [53] for the special relevant case $n=2$, and in [54] for general $n\ge 2$, Fonseca-Fusco-Marcellini addressed the study of the Jacobian determinant outside the traditional regularity space $W^{1,n}(\Omega,\mathbb{R}^n)$, precisely in $W^{1,p}(\Omega,\mathbb{R}^n)$, for p in the open interval of real numbers $(n-1,n)$, by the *relaxation formula,*

$$TV(\Omega,u):=\inf\left\{\begin{array}{l}\liminf\limits_{k\to+\infty}\displaystyle\int_\Omega|\det Du_k(x)|\,dx\ : \\ u_k\in W^{1,n}(\Omega,\mathbb{R}^n)\,,\ u_k\rightharpoonup u \text{ in } W^{1,p}(\Omega,\mathbb{R}^n)\end{array}\right\}. \tag{5.26}$$

A priori, definition (5.26) may depend on p, and we could denote by $TV_p(\Omega,u)$ this *relaxed total variation.* However, the representation formulas for $TV(\Omega,u)$ in [53], [54] turn out to be independent of p. It was also shown that weak convergence may be equivalently replaced by strong convergence in $W^{1,p}(\Omega,\mathbb{R}^n)$. The considered classes of functions was $u\in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\cap W^{1,p}(\Omega;\mathbb{R}^n)$, $n-1-p<n$, in particular for those u which are locally Lipschitz continuous away from a given point $x_0\in\Omega$, and thus with the Jacobian determinant $\det Du$ possibly singular only at x_0. Thus $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{x_0\};\mathbb{R}^n)$.

We limit ourselves here to consider the 2-d case; i.e., $n=2$. Let

$$v:[0,2\pi]\to\Gamma\subset\mathbb{R}^2$$

be a Lipschitz continuous map, satisfying $v(0)=v(2\pi)$, with the components $v(\vartheta)=(v^1(\vartheta),v^2(\vartheta))$ and with values on a set $\Gamma=v([0,2\pi])$, *boundary* of a simply connected domain D, analytically represented by

$$\Gamma=\{\xi+r(\vartheta)(\cos\vartheta,\sin\vartheta):\vartheta\in[0,2\pi]\}\,,$$

where $r(\vartheta)$ is a piecewise C^1-function such that $r(\vartheta)\ge r_0$ for every $\vartheta\in[0,2\pi]$ and for some positive r_0. Γ is the Lipschitz continuous boundary of a domain *starshaped* with respect to a point ξ in the *interior* of D.

Theorem 5.1. ([53]) *Let u be a function of class $W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^2)$ for some $p\in(1,2)$. Let $v:[0,2\pi]\to\Gamma$, be a Lipschitz continuous map, $v(0)=v(2\pi)$ such that $\lim_{\varrho\to 0}\|u(\varrho,\cdot)-v(\cdot)\|_{L^\infty((0,2\pi);\mathbb{R}^2)}=0$. If the tangential derivative $D_\tau u$ of u satisfies the bound*

$$\sup_{\varrho>0}\frac{1}{\varrho^{2-p}}\int_0^\varrho r^{1-p}\,dr\int_0^{2\pi}|u_\vartheta(r,\vartheta)|^p\,d\vartheta\le M_0$$

for a positive constant M_0, then the representation formula holds

$$TV(u,\Omega)=\int_\Omega|\det Du(x)|\,dx+\frac{1}{2}\left|\int_0^{2\pi}\left\{v^1v^2_\vartheta-v^2v^1_\vartheta\right\}d\vartheta\right|.$$

An example of application of the general 2-d result for the map

$$u(x)=v\Big(\frac{x}{|x|}\Big),\quad u\in W^{1,p}\left(\Omega;\mathbb{R}^2\right)\cap W^{1,\infty}_{\mathrm{loc}}\left(\Omega\setminus\{0\};\mathbb{R}^2\right),\quad\forall\,p<2,$$

where $v:[0.2\pi]\to S^1$ is the map $v(\vartheta)=(\cos g(\vartheta),\sin g(\vartheta))$, with $g:[0.2\pi]\to\mathbb{R}$ Lipschitz continuous function such that, for some $k\in\mathbb{Z}$, $g(2\pi)=g(0)+2k\pi$.

