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Remembering Hedy Attouch

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Dedicated to the memory of Hedy Attouch.

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An elementary, self-contained introduction of so-called M-convergence of subspaces and convex sets is provided, not available in the published literature so far. The aim of this description is to highlight the peculiar expository style of Hedy Attouch which combines insight and deepness with clarity and simplicity.

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Foreword for young readers

Some years ago I inherited a book from my father. The author was a great mathematician, Beppo Levi. In this book he explained simple elementary mathematics to his own grandchildren. Of course, he did that wonderfully.

This brings me to Hedy Attouch. Hedy as a master of explaining difficult mathematics in extremely clear and simple way. Ideally, Hedy is to me as Beppo Levi was to his grandchildren! In fact, in what follows, I will do my best to capture the essence of his work on some important aspects of his work related to the two fundamental concepts of *duality* and *convergence*, in a way as simple as his style is.

I should add that the present contribution to the special number of the Journal of Convex Analysis published in his memory provides a self-contained, elementary introduction to so-called *M-Convergence* not available in the printed literature. The paper is therefore of independent interest in itself.

This program is carried out in three main steps, denoted below as Chapter 1 (Euclidean spaces), Chapter 2 (Real Hilbert spaces), Chapter 3 (Complex Hilbert spaces).

A few additional comments on the vast, fundamental work by Attouch on variational theories and their applications, as developed by him, in particular, in the two references [1] and [2], will be added in our final Chapter 5.

Chapter 1. Euclidean spaces

We begin with a couple of intuitive elementary evidences, named, for lack of better words, *Fact 1* and *Fact 2*.

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We consider a right triangle of vertices **APB** and collocate a point **P** on the semi-circle lying above the diameter **AB**. While keeping **A** and **B** fixed, we move the point **P** clockwise along the semi-circle, starting from **A** and ending in **B**. We the observe two facts:

Fact 1. The triangle of vertices **APB** is a right triangle, hence the sides **PA** and **PB** are *orthogonal*.

Fact 2. As **P** converges to **A** (or to **B**), the triangle **APB** *degenerates into the segment* **AB**.

We now reformulate these two Facts, modulo a translation, in an equivalent form which better reveals the general property that underlies these two Facts. We call these remarks Fact 1' and Fact 2':

Fact 1'. The straight-line, or subspace **M** through the origin **O** and slope $|\mathbf{PC}|/|\mathbf{AC}|$, where **C** is the center of the circle, and the straight-line, or subspace **N** through the origin **O** and slope $|\mathbf{PC}|/|\mathbf{BC}|$, are mutually *orthogonal*.

Fact 2'. As **P** converges to **A** (or to **B**), the triangle **APB** *degenerates into the segment* **AB**.

The elementary constructions done so far bring into evidence the following two Properties:

Property 1. The plane can be decomposed as the vector sum of two *orthogonal* lines M and N through the origin. Here *orthogonal* means elementary perpendicularity.

Property 2. In the Euclidean plane, the two lines M and N of Proposition 1 *converge* to a limit line through the origin provided the *angle* between the lines shrinks to zero.

Now we ask ourselves how the previous elementary constructions and Property 1, Property 2 can be generalized to more general spaces than the plane. The answer is different in the two cases.

Property 1 keeps sense, without any change in its formulation, if we replace perpendicularity by orthogonality of vectors in Hilbert spaces. Instead, Property 2 requires to define what *convergence of vectors* means in a general Hilbert space. This is not so obvious to describe and we shall do it in the next Chapter 2 (real Hilbert spaces) and Chapter 3 (complex Hilbert spaces).

Chapter 2. Real Hilbert spaces

Definition 2.1. A *real pre-Hilbert space* H is defined to be a linear vector space over the real field, endowed with a strictly positive definite bilinear form over the reals, denoted (x, y) and called a (real) *inner product*:

(i) $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1(x_1, y) + \alpha_2(x_2, y)$ for every real numbers α_1, α_2 and any vectors $x_1, x_2, y \in H$

(ii) $(x, y) = (y, x) \quad \forall\, x, y \in H$

(iii) $(x, x) \geq 0 \quad \forall\, x \in H$ and $(x, x) = 0$ implies $x = 0$.

In any real pre-Hilbert space H a *norm*, denoted $||x||$, is defined by setting for every vector $x \in H$:

(iv) $||x|| = \sqrt{(x,x)}$.

Endowed with such $||x||$, H is a *normed space* according to the following definition:

Definition 2.2. A *normed space* X is a linear vector space X over the real field endowed with a *norm* $||x||$, defined by the properties:

(j) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$

(jj) $||x + y|| \leq ||x|| + ||y||$

(jjj) $||\alpha x|| = |\alpha|\ ||x||$, where $|\alpha|$ is the modulus of the real number α, i.e., $|\alpha| = \alpha$ if $\alpha \geq 0$, $|\alpha| = -\alpha$ if $\alpha < 0$.

We start by proving a fundamental inequality, namely, so called *Cauchy-Schwartz inequality*:

Proposition 2.3. (Cauchy-Schwartz Inequality) *If H is a real pre-Hilbert space, then the following Cauchy-Schwartz inequality holds for every $x, y \in H$:*

$$\text{(CS)} \qquad |(x,y)| \leq ||x||\,||y||\,.$$

The equality in this inequality is verified if and only if x and y are linear dependent, that is, $x = cy$ for some $c \neq 0$.

Proof. By (i) and (ii), for any real number α we have

$$(x + \alpha(x,y)y, x + \alpha(x,y)y) = ||x||^2 + 2\alpha|(x,y)|^2 + \alpha^2|(x,y)|^2||y||^2 \geq 0.$$

This is a polynomial in the indeterminate α which does not change sign when α runs from ∞ to $+\infty$. Therefore the coefficient must satisfy the condition

$$4|(x,y)|^4 - 4||x||^2|(x,y)|^2||y||^2 \leq 0,$$

which is to say

$$|(x,y)|^4 \leq ||x||^2|(x,y)|^2||y||^2\,.$$

Therefore,

$$|(x,y)| \leq ||x||||y||\,.$$

In the last step we have implicitly assumed, when dividing both members of the inequality by $|(x,y)|$, that $|(x,y)| \neq 0$. In the case that $(x,y) = 0$, the inequality (CS) reduces to $0 \leq ||x||||y||$ and then it holds trivially. Moreover, the last property in the statement of the Proposition 2.3 follows from the property (iii) of the inner product. □

An important consequence of Cauchy-Schwartz inequality is the following *triangle inequality*:

Proposition 2.4. (Triangle Inequality) *For every vectors x and y in a real pre-Hilbert space, the following triangle inequality holds*:

$$\text{(TI)} \qquad ||x + y|| \leq ||x|| + ||y||\,.$$

Proof. We start by developing $||x + y||^2$ as follows:

$$||x + y||^2 = (x + y, x + y) = ||x||^2 + (x,y) + (y,x) + ||y||^2 \leq ||x||^2 + 2|(x,y)| + ||y||^2.$$

By applying Cauchy-Schwartz inequality (CS) at the right hand side

$$||x||^2 + 2|(x,y)| + ||y||^2 \leq ||x||^2 + 2||x||||y|| + ||y||^2 = (||x|| + ||y||)^2,$$

therefore $||x+y||^2 \leq (||x|| + ||y||)^2$ from which the triangle inequality (TI) follows by taking square roots at both sides. □

Proposition 2.5. *A real pre-Hilbert space H with inner product (x,y) becomes a normed space if the norm $||x||$ in H is defined by*:

$$||x|| := \sqrt{(x,x)}.$$

Proof. We must verify that $||x||$, as defined above, verifies the properties (j), (jj), (jjj) of Definition 2.2. Property (j) is a reformulation of (iii) of Definition 2.2. Property (jj) is the triangle inequality established by Proposition 2.4. Property (jjj) is a special case of (i) of Definition 2.2 when $\alpha_1 = \alpha$ and $\alpha_2 = 0$. □

Now we prove that in every real pre-Hilbert space an important identity holds, called *parallelogram identity*:

Proposition 2.6. (The parallelogram identity) *For every vectors x, y in a pre-Hilbert space H the following parallelogram identity holds*

$$\text{(ParI)} \qquad ||x+y||^2 \, + \, ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

This identity can be equivalently written as follows:

$$\text{(ParI*)} \qquad ||\frac{x+y}{2}||^2 \, + \, ||\frac{x-y}{2}||^2 = \, \frac{1}{2}\left(||x||^2 \, + \, ||y||^2\right).$$

Proof. By the definition (iv) of a norm in a pre-Hilbert space, we have $||x-y||^2 = (x-y, x-y)$. Developing the inner product at the right hand side we get:

$$||x-y||^2 = (x-y, x-y) = (x,x) + (x,-y) + (-y,x) + (-y,-y) = ||x||^2 - 2(x,y) + ||y||^2 \, .$$

Here we have used (i) and (ii) of Definition 2.1, that is, that an inner product is symmetric and linear in each term.

Similarly, we expand $||x+y||^2 = (x+y, x+y) = ||x||^2 + 2(x,y) + ||y||^2$.

By adding these two equalities term by term, the inner product addends cancels and we get

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

which is (ParI). By multiplying both terms by $\frac{1}{4}$ we get

$$||\frac{x+y}{2}||^2 \, + \, ||\frac{x-y}{2}||^2 = \frac{1}{2}\left(||x||^2 \, + \, ||y||^2\right).$$

which is (ParI*). Conversely, by multiplying both terms of (ParI*) by 4, we get back to (ParI). □

We now give an important identity, called *polarization identity.*

Proposition 2.7. (The polarization identity) *Let H be a pre-Hilbert space H and let $||x|| = \sqrt{(x,x)}$ be the norm of H as introduced in* (iv) *of Definition* 2.1. *Then the following identity holds*:

$$(x,y) = \frac{1}{4}(||x+y||^2 - ||x-y||^2)\,. \tag{PoI}$$

Proof. As before, we develop the squared norms as in the proof of Proposition I.4:

$$||x+y||^2 = ||x||^2 + 2(x,y) + ||y||^2\,,$$

$$||x-y||^2 = ||x||^2 - 2(x,y) + ||y||^2\,.$$

By subtracting these two equalities term by term, we get

$$||x+y||^2 - ||x-y||^2 = 4(x,y)\,.$$

By dividing by 4, the identity (PoI) follows. □

The original result by M. Fréchet, J. von Neumann and P. Jordan mentioned before, in comparison with Proposition 2.6, is a kind of reverse Proposition 2.7. In Proposition 2.7 , we prove that in any *given* real pre-Hilbert space the polarization identity (PoI) holds. Now, in Proposition 2.8 below, we show that *given any real normed space X* with an assigned norm $||x||$, if we *define* a form (x,y) by means of the polarization identity (PoI), then (x,y) is indeed an inner product for X.

Proposition 2.8. (The F-vN-J result) *In any given real normed space X, if we define (x,y) by setting for every x and y of X :*

$$(x,y) := \frac{1}{4}(||x+y||^2 - ||x-y||^2)\,. \tag{*}$$

then (x,y) is a real inner product, that is, it satisfies the properties (i), (ii), (iii).

Proof. In order to prove that (x,y), as defined in (*), is an inner product on X, we must check that the following properties are satisfied:

(i) $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1(x_1,y) + \alpha_2(x_2,y)$ for every real numbers α_1, α_2 and any vectors $x_1, x_2, y \in X$;

(ii) $(x,y) = (y,x)\ \forall\ x,y \in X$;

(iii) $(x,x) \geq 0\ \forall\ x \in X$ and $(x,x) = 0$ implies $x = 0$.

Property (ii) is clear, because the values of the form (x,y) in (*) do not change if we exchange x and y.

We check property (iii). The property $(x,x) \geq 0$ is easily checked by replacing $y = x$ in (*). In fact, this gives

$$(x,x) = \frac{1}{4}(||x+x||^2 - ||x-x||^2) = ||x||^2 \geq 0\,.$$

This expression of (x,x) shows also that $(x,x) = 0$ implies $||x||^2 = 0$, hence $x = 0$.

It remains to prove (i), that is, that (x,y) is linear in both x and y. Because of the symmetry (ii), it suffices to show that (x,y) is linear in the first term x.

We start by proving that for each fixed $z \in X$ the form (x, z) is *additive* in the first term x, that is

$$\text{(ADD)} \qquad (x + y, z) = (x, z) + (y, z)\,, \qquad \forall\, x,\, y \in X\,.$$

We proceed as follows:

$$\begin{aligned}
&(x,z) + (y,z) = \text{ [by (*)] } 4^{-1}\,(||x+z||^2 - ||x-z||^2 + ||y+z||^2 - ||y-z||^2) \\
&= \text{ [by (ParI)] } 4^{-1}[-||x-z||^2 + 2||x||^2 + 2||z||^2 - ||y-z||^2 + 2||y||^2 + 2||z||^2 \\
&\quad + ||x+z||^2 - 2||x||^2 - 2||z||^2 + ||y+z||^2 - 2||y||^2 - 2||z||^2] \\
&= 4^{-1}||x+z||^2 - ||x-z||^2 + ||y+z||^2 - ||y-z||^2 = \text{by (*)] } (x,z) + (y,z)\,.
\end{aligned}$$

We rewrite the equality (ADD) obtained above as

$$\text{(ADD*)} \qquad (x, z) + (y, z) = 2(\frac{x+y}{2}, z)$$

valid for all $(x, y \in X$, where we used the linearity of the inner product in the first term. If in (ADD*) we take $y = 0$, we obtain

$$(x, z) = by(*) = 4^{-1}(||0 + z||^2 - ||0 - z||^2) = 0\,.$$

Therefore, we obtain $(x, z) + (y, z) = (x + y, z)$ which is indeed (ADD).

From (ADD) it follows in particular, by replacing $y = x$, that

$$(\alpha\, x, y) = \alpha\ (x, y)$$

for $\alpha = 2$ and, by iteration, for all rationals numbers $\alpha = m/2^n$, $m \geq 1$ and $n \geq 0$ natural integers. In a normed linear space, $||\alpha x + y||$ and $||\alpha x - y||$ are continuous in α. Hence, by (*), $(\alpha x, y)$ is continuous in α. Therefore, by the density of rational numbers into the reals, is proved for every real number α. This shows the linearity of (x, y) with respect to the first term, therefore, as already mentioned, by the symmetry of (x, y), (x, y) is linear in both terms. This concludes the proof of (i) and with this the proof of the Proposition 2.8. □

Remark 2.9. As mentioned before, in Proposition 1.6 we start with a normed space and then we *define* an inner product associated with this norm by means of (*). We observe that the *new* inner product obtained in this way may *not* coincide with the original inner-product to which the norm is associated. The two inner products, when we complete that spaces, have however something in common: we have got two Hilbert spaces which are mutually *unitary equivalent*, that is, they obey the relation $H = U^{-1}UH$ where $U : H \mapsto H$ is a so called *unitary operator* with $U^{-1} = U^*$, U^* denoting the so called *adjoint* operator of U. For Euclidean spaces of finite dimension n, this amounts to a change of inner product (which in this case is simply given by a by means of a unitary matrix $n \times n$), and in elementary geometric terms this corresponds, for example, to an orthogonal transformation of the Cartesian reference axes x, y, transforming ellipses into circles. □

Remark 2.10. The only norm satisfying a parallelogram identity are those obtained from an inner product by the relation $||x|| = \sqrt{(x, x)}$. We will see that the norms derived by an inner-product play a crucial role in establishing the existence results given later below, see e.g. Proposition 2.15. □

We now introduce the fundamental concept of *completeness* for Hilbert or Banach spaces. This property, as we shall see, is crucial in establishing *existence* results of various kind. Incidentally, this notion of distance $d(x, y)$ belongs to the theory of *metric spaces*. For our purposes in this chapter, it is enough to confine ourselves to define completeness in case of *pre-Hilbert spaces* or *pre-normed spaces*.

Complete pre-Hilbert spaces are simply called *Hilbert spaces* and *complete normed spaces* are called *Banach spaces*.

Definition 2.11. (Complete normed spaces) A normed space X endowed with a norm $||x||$ is called a *Banach space* if every *Cauchy sequence* $\{x_n\}$ in X satisfying the condition

$$||x_n - x_m|| \to 0 \ \ \forall \ m, n \to +\infty$$

converges to a vector $x \in X$ in the norm of X: $||x_n - x|| \to 0$ as $n \to +\infty$.

Definition 2.12. (Complete inner-product spaces) A pre-Hilbert space H is called a *Hilbert space*, if it is complete for the norm $||x|| := \sqrt{(x, x)}$ of H in the sense of Definition 2.1(iv). More explicitly, a pre-Hilbert space is a Hilbert space if for every sequence $\{x_n\}$ in X satisfying the condition

$$(x_n - x_m, x_n - x_m) \to 0 \quad \forall \ m, n \to +\infty$$

there exists a vector $x \in X$ such that $(x_n - x, x_n - x) \to 0$ as $n \to +\infty$.

Remark 2.13. We observe, incidentally, that the family of *metric spaces* is much larger that that of normed or inner-product spaces. For example, we can easily define a distance on a set which is not a vector space. □

Remark 2.14. Before proceeding, let us motivate the introduction of complete spaces by going back to the very construction of the real numbers out of the set of rational numbers Q. We recall that a rational number $w \in Q$ is defined as the ratio p/q of two integer numbers p and q, $q \neq 0$. This extension is fundamental to state the *Pythagorean theorem* for a right triangle ABC of unit right sides, that is, the property that the hypothenuse of such a triangle has a length given by $\sqrt{2}$. It is well known that $\sqrt{2}$ is an *irrational number*, which in decimal form has infinitely many non-zero decimal digits. This motivates the introduction of the field of *real numbers*, which indeed contains not only rational and irrational numbers, but also so-called *trascendental numbers* as well. For the construction of real numbers starting with the natural integers, see e.g. Rudin's book [9]. □

A first application of the completeness property, together with that of *closedness* introduced before, is the proof that there exists a point of minimum distance to a closed linear subspace M of a Hilbert space H from a closed linear subspace M of H. We state this result in the following

Proposition 2.15. (Existence of minimizers for the projection of vectors on subspaces of Hilbert spaces) *Let M be a closed, linear subspace of a (real) Hilbert space H. Then, for every vector $x \in H$ there exists a vector $u \in M$ such that*

$$\text{(MV)} \qquad ||x - u|| = \mathrm{Min}_{v \in M} ||x - v|| \,.$$

Such a minimizer u is unique and is called the orthogonal projection of x upon M and is denoted by $P_M x$.

Proof. Let δ be the *infimum* of $||x - v||$ over all $v \in M$. We have $\delta > -\infty$ as M is not empty, because the origin O of the space H belongs to any linear subspace of H. We put $\delta_n := ||x - u_n||$, where u_n is so chosen to have $||x - u_n|| \to ||x - u||$ as $n \to +\infty$, such a sequence $\{u_n\}$ being called a sequence of *minimizers* for the minimum problem (MV). Clearly, we have $\delta_n \to \delta$ as $n \to +\infty$.

We now apply the parallelogram identity of Proposition 2.6 in the form of (ParI*) to the vectors $x - u_n$ and $x - u_m$ for every n, m. By replacing these vectors in (ParI*), after cancellations we obtain, as $||x - u_n|| = \delta_n$ and $||x - u_m|| = \delta_m$:

$$||\frac{u_n - u_m}{2}||^2 + ||\frac{u_n + u_m}{2}||^2 = \frac{1}{2}(d_n^2 + d_m^2)\,.$$

Since $\dfrac{u_n + u_m}{2} \in M$, we have $||x - \dfrac{u_n + u_m}{2}|| \geq \delta$, therefore

$$||\frac{u_n - u_m}{2}||^2 \leq \frac{1}{2}\,(\delta_n^2 + \delta_m^2) - \delta^2\,,$$

hence
$$||u_n - u_m|| \to 0 \quad \text{as } n, m \to +\infty.$$

This shows that the sequence $\{u_n\}$ is a *Cauchy sequence* in H. Since H is *complete* and M is *closed*, this implies that $u \in M$ and $||x - u|| = \delta$. Therefore, the minimum is achieved and u is a point where the minimum is attained.

Proposition 2.16. (Characterization of minimizers) *Any vector $u \in M$ is a minimizer of the problem* (MV) *if and only if, u is a solution of the problem*

(MV) $$u \in M: \quad (x - u, w - u) \leq 0 \quad \forall\, w \in M\,.$$

Proof. We have $v = (1 - t)u + tw$ with $0 < t \leq 1$. Any such w belongs to M.