Since $v^1v^2_\vartheta-v^2v^1_\vartheta=g'$ and $\det Du(x)=\det Dv(x/|x|)=0$ for all $x\ne 0$,

$$TV(u,B_1)=\int_\Omega|\det Du(x)|\,dx+\frac{1}{2}\left|\int_0^{2\pi}g'(\vartheta)\,d\vartheta\right|=\frac{1}{2}\left|\int_0^{2\pi}g'(\vartheta)\,d\vartheta\right|=|k|\,\pi\,.$$

Note that here g not necessarily is a monotone function and that

$$TV(u,B_1)=\frac{1}{2}\left|\int_0^{2\pi}g'(\vartheta)\,d\vartheta\right|$$

has the *absolute value sign outside the integral sign*, and not inside. While, if $w(x)=|x|\;u(x)$ is the radially linear Lipschitz continuous extension of v, then we have

$$TV(w,B_1)=\frac{1}{2}\int_0^{2\pi}|g'(\vartheta)|\,d\vartheta.$$

instead. A *geometrical interpretation*: if $v:S^{n-1}\to S^{n-1}$ is a map of class C^1 onto S^{n-1}, locally invertible with local inverse of class C^1, and if $u(x):=v(x/|x|)$, then $TV(B_1,u)$ may be expressed in terms of the *topological degree* of the maps v and $\widetilde{v}$, where $\widetilde{v}:B_1\to B_1$ is any *Lipschitz continuous extension* of v to the unit ball B_1: $TV(u,B_1)=\omega_n|\deg v|=\omega_n|\deg\widetilde{v}|$.

Some relevant related articles are due to Fonseca-Leoni-Malý [55], Mingione-Mucci [93], DePhilippis [42]. In particular in [42, Section 6] Guido DePhilippis extended to the general context some results proved in [76],[77] (see also [57] and [53],[54]) for radial maps $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$, $n\ge 2$.

5.4.3. Schmidt [97], DeFilippis [37], DeFilippis-Stroffolini [41] and Gmeineder-Kristensen [65] partial regularity

A well known result by Evans [50] states that minimizers u of a class of *quasi-convex integrals* of the form $F(u)=\int_\Omega f(Du)\,dx$ are *partially regular*; i.e., their gradient Du is *locally Hölder continuous in* Ω *outside a closed subset of zero measure*. Later, about *relaxation* and *partial regularity* for variational problems in the *quasiconvex vector-valued case and* p, q*-growth conditions*, after the relevant work by Schmidt [97], more recently we refer to DeFilippis [37], DeFilippis-Stroffolini [41] and Gmeineder-Kristensen [65]. Their approach starts with an energy integral which a-priori is defined only for *smooth maps* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n, m \geq 2$, and they introduce a *relaxation formula* of the type described above in (4.3), (5.13), (5.22), (5.26).

We give here some details, starting from Thomas Schmidt's work. After some preliminary results obtained in 2008 when he did not use minimizers of relaxed energies, his main partial regularity result maybe is Theorem 6.4 in [97], which instead is correctly related to minimizers of the *relaxed energy functional* (defined as above). More precisely:

$$\begin{aligned} &F(u) = \int_\Omega f(Du)\, dx\,, \quad u \in W^{1,q}(\Omega, \mathbb{R}^m)\,, \\ &f \in C^2\left(\mathbb{R}^{m\times n}\right) \text{ quasiconvex in } \xi, \text{ satisfying some } p, q\text{-growth conditions.} \\ &\text{Let } u \text{ be a local minimizer of} \\ &\overline{F}(u) := \inf \left\{ \begin{array}{l} \liminf_k \; F(u_k) : \; u_k \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cap W^{1,p}(\Omega, \mathbb{R}^m) \\ u_k \rightharpoonup u \; \text{ in } W^{1,p}(\Omega, \mathbb{R}^m) \end{array} \right\}. \\ &\text{If } p \geq 2 \text{ and } \frac{q}{p} < 1 + \frac{1}{np}\,, \text{ or } 1 < p < 2 \text{ and } \frac{q}{p} < 1 + \frac{1}{n}\,, \text{ there exists} \\ &\text{an open set } \Omega_u, \text{ with } \Omega \backslash \Omega_u \text{ of zero Lebesgue measure, such that} \\ &\text{the } m \times n \text{ gradient matrix } Du \text{ is } \textit{Hölder continuous in } \Omega_u; \\ &\textit{i.e. } Du \in C^{0,\alpha}_{\mathrm{loc}}\left(\Omega_u; \mathbb{R}^{m\times n}\right) \text{ for all } \alpha \in (0,1)\,. \end{aligned} \tag{5.27}$$

Cristiana DeFilippis in [37] considers energy integrals, defined on an open bounded set $\Omega \subset \mathbb{R}^n$, of the type

$$\begin{aligned} &F(u) = \int_\Omega \{f(Du) - b(x)\, u\}\, dx\,, \quad u \in W^{1,q}(\Omega, \mathbb{R}^m)\,, \\ &f \text{ quasiconvex in } \xi\,, \quad \text{satisfying some } p, q\text{-growth conditions} \\ &b = b(x) \;\text{ in the } \textit{Lorentz space } L(n,1)\,. \end{aligned} \tag{5.28}$$

Recall in $(5.28)_3$ that $L(n,1) \subset L^n$ and $L^r \subset L(n,1)$ for all $r > n$. The relaxed energy $\overline{F}(u)$ in [37, formula (1.2)] is defined, similarly above in this manuscript, for all $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, by

$$\overline{F}(u) := \inf\left\{\liminf_k \; F(u_k) : u_k \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)\,,\; u_k \rightharpoonup u \;\text{ in } W^{1,p}(\Omega, \mathbb{R}^m)\right\}. \tag{5.29}$$

One of the main results by DeFilippis is the following.

Theorem 5.2. ([37, Theorem 2]) *We consider the integral*

$$\int_\Omega \{f(Du) - b(x)\, u\}\, dx$$

in $(5.28)_1$ *under some qualified* p, q*-growth conditions* ($p \geq 2$) *on the (strictly) quasi-convex* C^2*-function* $f(\xi)$ *(see details in* (2.4), (2.6) *of* [37]) *and with the summability assumption* $b \in L^r(\Omega; \mathbb{R}^m)$ *for some* $r > n$*. Let* u *be a local minimizer of the relaxed energy functional* (5.29)*. If*

$$p \leq q < p + \frac{1}{n}, \tag{5.30}$$

there exists an open set Ω_u*, with* $\Omega \backslash \Omega_u$ *of zero Lebesgue measure, such that the* $m \times n$ *gradient matrix* Du *is Hölder continuous in* Ω_u*; i.e.* $Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u; \mathbb{R}^{m \times n})$ *for some* $\alpha \in (0, 1)$*.*

Remark 5.3. It is interesting to connect the DeFilippis' Theorem 5.2 to the model example of Sections 2,4 above, with the energy integral, for $m = n \geq 2$,

$$F(u) = \int_\Omega f(Du(x))\, dx = \int_\Omega \{|Du(x)|^p + g(|\det Du(x)|)\}\, dx \tag{5.31}$$

where $g : [0, +\infty) \to [0, +\infty)$ is a real nonnegative function such that $g(t) \leq c(1 + t)$ for a positive constant c and all $t \geq 0$. Then, as in Sections 2,4, we are in presence of p, q*-growth conditions with* $q = n$

$$|Du(x)|^p \leq f(Du(x)) \leq M(1 + |Du(x)|^n), \tag{5.32}$$

precisely $M > 0$ and $p < q = n$.