Therefore $\quad ||x - u|| \leq ||x - [(1 - t)u + tw]|| = ||(x - u) - t(w - u)||\,.$

Hence $\quad ||x - u||^2 \leq ||x - [(1 - t)u + tw]||^2 = ||(x - u) - t(w - u)||^2.$

From this it follows $||x - u||^2 \leq ||x - u||^2 - 2t(x - u, w - u) + t^2||w - u||^2$, hence

$$2t(x - u, w - u) \leq t^2||w - u||^2\,.$$

By dividing both terms by a strictly positive $t > 0$, we get

$$2(x - u, w - u) \leq t||w - u||^2$$

By letting $t \to 0$ at the right term, we obtain $2(x - u, w - u) \leq 0$, hence also $(x - u, w - u) \leq 0$. Then

$$2(x - u, w - u) \leq 0,$$
$$2(x - u, w - u) - ||u - w||^2 \leq 0,$$
$$2(x - u, w - x + x - u) - ||u - w||^2 \leq 0.$$

Since $w \in M$ was arbitrary, this proves (MV).

Conversely, let $u \in M$ verify (MV), that is, let $u \in M$ be such that

$$(x - u, w - u) \leq 0 \quad \text{for all } w \in M.$$

Then for every $w \in M$ we have

$$||u - x||^2 - ||w - x||^2 = 2(x - u, w - u) - ||u - w||^2 \leq 0\,,$$

Since by assumption $(x - u, w - u) \leq 0$,

this implies $\qquad ||u - x||^2 - ||w - x||^2 \leq 0\,,$

hence the conclusion $\qquad ||u - x||^2 \leq ||w - x||^2$

for every $w \in M$ follows. This concludes the proof of Proposition 2.16. □

Proposition 2.17. (Uniqueness of minimizers) *Let u_1 and u_2 be two minimizers, that is, two vectors where the minimum in problem* (MV) *is achieved. Then $u_1 = u_2$.*

Proof. By the characterization of minimizers given in Lemma Proposition 2.16, any two minimizers u_1 and u_2 satisfy the inequalities below

$$(x - u_1, w - u_1) \leq 0, \quad (x - u_2, w - u_2) \leq 0$$

for every $w \in M$. By replacing $w = u_2$ in the first inequality and $w = u_1$ in the latter, we get

$$(x - u_1, u_2 - u_1) \leq 0, \quad (x - u_2, u_1 - u_2) \leq 0\,.$$

By adding the two inequalities term by term, we then get

$$(x - u_1, u_2 - u_1) + (x - u_2, u_1 - u_2) \leq 0\,.$$

The terms containing x cancel each other, so that we obtain $||u_1 - u_2||^2 \leq 0$, hence $u_1 = u_2$. □

Proposition 2.18. (Perpendicular space) *For every linear subspace M of H we define its* orthogonal subspace*, denoted $M^\perp$, by setting*

$$(\mathrm{M}^\perp) \qquad M^\perp := \{u \in H : \ (u, v) = 0, \ \forall\, lv \in M\}\,.$$

Then, $M^\perp$ is a closed, linear subspace of H.

Proof. We check that $M^\perp$ is indeed a *linear* subspace of H. In fact, let $au_1 + bu_2$, with a, b real number and $u_1, u_2 \in M$. By the linearity of the (symmetric) inner product in the second argument, we have

$$(x - u, au_1 + bu_2) = a(x - u, u_1) + b(x - u, u_2) = a \times 0 + b \times 0 = 0\,.$$

In a similar way we check that $M^\perp$ is *closed* in H. In fact, let $\{w_n\}$ be a sequence with $w_n \in M^\perp$ for every n, and let $z \in H$ be such that $w_n \to w$ in H. By the continuity in H of the (symmetric) inner product of H in the second argument, we have

$$(x - u, w) = (x - u, \lim w_n) = \lim(x - u, w_n) = 0\,,$$

hence $u \in M^\perp$. □

We summarize our main results established so far in the following theorem.

Theorem 2.19. *Let H be a real Hilbert space (with its topological dual space H' identified with H). Let M be a closed linear subspace of H. Then, for every $x \in H$ there exists a minimizing vector $u \in M$ for the minimum problem* (MV)*. Moreover, such a minimizer is unique and it satisfies the property $(x - u, v) = 0$ for every $v \in M$, that is, $x - u \in M^\perp$.*

Proof. For $x \in H$, let $u \in M$ be the minimizer in the problem

$$||x-u|| = \mathrm{Min}_{v\in M} ||x-v|| \,.$$

By Proposition 2.15, such a minimizer exists and it satisfies

$$u \in M : \quad ||x-u|| \le ||x-w|| \quad \forall\, w \in M \,.$$

By Proposition 2.17 the minimizer u is unique, and by Proposition 2.16 it is characterized by

$$(x-u, w-u) \le 0 \quad \forall \; w \in M \,.$$

By replacing $w = u + 2v$ we get $(x - u, 2v) \le 0$. By replacing $w = u - 2v$ we get $(x - u, -v) \le 0$, which is the same as $-2(x - u, v) \le 0$. Therefore we have $(x-u, v)-(x-u, v \le 0$, hence $(x-u, v) = 0$. Since v is arbitrary in M, this proves that $x - u \in M^\perp$. □

Definition 2.20. (Projection operators in real Hilbert spaces) For every closed linear subspace M of a real Hilbert space H, we define the linear operator P_M from H into H, by setting $u = P_M x$ for every $x \in M$, where u is the solution of the minimum problem (MV).

Remark 2.21. We observe that this definition is well posed, because we know from Proposition 2.15 above that such a minimizer does indeed exist and, moreover, it is unique as shown in Theorem 2.19. □

The main properties of the projection operators are established in the following:

Theorem 2.22. (Projection Operators) *The operator P_M introduced in Definition* 2.20 *satisfies the* norm contractive property:

(NC) $$||P_M x_1 - P_M x_2|| \le ||x_1 - x_2|| \qquad \forall\, x_1,\, x_2 \in H.$$

Furthermore, the vector $u = P_M x$ is characterized by the system of equations:

(ORTH) $$u \in M : \quad (x - P_M x, w) = 0 \;\; \forall; w \in M \,,$$

that is, $u = P_M x$ if and only if $x - u \in M^\perp$.

Proof. Let $u_1 = P_M x_1$ and $u_2 = P_M x_2$. We have

$$(x_1 - u_1, w - u_1) \le 0 \;\; \text{and} \;\; (x_2 - u_2, w - u_2) \le 0.$$

By replacing $w = u_2$ in the first inequality and $w = u_1$ in the second, inequality, we obtain

$$(x - u_1, u_2 - u_1) \le 0, \quad (x - u_2, u_1 - u_2) \le 0 \,.$$

By adding the two inequalities term by term, we then get

$$||u_1 - u_2||^2 \le ||x_1 - x_2||^2 \,.$$

In terms of the projection operators, this can be rewritten as

$$||P_M x_1 - P_M x_2|| \leq ||x_1 - x_2||,$$

which is the desired contractive norm inequality (NC). $\square$

We now move to the proof of the last sentence in the statement of Theorem 2.22. Thus, our claim now is that $u = P_M x$ if and only if $x - u \in M^{\perp}$. We proceed as follows:

First part. Let $u = P_M x$. Then, the property $x - u \in M^{\perp}$ has been established in Theorem 2.19.

Second part. We now prove the second part of the claim, that is, we prove that if $x - u \in M^{\perp}$, then $u = P_M x$.

We proceed to this proof as follows.

We recall that, by the definition of $P_M x$ in Definition 2.20, the vector $u = P_M x$ is defined to be the solution u of the minimum problem:

$$\text{(MV)} \qquad ||x - u|| = \mathrm{Min}_{v \in M} ||x - v|| \,.$$

Moreover, by Proposition 2.4, we have proved that such a minimizer u does indeed exists, and by Proposition 2.6 it is unique.

By applying this result to the vector $u = P_M x$, we obtain that $u = P_M x$ is indeed the solution of the minimum problem (MV), that is, it satisfies (MV), which is what we had to prove. $\square$

Theorem 2.23. (Decomposition of real Hilbert spaces into orthogonal subspaces) *Any vector $x \in H$ can be uniquely decomposed in the form*

$$\text{(Y)} \qquad x = m + n, \quad \textit{where} \;\; m \in M \;\; \textit{and} \;\; n \in M^{\perp}.$$

Moreover, we have $x = P_M x + P_N x$ for every $x \in H$, that is, $I = P_M + P_N$ as maps from H into H.

Proof. We start by proving (Y).

We already proved that $N := M^{\perp}$ is a closed, linear subspace of H. Moreover, the decomposition is clearly unique, since a vector orthogonal to itself is the zero vector. We may assume that $M \neq H$ and $x \in m$, for, if $x \in m$, we have the trivial decomposition $m = x$ and $n = 0$. To prove the decomposition (Y), we may assume that $M \neq H$ and $x \in M$, because if $x \in M$ we can write the decomposition trivially with $m = x$ and $n = 0$.

Since M is closed and $x \in M$, we have $d := \mathrm{Min}_{m \in M} ||x - m|| > 0$.

Let $\{m_n\} \subset M$ be a *minimizing sequence*, i.e., $\lim ||x - m_n|| = d$ as $n \to +\infty$. Then $\{m_n\}$ is a Cauchy sequence. In fact, by the identity

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

proved in Proposition 2.6 for any real pre-Hilbert space, we obtain

$$\begin{aligned}||m_k - m_n||^2 &= ||(x-m_n) - (x-m_k)||^2 \\ &= 2(||x-m_n||^2 + ||x-m_k||^2) - ||2x - m_n - m_k||^2 \\ &= 2(||x-m_n||^2 + ||x-m_k||^2) - 4||x - (m_n+m_k)/2||^2 \\ &\leq 2(||x-m_n||^2 + ||x-m_k||^2 - 4d^2) \to 2(d^2+d^2) - 4d^2 = 0\end{aligned}$$

as $k, n \to \infty$. In the last step, we used that $(m_n + m_n)/2 \in M$. By the completeness of or Hilbert space H, there exists an element $m \in H$ such that $s - \lim m_n = m$ as $n \to +\infty$. Here by s–lim we denote the limit in the norm of the space H, usually called the *strong limit* of vectors of H. That is, we have $m \in M$, since M is closed, and $||m_n - m|| \to 0$ as $n \to +\infty$, where the norm is the one of H. We also recall that the norm is *continuous*, that is

$$|\,||x|| - ||y||\,| \leq ||x-y||\,.$$

Therefore, above we obtain $||x - m|| = d$.

Write $x = m + (x-m)$. Putting $n := x - m$, we have to show that $n \in M^{\perp}$. For any $m' \in M$ and any real number α, we have $(m + \alpha m') \in M$ and so

$$d^2 \leq ||x - m - \alpha m'||^2 = (n - \alpha m', n - \alpha m') = ||n||^2 - 2\alpha(n, m') + \alpha^2 ||m'||^2\,.$$

Since $||n|| = d$, we get $\qquad 0 \leq -2\alpha(n,m') + \alpha^2||m'||^2$

for every real α, and by dividing both terms by $\alpha > 0$, we obtain

$$2(n, m') \leq \alpha ||m'||^2\,.$$

By letting $\alpha \to 0$, this implies $(n, m') = 0$, which has been obtained for all $m' \in M$. Therefore, we finally get $n \in N \subset M^{\perp}$. This show that (Y) holds. □

Now we prove the other properties stated in Theorem 2.23. The first one of these property is that for every $x \in h$ we have

$$x = P_M x + P_N x$$

for every $x \in H$, where $N := M^{\perp}$. The proof is as follows.

We know by (Y) that $x = m + n$ for $m \in M$ and $n \in M^{\perp}$.

Since $m \in M$, clearly $m = P_M m$. Also, since $n \in M^{\perp}$: $n \in P_N x$ and $N := M^{\perp}$.

This gives the decomposition $x = P_M x + P_N x$ which we had to prove.

Finally, we observe that the previous decomposition can be clearly reformulated by saying that $I = P_M + P_N$ as maps from H into H.

This concludes the proof of Theorem 2.23 in the case of a *real* Hilbert space H. □

Remark 2.24. The extension of Theorem 2.23 to the case of a *complex* Hilbert space H requires a slightly modified proof. This is due to the fact that now the inner product (x, y) of H is not a real number, it is a complex number satisfying the properties of a *complex inner product* as we shall define it as said in the following Chapter 3.

Chapter 3. Complex Hilbert spaces

This chapter is dedicated to extend the content of Chapter 2 for *real* Hilbert and Banach spaces to the case of *complex* spaces.

We recall here that if $z = x + iy = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$ is a complex number, then we call $|z| := \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}$ the *modulus* of z. Here i is the complex unit, $i^2 = -1$, $(-i)^2 = -1$. In particular, when $z = i$, we have $i^2 = (-i)^2 = -1$, $\mathrm{Re}\, i = 0$, $\mathrm{Im}\, i = 1$, $|i| = \sqrt{(\mathrm{Re}\, i)^2 + (\mathrm{Im}\, i)^2} = 1$.

We begin by defining in general *complex normed spaces* and *complex inner-product spaces*. Occasionally, in place of inner-product space we say *pre-Hilbert space*.

Definition 3.1. A *real or complex normed space* X is a linear vector space over the real or complex field on which a real number $||x||$, called the *norm* of the vector x, such that:

(j) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$

(jj) $||x + y|| \leq ||x|| + ||y||$

(jjj) $||\alpha x|| = |\alpha|\ ||x||$

where $|\alpha|$ is the modulus of the complex number α.

Definition 3.2. A real or complex normed space X is called a *pre-Hilbert space* if its norm satisfies the condition

$$||x + y||^2 + ||x - y||^2 = 2\,(||x||^2 + ||y||^2)\,. \tag{1}$$

Theorem 3.3. (M. Frechet, J. von Neumann, P. Jordan)
In a real pre-Hilbert space X we define

$$(x, y) := 4^{-1}(||x + y||^2 - ||x - y||^2)\,. \tag{2}$$

Then we have the properties:

$$(\alpha x, y) = \alpha(x, y) \quad \text{for every real } \alpha \tag{3}$$
$$(x + y, z) = (x, z) + (y, z) \tag{4}$$
$$(x, y) = (y, x) \tag{5}$$
$$(x, x) = ||x||^2\,. \tag{6}$$

Proof. (5) and (6) are clear. We have, from (1) and (2), the following equality:

$$\begin{aligned}(x, z) + (y, z) &= 4^{-1}(||x + z||^2 - ||x - z||^2 + ||y + z||^2 - ||y - z||^2) \\ &= 2^{-1}(||\frac{x+z}{2} + z||^2 - ||\frac{x+y}{2} - z||^2) = 2(\frac{x+y}{2}, z)\,.\end{aligned} \tag{7}$$

If we take $y = 0$, we obtain $(x, z) = (2\frac{x}{2}, z)$, because $(0, z) = 0$ by (2). Hence, by (7), we obtain (4). Thus we see that (3) holds for rational numbers α of the form $\alpha = m/2^n$. In a normed linear space, $||\alpha x + y||$ and $||\alpha x - y||$ are continuous in α. Hence, by (2), $(\alpha x, y)$ is continuous in α. Therefore (3) is proved for every real number α. $\square$

Corollary 3.4. (J. von Neumann, P. Jordan) *In a complex normed space X, if we define (x,y) by setting for every x and y of X:*

$$(x,y) := (x,y)_1 + i(x,iy)_1\,, \tag{8}$$

where $i = \sqrt{-1}$ *and* $(x,y)_1 = \dfrac{1}{4}\,(||x+y||^2 - ||x-y||^2)\,.$

Then we have (4), (6) *and* $$(\alpha x, y) = \alpha(x,y) \tag{3'}$$

for every complex α. Moreover $$(x,y) = \overline{(y,x)} \tag{5'}$$

where $\overline{(y,x)} = x - iy$ *is the complex conjugate of* $(x,y) = x+iy$.

Proof. The space X is, in particular, a real pre-Hilbert space, therefore (4) and (3') for a real α holds, as seen in Proposition 2.8 of Chapter 2.

We have, by (8), $\quad (y,x)_1 = (x,y)_1, \quad (ix,iy)_1 = (x,y)_1$

and hence $\quad (y,ix)_1 = (-iiy,ix)_1 = -(iy,x)_1 = -(x,iy)_1$

therefore $\quad (y,x) = (y,x)_1 + i(y,ix)_1 = (x,y)_1 - i(x,iy)_1 = \overline{(x,y)}\,.$

Similarly, we have

$$(ix,y) = (ix,y)_1 + i(ix,iy)_1 = -(x,iy)_1 + i(x,y)_1 = i(x,y)_1$$

and therefore we have proved (3').

Finally we have (6), because $(x,x)_1 = ||x||^2$ and

$$(x,ix)_1 = 4^{-1}\,(|1+i\,|^2 - |1-i|^2)\,||x||^2 = 0\,. \qquad \square$$

Proposition 3.5. *A (real or) complex space X is a (real or) complex pre-Hilbert space, if for every pair pair of elements $x,y \in X$ there is associated a (real or) complex number (x,y) satisfying* (3'), (4) (5') *and*

$$(x,x) \geq 0 \text{ and } (x,x) = 0 \text{ if and only if } x = 0\,. \tag{9}$$

Proof. For any real number α, we have by (3'), (4) and (5')

$$(x + \alpha(x,y)y, x + \alpha(x,y)y) = ||x||^2 + 2\alpha|(x,y)|^2 + |\alpha|^2|(x,y)|^2||y||^2 \geq 0$$

where $||x|| = \sqrt{(x,x)}$, so that we have

$$|(x,y)|^4 - ||x||^2\,|(x,y)|^2\,||y||^2 \leq 0.$$

Hence we obtain the Schwartz' inequality

$$|(x,y)| \leq ||x||\,||y||$$

where the equality is satisfied iff x and y are linearly dependent. The latter part of (10) is clear from the latter part of of (9).

We have, by (10), the triangle inequality for $||x||$:

$$||x+y||^2 = (x+y,x+y) = ||x||^2 + (x,y) + (y,x) + ||y||^2 \leq (||x|| + ||y||)^2\,.$$

Finally, the equality (1) is verified easily. $\square$

We are now in a good position for proving the *existence* result of the minimum problems and of the associated minimizers. However, as in Chapter 2, we must first introduce the notions of *completeness* and the related definitions of *complex Banach spaces* and *complex Hilbert spaces*:

Definition 3.6. (Complete complex normed spaces) A complex normed space X, endowed with a norm $||x||$, is called a *Banach space* if every *Cauchy sequence* $\{x_n\}$ in X satisfying the condition

$$||x_n - x_m|| \to 0 \;\; \forall \;\, m, n \to +\infty$$

converges to a vector $x \in X$ in the norm of X: $||x_n - x|| \to 0$ as $n \to +\infty$.

Definition 3.7. (Complete complex inner-product spaces) A pre-Hilbert space H is called a *Hilbert space*, if it is complete for the norm of H defined in Definition 2.1(iv). More explicitly, if every sequence $\{x_n\}$ in H satisfying the condition

$$(x_n - x_m, x_n - x_m) \to 0 \;\; \forall \;\, m, n \to +\infty$$

converges to a vector $x \in H$ in the inner product of H, that is: $(x_n - x, x_n - x) \to 0$ as $n \to +\infty$.

We omit to show why the notion of completeness is related to the construction of the real numbers from the rational numbers, and we refer for this to the Comments of Chapter 2.

We now extend the existence results of Chapter 2 for minimum problems and the related properties of minimizers to complex Hilbert spaces.

Proposition 3.8. (Existence of minimizers for the projection of vectors on subspaces of Hilbert spaces) *Let M be a closed, linear subspace of a (real) Hilbert space H. Then, for every vector $x \in H$ there exists a vector $u \in M$ such that*

$$\text{(MV)} \qquad ||x - u|| = \mathrm{Min}_{v \in M} \, ||x - v|| \, .$$

Such a minimizer u is unique and is called the orthogonal projection of x upon M and is denoted by $P_M x$.

Proof. Let δ be the *infimum* of $||x - v||$ over all $v \in M$. We have $\delta > -\infty$ as M is not empty, because the origin O of the space H belongs to any linear subspace of H. We put $\delta_n := ||x - u_n||$, where u_n is so chosen to have $||x - u_n|| \to ||x - u||$ as $n \to +\infty$, such a sequence $\{u_n\}$ being called a sequence of *minimizers* for the minimum problem (MV). Clearly, we have $\delta_n \to \delta$ as $n \to +\infty$.