The bound (5.30) for the partial regularity of Theorem 5.2, when $p < q$ and $q = n$ reads $p < n < p + \frac{1}{n}$; i.e.

$$n - \frac{1}{n} < p < n, \quad \text{with } n \geq 2, \tag{5.33}$$

which is comparable with the bounds for lower semicontinuity and relaxation. For instance with $\frac{n^2}{n+1} < p < n$ in (4.4), which is less strict than (5.33); in fact the inequality $\frac{n^2}{n+1} < n - \frac{1}{n}$ is equivalent to $n - 1 + \frac{1}{n+1} < n - \frac{1}{n}$ and to

$$\frac{1}{n} < 1 - \frac{1}{n+1} = \frac{n}{n+1};$$

i.e., $n + 1 < n^2$, which is satisfied by all $n \geq 2$. Of course we also have

$$n - 1 < n - \frac{1}{n} < p < n,$$

for all $n \geq 2$, which make comparable (5.33) with the Gmeineder-Kristensen's [65, Theorem 2.2] bound $p \in (n-1, n)$, described below and obtained for partial regularity when $p > 1$ and $q = n$. A natural question is if these partial regularity theorems hold (DeFilippis [37, Theorem 2], Gmeineder-Kristensen [65, Theorem 2.2] and Schmidt [97] too) with the larger bound in (4.4) $p \in [n-1, n)$ when $p > 1$ and $q = n$. The discussion in [65] seems to give a negative answer.

Inspired by the cavitation phenomenon [77] introduced in nonlinear elasticity by Ball [6], further questions remain about the model energy integral (5.31), in particular on the function g with linear behavior at $+\infty$, i.e. $0 \leq g(t) \leq c(1 + t)$ for a constant c and all $t \geq 0$.

If $g(t)$ is convex for all $t \geq 0$ and we consider a minimizer of the relaxed energy functional $\overline{F}(u)$ in (5.29) in the ball $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, with a radial boundary condition $u_0(x) := \lambda x$ on $\partial\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$. Is the map u, defined equal to $u(x) := \lambda x$ in all the ball Ω, also a minimizer of the original energy integral $F(u)$ in (5.31)? Yes, it is! By quasiconvexity! Therefore, does the phenomenon of cavitation appears only for large values of λ and only when $g(t)$ is not convex for large values of t (for instance as in the Figure 2 of [77])?

We do not expect that these partial regularity results hold for minimizers of (5.31), or of the associated relaxed energy functional, when $g(t)$ is not convex for large values of $t > 0$. Can we expect everywhere regularity for minimizers of the model case with a convex g? $\square$

The approach by F. Gmeineder and J. Kristensen [65] is also related to a quasiconvex integrand satisfying some p, q-growth conditions. A main difference with the described paper by DeFilippis [37] is that the p, q-growth by Gmeineder and Kristensen allows $p = 1$. Their analysis include the space of *maps of bounded variation* $BV(\Omega; \mathbb{R}^m)$ and their representations formula for the relaxed energy $\overline{F}(u)$ in (5.29) is also based on the *recession function,* as in (4.5) and (5.23),(5.24) above.

A further difference of the Gmeineder-Kristensen approach with respect to [37] is their Theorem 2.2, which is related to the case $p > 1$. In this case their bound for partial regularity is given by

$$1 < p \le q < \min\{\frac{np}{n-1}, p+1\}.$$

Similarly as done in Remark 5.3 when $p < q$ and $q = n$, if we compare this bound to the model energy integral of Sections 2,4, we obtain $p < n < \min\{\frac{np}{n-1}, p+1\}$, which is equivalent to the *sharp conditions* $n - 1 < p < n$.

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