We now apply the parallelogram identity of Proposition 2.6 in the form of (ParI*) to the vectors $x - u_n$ and $x - u_m$ for every n, m. By replacing these vectors in (ParI*), after cancellations we obtain, as $||x - u_n|| = \delta_n$ and $||x - u_m|| = \delta_m$:

$$||\frac{u_n - u_m}{2}||^2 \, + \, ||\frac{u_n + u_m}{2}||^2 \; = \; \frac{1}{2}(d_n^2 \, + \, d_m^2) \, .$$

Since $\dfrac{u_n + u_m}{2} \in M$, we have $||x - \frac{u_n+u_m}{2}|| \geq \delta$, therefore

$$||\frac{u_n - u_m}{2}||^2 \leq \frac{1}{2} \, (\delta_n^2 + \delta_m^2) - \delta^2$$

hence $||u_n - u_m|| \to 0$ as $n, m \to +\infty$.

This shows that the sequence $\{u_n\}$ is a *Cauchy sequence* in H. Since H is *complete* and M is *closed*, this implies that $u \in M$ and $||x-u|| = \delta$. Therefore, the minimum is achieved and u is a point where the minimum is attained. □

Proposition 3.9. (Characterization of minimizers) *Any vector $u \in M$ is a minimizer of the problem* (MV) *if and only if, u is a solution of the problem*

$$u \in M : \quad (x-u, w-u) \leq 0 \quad \forall\, w \in M\,. \tag{MV}$$

Proof. We have $v = (1-t)u + tw$ with $0 < t \leq 1$. Any such w belongs to M.

Therefore $\quad ||x-u|| \leq ||x - [(1-t)u + tw]|| = ||(x-u) - t(w-u)||\,.$

Hence $\quad ||x-u||^2 \leq ||x - [(1-t)u + tw]||^2 = ||(x-u) - t(w-u)||^2\,.$

From this it follows

$$||x-u||^2 \leq ||x-u||^2 - 2t(x-u, w-u) + t^2||w-u||^2$$

hence

$$2t(x-u, w-u) \leq t^2||w-u||^2\,.$$

By dividing both terms by a strictly positive $t > 0$, we get

$$2(x-u, w-u) \leq t||w-u||^2.$$

By letting $t \to 0$ at the right term, we obtain $2(x-u, w-u) \leq 0$, hence also $(x-u, w-u) \leq 0$. Then

$$2(x-u, w-u) \leq 0,$$
$$2(x-u, w-u) - ||u-w||^2 \leq 0,$$
$$2(x-u, w-x+x-u) - ||u-w||^2 \leq 0.$$

Since $w \in M$ was arbitrary, this proves (MV).

Conversely, let $u \in M$ verify (MV), that is, let $u \in M$ be such that

$$(x-u, w-u) \leq 0$$

for all $w \in M$. Then for every $w \in M$ we have

$$||u-x||^2 - ||w-x||^2 = 2(x-u, w-u) - ||u-w||^2 \leq 0\,,$$

Since by assumption $(x-u, w-u) \leq 0$, this implies

$$||u-x||^2 - ||w-x||^2 \leq 0\,,$$

hence the conclusion $||u-x||^2 \leq ||w-x||^2$ follows for every $w \in M$. This concludes the proof of Proposition 3.9. □

Proposition 3.10. (Uniqueness of minimizers) *Let u_1 and u_2 be two minimizers, that is, two vectors where the minimum in problem (MV) is achieved. Then $u_1 = u_2$.*

Proof. By the characterization of minimizers given in Proposition 3.9, any two minimizers u_1 and u_2 satisfy the inequalities below

$$(x-u_1, w-u_1) \leq 0, \quad (x-u_2, w-u_2) \leq 0$$

for every $w \in M$. By replacing $w = u_2$ in the first inequality and $w = u_1$ in the latter, we get

$$(x - u_1, u_2 - u_1) \leq 0, \quad (x - u_2, u_1 - u_2) \leq 0\,.$$

By adding the two inequalities term by term, we then get

$$(x - u_1, u_2 - u_1) + (x - u_2, u_1 - u_2) \leq 0\,.$$

The terms containing x cancel each other, so we obtain

$$||u_1 - u_2||^2 \leq 0, \quad \text{hence} \;\; u_1 = u_2. \qquad \square$$

In complex Hilbert spaces, as well as in real Hilbert spaces as seen in Chapter 2, the notion of *perpendicular*, or *orthogonal* vectors and linear subspaces, play an important role in many theories.

Proposition 3.11. (Orthogonal subspace) *For every linear subspace M of H we define its orthogonal subspace, denoted $M^\perp$, by setting*:

$$(\mathrm{M}^\perp) \qquad M^\perp := \{u \in H: \;\; (u, v) = 0, \; \forall\, v \in M\}\,.$$

Then, $M^\perp$ is a closed, linear subspace of H.

Proof. We check that $M^\perp$ is indeed a *linear* subspace of H. In fact, let $au_1 + bu_2$, with a, b real number and $u_1, u_2 \in M$. By the linearity of the (symmetric) inner product in the second argument, we have

$$(x - u, au_1 + bu_2) = a(x - u, u_1) + b(x - u, u_2) = a \times 0 + b \times 0 = 0\,.$$

In a similar way we check that $M^\perp$ is *closed* in H. In fact, let $\{w_n\}$ be a sequence with $w_n \in M^\perp$ for every n, and let $z \in H$ be such that $w_n \to w$ in H. By the continuity in H of the (symmetric) inner product of H in the second argument, we have

$$(x - u, w) = (x - u, \lim w_n) = \lim(x - u, w_n) = 0\,,$$

hence $u \in M^\perp$.

We summarize the main results established so far in the following:

Theorem 3.12. *Let H be a real Hilbert space (with its topological dual space H' identified with H). Let M be a closed linear subspace of H. Then, for every $x \in H$ there exists a minimizing vector $u \in M$ for the minimum problem* (MV) *of Proposition* 2.15*. Moreover, such a minimizer is unique and it satisfies the property $(x - u, v) = 0$ for every $v \in M$, that is, $x - u \in M^\perp$.*

Proof. For $x \in H$, let $u \in M$ be the minimizer in the problem

$$||x - u|| = \mathrm{Min}_{v \in M}\, ||x - v||\,.$$

By Proposition 3.8, such a minimizer exists and it satisfies

$$u \in M: \quad ||x - u|| \leq ||x - w|| \quad \forall\, w \in M\,.$$

Following Proposition 3.9 the minimizer u is unique, and by Proposition 3.8 is characterized by

$$(x-u, w-u) \leq 0 \quad \forall\, w \in M\,.$$

By replacing $w = u + 2v$ we get $(x-u, 2v) \leq 0$. By replacing $w = u - 2v$ we get $(x-u, -v) \leq 0$, which is the same as $-2(x-u, v) \leq 0$. Therefore

$$(x-u, v) - (x-u, v \leq 0, \quad \text{hence} \quad (x-u, v) = 0.$$

Since v is arbitrary in M, this proves that $x - u \in M^{\perp}$. □

We now define the *projection operators* associated with (closed, linear) subspaces:

Definition 3.13. (Projection operators) For every closed linear subspace M of a complex Hilbert space H, we define the linear operator P_M from H into H, by setting $u = P_M x$ for every $x \in M$, where u is the solution of the minimum problem (MV).

Remark 3.14. We observe that this definition is well posed, because we know from Theorem 3.2 above that such a minimizer does indeed exists and is unique..

The main properties of the projection operators are summarized in the following:

Theorem 3.15. (Projection operators) *The operator P_M introduced by Definition* 3.13 *satisfies the norm contractive property:*

$$\text{(NC)} \qquad ||P_M x_1 - P_M x_2|| \leq ||x_1 - x_2|| \quad \forall\, x_1,\, x_2 \in H.$$

Furthermore, the vector $u = P_M x$ is characterized by the system of equations:

$$\text{(ORTH)} \qquad u \in M: \;\; (x - P_M x, w) = 0 \;\; \forall\, w \in M\,,$$

that is, $u = P_M x$ if and only if $x - u \in M^{\perp}$.

Proof. Let $u_1 = P_M x_1$ and $u_2 = P_M x_2$. We have

$$(x_1 - u_1, w - u_1) \leq 0 \quad (x_2 - u_2, w - u_2) \leq 0\,.$$

By replacing $w = u_2$ in the first inequality and $w = u_1$ in the second, inequality, we obtain

$$(x - u_1, u_2 - u_1) \leq 0, \quad (x - u_2, u_1 - u_2) \leq 0\,.$$

By adding the two inequalities term by term, we then get

$$||u_1 - u_2||^2 \leq ||x_1 - x_2||^2\,.$$

In terms of the projection operators, this can be rewritten as

$$||P_M x_1 - P_M x_2|| \leq ||x_1 - x_2||$$

which is the desired contractive norm inequality (NC).

We now prove property (ORTH), stated explicitly in the last sentence in the statement of Theorem 3.12:

$$u = P_M x \;\; \text{if and only if} \;\; x - u \in M^{\perp}.$$

We prove first that if $u = P_M x$, then $x - u \in M^{\perp}$. This has already been established in Theorem 3.12.

Now we prove the opposite implication, that is, if $x - u \in M^\perp$, then $u = P_M x$.

In fact, the vector $u = P_M x$ is defined in Definition 2.20 as the solution u of the minimum problem:

$$\text{(MV)} \qquad ||x - u|| = \text{Min}_{v \in M} ||x - v||.$$

We recall that in Proposition 2.15 we have proved that such a minimizer u does indeed exists and in Proposition 2.17 that it is unique and is denoted by $u = P_M x$. Now our assumption is that $x - u \in M^\perp$. By Definition 2.20, this means that u is the (unique) solution of the minimum problem (MV), which is what we had to prove. $\square$

Theorem 3.16. (Decomposition of complex Hilbert spaces into orthogonal subspaces) *Any vector $x \in H$ can be uniquely decomposed in the form*

$$(\text{Y}_1) \qquad x = m + n, \quad \textit{where } m \in M \ \textit{ and } n \in M^\perp.$$

Moreover, we have

$$(\text{Y}_2) \qquad x = P_M x + P_N x$$

for every $x \in H$, that is,

$$(\text{Y}_3) \qquad I = P_M + P_N$$

as maps from H into H.

Proof. A preliminary remark. A vector x that belongs to both M and $N = M^\perp$ is orthogonal to itself, that is, $(x, x) = 0$, hence it is the null vector $x = 0$. Moreover, we already proved by Proposition 2.18 that $N := M^\perp$ is a closed, linear subspace of H. Moreover, the decomposition is *unique*, because if the same vector x has two decompositions of this kind, by taking differences, we would have the decomposition $0 = m + n$, hence $m = -n$ with $(m, -n) = 0$ and both m and $-n$ in the same subspace $M^\perp$, what implies that $m = 0 = n$.

In carrying on the proof of (Y_1), we may also assume that $M \neq H$. In fact, if $M = H$ then $M^\perp = \{0\}$ and (Y_1) holds trivially with $m = x$ and $n = 0$.

We may also assume in addition $x \in M$, because if $x \in M$ we have the trivial with $m = x$ and $n = 0$.

We now proceed with the proof of (Y_1).

Since M is closed and $x \in M$, we have

$$d := \text{Min}_{m \in M} ||x - m|| > 0\,.$$

Let $\{m_n\} \subset M$ be a *minimizing sequence*, i.e.,

$$\lim ||x - m_n|| = d$$

as $n \to +\infty$. Then $\{m_n\}$ is a Cauchy sequence. In fact, by the identity

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$$

proved in Proposition 2.6 for any complex pre-Hilbert space, we obtain

$$\begin{aligned} ||m_k - m_n||^2 &= ||(x - m_n) - (x - m_k)||^2 \\ &= 2(||x - m_n||^2 + ||x - m_k||^2) - ||2x - m_n - m_k||^2 \end{aligned}$$

$$= 2(||x - m_n||^2 + ||x - m_k||^2) - 4||x - (m_n + m_k)/2||^2$$
$$\le 2(||x - m_n||^2 + ||x - m_k||^2 - 4d^2) \to 2(d^2 + d^2) - 4d^2 = 0$$

as $k, n \to \infty$. In the last step, we used that $(m_n + m_n)/2 \in M$. By the completeness of the Hilbert space H, there exists an element $m \in H$ such that $s - \lim\ m_n = m$ as $n \to +\infty$. Here by *s-lim* we denote the limit in the norm of the space H, usually called the *strong limit* of vectors of H. That is, we have have $m \in M$, since M is closed, and $||m_n - m|| \to 0$ as $n \to +\infty$, where the norm is the one of H. We also recall that the norm is *continuous*, that is

$$|\,||x|| - ||y||\,| \le ||x - y||, \quad \text{hence} \quad ||x - m|| = d.$$

Write $x = m + (x - m)$. Putting $n := x - m$, we have to show that $n \in M^\perp$. For any $m' \in M$ and any real number α, we have $(m + \alpha m') \in M$ and so

$$d^2 \le ||x - m - \alpha m'||^2 = (n - \alpha m', n - \alpha m') = ||n||^2 - 2\alpha(n, m') + \alpha^2||m'||^2 .$$

Since $||n|| = d$, we get $\qquad 0 \le -2\alpha(n, m') + \alpha^2||m'||^2$

for every real α, and by dividing both terms by $\alpha > 0$, we obtain

$$2(n, m') \le \alpha||m'||^2 .$$

By letting $\alpha \to 0$, this implies $(n, m') = 0$, which has been obtained for all $m' \in M$. Therefore, we finally get $n \in N \subset M^\perp$. This show that (Y_1) holds.

Now we prove property (Y_2). Property (Y_2) states that for every $x \in H$ we have

$$x = P_M x + P_N x$$

where $N := M^\perp$. The proof is as follows. We know by (Y_1) that

$$x = m + n, \;\; m \in M, \quad n \in M^\perp .$$

Since $m \in M$, then clearly $m = P_M m$, moreover, since $n \in M^\perp$, then $n \in P_N x$, $N := M^\perp$. This gives the decomposition (Y_1) $x = P_M x + P_N x$ which we had to prove.

Finally, we prove (Y_3). Property (Y_3) is simply a reformulation of (Y_2) in terms of mapping from H into H.

This concludes the proof of Theorem 3.16, in the case of a complex Hilbert space H. □

Chapter 4. Convergence of vectors and subspaces

This chapter is dedicated to examine the various notions of *convergence of vectors* and *convergence of linear subspaces* in both (complex) Banach and Hilbert spaces.

We start by defining the *strong convergence* of vectors as follows:

Definition 4.1. (Strong convergence of vectors in normed spaces) Let $\{u_n\}$ be a sequence of vectors in a (real or complex) normed space X, and let u be a vector of X. Then we say that u_n *converges strongly* to u in X as $n \to +\infty$, if:

$$||u_n - u|| \to 0, \quad as \quad n \to +\infty.$$

Remark 4.2. The definition of weak convergence of vectors in normed spaces will be considered later.

Remark 4.3. It is easy to see that any strongly converging sequence according to Definition 2.11 is *bounded* in X, that is, there exists some constant $c_0 > 0$ such that

$$||x_n|| \leq c_0 \quad \text{for all integers } n\,.$$

In fact, given any constant $c_0 > 0$ there is an index $\overline{n}$, possibly depending on c_0, such that $||u_n - u|| \leq c_0$ for all $n \geq \overline{n}$. Therefore, by triangle inequality

$$||u_n|| \leq ||u - u_{\overline{n}}|| + ||u_{\overline{n}}|| \leq c_0 + c_0 = 2c_0\,.$$

However, such an upper bound c_0 may depend on the specific sequence at hand.

Definition 4.4. (Weak convergence of vectors in pre-Hilbert spaces) Let $\{u_n\}$ be a sequence of vectors in a (real or complex) pre-Hilbert space H, and let u be a vector of H. Then we say that u_n *converges weakly* to u in H as $n \to +\infty$, if for every $v \in X$:

$$(u_n, v) \to (u, v) \quad as \quad n \to +\infty\,.$$

Remark 4.5. In view of what we have seen above in the case of normed spaces, weakly converging sequences in pre-Hilbert spaces are bounded in the norm $||u|| = \sqrt{(u,u)}$ of the space. However, the upper bound will depend not only on the sequence, but also on the vector z entering the definition of weak convergence itself.

An important tool in Hilbert space theory is played by so called *complete orthonormal sets* of vectors, abbreviated *c.o.n.s.*:

(a) $(u_n, u_m) = \delta_n^m$ for the integers m, n, where δ_n^m is the Kronecker symbol defined by setting $\delta_n^n = 1$ if $m = n$ and $\delta_n^m = 0$ if $m \neq n$;

(b) every vector $u \in H$ can be developed as the limit

$$u = \sum_{n=1}^{\infty} (u, u_n) u_n = \lim_{N \to +\infty} \sum_{n=1}^{N} (u, u_n) u_n\,,$$

where the limit of the series is taken as a strong limit in the norm of H:

$$|| \sum_{n=1}^{\infty} (u, u_n) u_n - \sum_{n=1}^{N} (u, u_n) u_n || \to 0 \quad as \quad N \to +\infty\,.$$

Remark 4.6. If $\{u_n\}$ is an orthonormal sequence in H, then for every u in H and any integer n, the following *Bessel inequality* is satisfied;

$$\sum_{n=1}^{\infty} |(u, u_n)|^2 \leq ||u||^2\,.$$

In fact, we have, by setting $a_n = (u, u_n)$,

$$0 \leq ||u - \sum_{n=1}^{N} a_n u_n||^2 = (u,u) - 2\sum_{n=1}^{N} a_n (u, u_n) + \sum_{n=1}^{N} a_n^2 (u_n, u_n) = (u,u) - \sum_{n=1}^{N} a_n^2$$

Remark 4.7. An orthonormal system of vectors $\{u_n\}$ is *complete* if and only if the zero vector is the only one which is orthogonal to every u_n.

In fact, lets us assume first that orthonormal system of vectors $\{u_n\}$ is *complete* and prove that 0 is orthogonal to every vector u_n: this is obvious, as clearly $(0, z) = 0$ for every vector $z \in H$.

Now let us prove the opposite implication, that is, if zero is the *only* vector orthogonal to all u_n, the orthonormal system of vectors $\{u_n\}$ is *complete.* In fact, let us assume that the orthonormal system is *not* complete. Since (a) holds, this means that (b) fails, that is, there exists a vector $\overline{u}$ which *cannot* be developed as the (strong) limit

$$||\sum_{n=1}^{\infty}(\overline{u},u_n)u_n - \sum_{n=1}^{N}(\overline{u},u_n)u_n|| \to 0 \quad as \quad N \to +\infty\,.$$

By extracting possibly a further sub-sequence, still denoted by the index n, we would have

$$\limsup_{N\to+\infty}\sum_{n=1}^{N}||(\overline{u},u_n)u_n|| = +\infty\,,$$

which can be rewritten, since $||u_n|| = 1$, as

$$\limsup_{N\to+\infty}\sum_{n=1}^{N}|(\overline{u},u_n)| = +\infty\,.$$

This would imply, up to possibly extracting a further sub-sequence, that the sequence $\{|(\overline{u},u_n)|^2\}$ is *unbounded*, but this would contradict Bessel inequality:

$$\limsup_{N\to+\infty}\sum_{n=1}^{N}|(\overline{u},u_n)|^2 \le \sum_{n=1}^{\infty}|(\overline{u},u_n)|^2 \le ||\overline{u}||^2 \;< +\infty\,.$$

This implies that the sequence $\{|(\overline{u},u_n)|^2\}$ is *bounded.* To see that we observe that if $|(\overline{u},u_n)| \ge 1$, then $|(\overline{u},u_n)| \le |(\overline{u},u_n)|^2$, while if $|(\overline{u},u_n)| < 1$ there is nothing to modify, therefore

$$\mathrm{Sup}_n\{|(\overline{u},u_n)|\} \le \max\{\mathrm{Sup}_n\{|(\overline{u},u_n)|\},1\}\,.$$

An important property of continuous linear functionals on a Hilbert space H is the following result by F. Riesz:

Theorem 4.8. (F. Riesz Representation Theorem) *For every continuous linear functional $F(v)$ defined on a real Hilbert space H, there exists a vector $u \in H$ such that*

$$F(v) = (u,v)_H \quad \forall\, v \in M\,.$$

Such a vector u is unique.

Proof. In this proof we omit for simplicity the suffix H in the notation of the inner product $(u,v)_H$. Let $\{v_i\}$ be an orthonormal sequence in H, and set $u := \Sigma F(v_i)\, v_i$.

Then for any vector $v = \Sigma(v,v_i)\; v_i$ we have

$$F(v) = \Sigma(v,v_i)\,(u,v_i) = \Sigma(F(v_i)\,v_i,v) = (u,v) \quad \forall\, v \in H\,,$$

where we have used only the linearity of the inner product with respect to its first argument. □

The following Theorem 4.9 is a special case of a more general *uniform boundedness Banach-Steinhaus theorem.* Here we state this theorem not in its full form, but only in the special case of a real or complex Hilbert space we are dealing with.

Theorem 4.9. *If a sequence $\{u_n\}$ has the property that $|(u_n, v)|$ is bounded for every $v \in H$, then $\{u_n\}$ is bounded: i.e., the sequence of norms $||u_n||$ is bounded.*

In this case, a direct proof has been given by S. H. Gould in his book [3], where it is stated as Lemma II of paragraph 4 of Chapter VII. For the reader's convenience, we reproduce Gould's proof below:

Proof of Theorem 4.9 according to [3]. Let us assume that there exists an unbounded sequence $||u_n||$ for which $|(u_n, v)| = |u_n(v)|$ is bounded for every $v \in H$. Then, in every sphere $S(v_0, \epsilon)$ with center v_0 and radius ϵ, the sequence $\{u_n(v)\}$ is unbounded for some v; that is, for every c, there exists a v in S and an index $\overline{n}$ such that $|u_{\overline{n}}(v)| > \epsilon$.

For, if we had $|u_{\overline{n}}| \leq c$ for all v in $S(v_0, \epsilon)$, and therefore, by the continuity of $u_{\overline{n}}(v)$, for all v on the surface of of S, then we would have $|u_{\overline{n}}(v)|2c/\epsilon||v||$ for all v in H, as may be proved as follows.

Let $v_\epsilon = \epsilon v/||v||$ and $v_s = v_0 + v_\epsilon$; then $u_n(v_\epsilon) = u_n(v_s) - u_n(v_0)$; but $|u_n(v_s)| \leq c$ and $|u_n(v_0)| \leq c$, so that $|u_n(v_\epsilon)| \leq 2c$ and therefore

$$|u_n(v)| = \frac{||v||}{\epsilon}|u_n(v)| \leq \frac{2c}{\epsilon}||v||$$

from which, putting $v = u_n$ we would have $||u_n|| \leq \frac{2c}{\epsilon}$, contrary to assumption. Consequently we may choose n_1 and v_1 such that $u_{n_1}(v_1) > 1$.

Then, since u_{n_1} is continuous, we may take $\epsilon_1 < \frac{1}{2}$ such that $|u_{n_1}(v)| > 1$ for all v in the sphere $S(v_1, v_1)$. Now, for successive $m = 2, 3, 4, ...$ let us choose an index n_m and an element v_m in the interior of the sphere $S_{m-1}(v_{m-1}, \epsilon_{m-1})$ such that $|u_{n_m}(v_m)| > m$ and take $\epsilon_m < \frac{1}{2}m$ such that $S_m(v_m, \epsilon_m)$ lies entirely inside S_{m-1} and

$$|u_{n_m}(v)| > m \quad \forall\, v \in S_m.$$

Then $\{v_m\}$ is a Cauchy sequence and so must converge, by Property 3.4, to a vector w, which clearly lies inside every S_m. But then

$$u_{n_1}(w) > 1, \quad u_{n_2}(w) > 2, \; ...., u_{n_m}(w) > m, \; ....,$$

which is contrary to the hypothesis that the sequence $\{u_n(w)\}$ is bounded. □

Lemma 4.10. *Every weakly Cauchy sequence $\{u_n\}$ is weakly convergent.*

Once again, a direct proof of Lemma 4.1 has been given by Gould in his book already mentioned, where it is stated as Lemma III of Paragraph 4 of Chapter 7. Our proof below of Lemma 4.1 is the original proof by Gould, reproduced here again for the reader's convenience:

Proof of Lemma 4.10 according to [3]. By Theorem 4.9 above, there exists an M for which $||u_n|| < M$ for all u_n in $\{u_n\}$. Let us set $f(v) = \lim(u_n, v)$, where the limit exists since $\{u_n\}$ is weakly Cauchy. But $|(u_n, v)| \leq ||u_n||\,||v|| \leq M\,||v||$, so that $f(v)$ is bounded and therefore, by Riesz' Representation Theorem 4.8, $f(v)$ can be written in the form

$$f(v) = \lim_{n \to +\infty}(u_n, v) = (u, v)\,.$$

Thus $u_n \to u$ weakly in H, as desired. □

Lemma 4.11. *If u_n converges strongly to u and v_n converges weakly to v, then (u_n, v_n) converges to (u, v) as $n \to +\infty$.*

Again, a direct proof of Lemma 4.11 is given by Gould in his book, stated there as the Lemma IV of Paragraph 4 of Chapter 7. Here is his proof:

Proof of Lemma 4.11 according to [3]. We may write

$$|(u_n, v_n) - (u, v)| = |(u_n - u, v_n) + (u, v_n - v)| \leq |(u_n - u, v_n)| + (u, v_n - v)|$$

and since v_n converges weakly to v, there exists, by Theorem 4.9, a constant b, such that $||v_n|| \leq b$. Also, since $u_n \to u$ strongly, we have $||u_n - u|| \to 0$. Thus

$$|(u_n - u, v_n)| \leq ||u_n - u|| \, ||v_n|| \to 0$$

and $$|(u, v_n - v)| = |(u, v_n) - (u, v)| \to 0\,.$$

But $$|(u, v_n - v)| = |(u, v_n) - (u, v)| \to 0$$

since v_n converges weakly to v. Thus $(u_n, v_n) \to (u, v)$ as desired. □

Next we prove a fundamental result, Theorem 4.12 below, which is of great importance in many existence properties occurring in the applications.

Theorem 4.12. (Sequential weak compactness of the unit ball in separable Hilbert spaces) *The unit ball $||u|| \leq 1$ in a separable Hilbert space is weakly sequentially compact.*

Proof of Theorem 4.12 according to [3], page 125. We must prove that every sequence $\{u_n\}$ in the unit ball contains a sub-sequence weakly convergent to a vector u in the unit ball.

Let $\{p_m\}$ be an arbitrary c.o.n.s. Then, since $\{p_n\}$ is bounded, it follows from Schwarz inequality that the sequence (p_m, u_n) is bounded for every p_m.

Thus we may choose a sub-sequence $\{u_{1,n}\}$ of $\{u_n\}$ such that $\{(p_1, u_{1,n})\}$ converges weakly, and for $m > 1$ we may take $\{u_{m,n}\}$ as a sub-sequence of $\{u_{n,m-1}\}$ in such a way that $\{(p_m, u_{n,m}\}$ converges weakly. Then, for each fixed m, the diagonal sequence $\{v_n\} = \{u_{n,n}\}$, at least after its first $(m-1)$ elements have been deleted, is a (diagonal) sub-sequence of the sequence $\{u_{n,m}\}$ that converges weakly. Thus, (p, m, v_n) is weakly convergent for each fixed m. Therefore, the sequence (p, v_n) is a weakly Cauchy sequence, so by Lemma 4.10, it is weakly convergent, hence there exists a vector u such that $v_n \to u$ weakly as desired.

Also $||u|| \leq 1$, so that u belongs to the unit ball; for, if we suppose $||u|| > 1$, then since the v_n are normalized, we would have by Schwarz inequality

$$|(u, v_n)| \leq ||u|| \, ||v_n|| \leq ||u|| < ||u||^2 = (u, u)$$

for all v_n, which is impossible, since $v_n \to u$ weakly and therefore $(u, v_n) \to (u, u)$. The proof is complete. □

Theorem 4.13. (Separation of a point from a subspace) *If M is a (closed linear) subspace of a (real or complex) Hilbert space H, if x is a vector of H and if*

$$\delta := \inf\{||y-x|| : y \in M\},$$

then there exists a vector $y_0 \in M$ such that $||y_0 - x|| = \delta$.

Proof. Let $\{y_n\}$ be a sequence of vectors in M such that $||y_n - x|| \to \delta$. By the parallelogram identity, we then have for every n and m

$$||y_n - y_m||^2 = 2||y_n - x||^2 + 2||y_m - x||^2 - 4||\frac{1}{2}(y_n + y_m) - x||^2 .$$

Since $\frac{1}{2}(y_n + y_m) \in M$, it follows that

$$||\frac{1}{2}(y_n + y_m) - x||^2 \geq \delta^2 ,$$

hence that $\qquad ||y_n - y_m||^2 \leq 2||y_n - x||^2 + 2||y_m - x||^2 - 4\delta^2.$

In the limit as $n \to \infty$ and $m \to \infty$, the right side of the last inequality tends to $2\delta^2 + 2\delta^2 - 4\delta^2 = 0$; therefore $\{y_n\}$ is a Cauchy sequence. Thus, by the completeness of the space, there exists a limit vector y_0. Such a vector belongs to M because M is closed, therefore, by the continuity of the norm,

$$||y_0 - x|| = \lim_{n\to\infty} ||y_n - x|| = \delta.$$

Theorem 4.14 (Separation of two subspaces) *If M and N are subspaces such that $M \subseteq N$ and $M \neq N$, then there exists a non-zero vector z in N such that $z \perp M$.*

Proof. Let x be any vector in N which is not in M and write

$$\delta = \inf\{||y-x|| : y \in M\}.$$

By Proposition 2.15, there exists a vector y_0 in M such that $||y_0 - x|| = \delta$, and write $x = y_0 - x$. The fact that $z \neq 0$ follows from the fact that x is not in M. Since $y_0 + \alpha y \in M$ for every vector y in M and every complex number α, it follows that

$$||z + \alpha y|| = ||(y_0 + \alpha y) - x|| \geq \delta$$

and hence that

$$0 \leq ||x + \alpha\, y||^2 - ||z||^2 = \overline{\alpha}\,(z, y) + \alpha\,(y, z) + |\alpha|^2\, ||y||^2 .$$

If, in particular, $\alpha = \beta\,(x, y)$ for any real number β, then

$$0 \leq 2\beta\,|(y, z)|^2 \,+\, \beta^2\,|(y, z)|^2\, ||y||^2 .$$

The validity of this inequality for small negative value of β implies the vanishing of the coefficient of the linear term. We conclude that $x \perp y$ and hence, since y is an arbitrary vector in M, that $z \perp M$.

Chapter 5. Convergence of vectors and subspaces

We start this Chapter 5 by describing the notion of so-called M-convergence of sequences of closed linear subspaces in a separable (complex) Hilbert spaces, originally introduced by the author in his papers [4], [6].

This notion has been found very useful in a variety of applications.

In order to give the definition of this convergence, we need two preliminary definitions, Definition 5.1 and Definition 5.2 below:

Definition 5.1. Given a sequence of closed, linear subspaces $\{M_n\}$ of an Hilbert space H, we define

$$s-\text{Limsup}\, M_n = \{u \in H: \ \exists\, u_n \in M_n \text{ such that } ||u - u_n|| \to 0 \text{ as } n \to +\infty\}.$$

Definition 5.2. Given a sequence of closed, linear subspaces $\{M_n\}$ of an Hilbert space H, we define

$$w-\text{Liminf}\, M_n = \left\{ v \in H: \begin{array}{l} v = \text{weak-}\lim v_j^n, \ v_j^n \in M_j^n, \\ \{M_j^n\}_j \text{ is a subsequence of } \{M_n\} \end{array} \right\}.$$

We now define M-convergence as follows:

Definition 5.3. We say that a sequence $\{M_n\}$ of linear subspaces of a Hilbert space H *M-converges* to a (closed, linear) subspace M of H, and write

$$M - \text{Lim}_{n\to+\infty} M_n = M \quad (\text{or} \ \ M = M - \text{Lim}_{n\to+\infty} \ M_n),$$

if *both* the following set-inclusions hold simultaneously:

$$s - \text{Limsup} \ \ M_n \subset M \subset w - \text{Liminf} \ \ M_n\,.$$

We point out that the fundamental property of *M-convergence* is to be *stable* under *orthogonal complementation* $M \to M^\perp$ in H. We have in fact the following:

Theorem 5.4. *A sequence of subspaces $\{M_n\}$ in a (separable) Hilbert space H M-converges to a subspace M if and only if the sequence of the orthogonal subspaces $\{M_n^\perp\}$ M-converges to the orthogonal complement $M^\perp$.*

The proof of Theorem 5.4 will be carried out below in several steps.

Proof. We must prove that as $n \to +\infty$:

(g_1) $\quad M = M - \text{Lim}_{n\to+\infty} M_n$ implies $s - \text{Limsup}\, M_n \le M$,

(g_2) $\quad M = M - \text{Lim}_{n\to+\infty} M_n$ implies $M \le w - \text{Liminf}\, M_n$.

We shall prove first (g_1). We recall, from Definition 5.1, that

$$s - \text{Limsup} \ M_n = \{u \in H: \ \exists\, u_n \in M_n \text{ such that } ||u - u_n|| \to 0, as \ \ n \to +\infty\}.$$

So, let $u \in H$ be given. We must find $u_n \in M_n$, such that $||u - u_n|| \to 0$ as $n \to +\infty$.

We choose $u_n := P_{M_n} u$ to be the orthogonal projection of u onto the subspace M_n.

We must check whether $||u - u_n|| \to 0$ as $n \to +\infty$.

This must come from our assumption $M = M - \mathrm{Lim}_{n\to+\infty} M_n$. By Definition 5.3, this means that both the following set-inclusions hold and are part of the present assumptions:

$$s - \mathrm{Limsup}\, M_n \subset M \subset w - \mathrm{Liminf}\, M_n\,.$$

In the case that $u \in M$, the second inclusion gives that $u \in w - \mathrm{Liminf}\, M_n$. By Definition 5.2, this means that u belongs to the set

$$w - \mathrm{Liminf}\, M_n = \{v \in H :\ v = w - \lim v_j^n \ \text{ as } \ j \to +\infty$$

$$\text{for every subsequence } \{M_j^n\} \text{ of } \{M_n\} \text{ and every } v_j^n \in M_j^n\,.$$

therefore $u = w - \lim u_n^j$ as $j \to +\infty$ for every subsequence $\{M_n^j\}$ of $\{M_n\}$. By the Banach-Steinhaus Theorem 4.9, this implies that the sequence $\{u_n\}$ is bounded, that is $||u_n|| \le C$ for all n with a constant C that does not depend on n and that $u = w - \lim_n u_n$.

Therefore, u_n converges *strongly* in the H norm to u, by the Theorem asserting that weak convergence of a sequence plus convergence of norms implies strong convergence, which can be deduced from the equality

$$||u||^2 = ||P_{M_n} u||^2 + ||P_{M_n^\perp} u||^2.$$

It remains to consider the case when $u \notin M$. Remember that our claim is that for every n there exist $u_n \in M_n$, such that $||u - u_n|| \to 0$ as $n \to +\infty$.

We apply Theorem 4.14 with $N = H$. We then get that there exists a vector $z \in H$, $z \neq 0$, such that $z \perp M$. We choose $u_n = z$ for all n. We then compute

$$||u - P_{M_n} z||^2 = -2(u, P_{M_n} z) + ||u||^2 + ||P_{M_n} z||^2$$

as well as $$||u + P_{M_n} z||^2 = +2(u, P_{M_n} z) + ||u||^2 + ||P_{M_n} z||^2.$$

Therefore $$||u - P_{M_n} z||^2 \le -2(u, P_{M_n} z)$$

as well as $$||u + P_{M_n} z||^2 \ge +2(u, P_{M_n} z)\,.$$

Let us define $$u_n := P_{M_n} z\,.$$

Then the two preceding two inequalities become

$$||u - u_n||^2 \le -2(u, u_n) \quad \text{and} \quad ||u + u_n||^2 \ge +2(u, u_n)\,.$$

We now distinguish two cases.

First case: $(u, u_n) = 0$ for every n. Then the first inequality gives $||u - u_n|| = 0$ for every n, hence obviously $||u - u_n|| \to 0$ as $n \to +\infty$ and our claim is proved. Note that the second inequality just says that $||u + u_n||^2 \ge 0$, which is trivial.

Second case: there is an index $n = \overline{n}$ such that $(u, u_{\overline{n}}) \neq 0$. We rewrite the two inequalities for such an index $\overline{n}$:

$$\text{(a)} \quad ||u - u_{\overline{n}}||^2 \le -2(u, u_{\overline{n}})$$

$$\text{(b)} \quad ||u + u_{\overline{n}}||^2 \ge +2(u, u_{\overline{n}})$$

and examine what could happen.

Two possibilities: $(u, u_{\overline{n}}) > 0$, or $(u, u_{\overline{n}}) < 0$, which we now examine separately.

Let $(u, u_{\overline{n}}) > 0$. Then, $-2(u, u_{\overline{n}}) < 0$ and the inequality (a) above would give:

$$0 \leq ||u - u_{\overline{n}}||^2 < -2(u, \overline{n}) < 0$$

which is absurd. The inequality (b) would give

$$||u + u_{\overline{n}}||^2 \geq +2(u, u_{\overline{n}}) > 0$$

that only implies $u \neq -u_{\overline{n}}$. This is indeed possible, because under the present assumptions $u \in H$, and $u_{\overline{n}} \in M_n$ for every n and both H and M_n being linear spaces.

The conclusion at this point is that the first possibility $(u, u_{\overline{n}}) > 0$ leads into absurdity. Therefore only the second possibility remains, namely $(u, u_{\overline{n}}) \leq 0$, which we are going to consider below.

Let $(u, u_{\overline{n}}) < 0$. Then, $-2(u, u_{\overline{n}}) > 0$ and the first inequality (a) above would give:

$$||u - u_{\overline{n}}||^2 \leq -2(u, u_{\overline{n}}) > 0\,.$$

Since $||u - u_{\overline{n}}||^2 \geq 0$, we then would have

$$0 \leq ||u - u_{\overline{n}}||^2 \leq -2(u, u_{\overline{n}}) > 0\,.$$

As before, this is indeed possible and only says that $u \neq u_{\overline{n}}$ (M_n and H are linear subspaces .

The second inequality (b) above would give

$$||u + u_{\overline{n}}||^2 \geq +2(u, u_{\overline{n}}) < 0$$

which is absurd.

This concludes the proof of Theorem 5.4. □

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We observe that the long, detailed exposition done in this paper in the aim of providing our unfortunate reader with a completely self contained text may appear, and probably is, pedantic and without style. That is too bad, I was not able to do better.

But probably the real question at stake is that value, content, precision, impact are in the end all a matter of style.

Style is probably the key word in thinking of Hedy Attouch. Style was certainly his most deep, precious interior quality and the root of his charisma, as a mathematician and as a wonderful person.

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The work on M-convergence by Hedy Attouch is only a special case of a much more comprehensive theory developed by him in the more general contest of *Moreau inf-convolution* and *Fenchel duality* theories for extended-value convex functionals in Hilbert and reflexive Banach spaces. Extended-value functionals are important to consider because they appear, in particular, in the theory of variational inequalities with obstacles.

This theory is based substantially on the seminal paper [1] of Attouch.

But Attouch's mathematical legacy goes well beyond this paper. In his masterful book [2] he provides an extraordinary survey of the most fundamental properties of variational convergence for functions and operators, encompassing so many aspects of homogenization theories, convex analysis, general calculus of variations and their important applications to ordinary and partial differential equations. Hedy Attouch's book will surely be cherished and used by many generations of students and researchers to come in science and engineering.

References

[1] H. Attouch: *Familles d'opérateurs maximaux monotone et mesurabilité*, Ann. Mat. Pura Appl., Serie IV 120 (1979) 35–111.

[2] H. Attouch: *Variational Convergence for Functions and Operators*, Pitman Advanced Publishing Program 19, Pitman, London (1984).

[3] S. H. Gould: *Variational Methods for Eigenvalue Problems*, Mathematical Expositions Vol. 10, University of Toronto Press, Toronto (1966).

[4] U. Mosco: *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. 3/4 (1969) 510–585.

[5] U. Mosco: *On the continuity of the Young-Fenchel transformations*, J. Math. Anal. Appl. 35 (1971) 518–535.

[6] U. Mosco: *An Introduction to the Approximate Solutions of Variational Inequalities*, Lecture Notes CIME, Erice 1971, Cremonese, Roma (1973).

[7] U. Mosco: *An elementary introduction to fractal analysis*, in: *Nonlinear Analysis and Applications to Physical Sciences*, V. Benci et al. (eds.), Lectures from the Summer School at Pistoia (2002), Springer, Milano (2004) 51–90.

[8] U. Mosco, F. Scarpini: *Complementarity systems and approximation of variational inequalities*, RAIRO R-1 (1975) 83–104.

[9] W. Rudin: *Principles of Mathematical Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York (1953).