Journal of Convex Analysis
Volume 33 (2026), No. 3&4, 1013–1048

Convergence of Subdifferentials versus Convergence of Slopes

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Dedicated to the memory of Hédy Attouch.

Received: May 16, 2025
Accepted: October 7, 2025

Given a sequence of lower semicontinuous proper convex functions on a Banach space, the graph convergence of their subdifferentials as well as the epigraphical convergence of their slopes are investigated. Several (counter-)examples showing that equivalence between these convergences is uncommon in infinite dimension are provided. However, under suitable additional conditions it is possible to establish some relations of this kind. Herein, under a compactness type assumption it is shown that bounded mixed convergence of subdifferentials of the sequence of functions is equivalent to the convergence of subdifferentials in the sense of Painlevé-Kuratowski, which turns to be equivalent to the epigraphical convergence of slopes of the functions in separable Banach spaces.

Keywords: Convex function, slope, slope determination of convex functions,convergence of slopes, convergence of slope under renorming, convergence of subdifferentials, bounded mixed way convergence of sets, Zarankiewicz's type set convergence results, perfect square criterion for maximality and compactness type assumptions.

2020 Mathematics Subject Classification: Primary 49J52; secondary: 47J22, 58E30.

1. Introduction

Attouch's theorem relates the graphical convergence of subdifferentials of a sequence of convex functions with the epigraphical convergence of the functions in reflexive Banach spaces, see [2]. Since the slope of a convex function turns out to be a good parameter describing the function, it is natural to link the graphical convergence of a sequence of subdifferentials of convex functions with the epigraphical convergence of their slopes; this is essentially the objective of our paper. In finite dimensions the coincidence between the two latter convergences has been carried

ISSN 0944-6532 / $ 2.50

out and shown in [11] under a minorization condition; as one of our aims we furnish herein a short proof of that coincidence in Theorem 3.2 in such a finite-dimensional setting. Unlike, in spaces of infinite dimension this type of results fails in general; investigating the infinite dimensional situation is then our principal aim. First, we provide a list of simple examples, see Section 6, of sequences of convex functions defined on commonly known Banach spaces for which it is not possible to obtain equivalences between the graphical convergence of subdifferentials and the epigraphical convergence of slopes. It seems that the main difficulty in obtaining this type of relationship is the lack of good conditions ensuring the maximal monotonicity of the limit set of a convergent sequence of subdifferentials. There are two ways to circumvent this inconvenience: first, assume that the limit set is maximally monotone, see for example [2, Proposition 3.59] and second, impose conditions ensuring this feature, see for example [1, Theorem 3.1]. A third way could be to repeat Attouch's reasoning by imposing normalization conditions, see [2, Theorem 3.66]. In this work, we decided to show the relationship between the convergences discussed above by imposing conditions such that the limit sets are maximally monotone. Our study is guided in two ways, first by ensuring the maximal monotonicity through compactness type assumption, see condition (CTA) from Section 7. In finite dimensional spaces they are always satisfied. In infinite-dimensional Banach spaces, the condition (CTA) can be verified in some important cases, see for example [1, Remark 3.1 (b) or (c)]. Comparing the condition (CTA) with that from [1, Theorem 3.1], we see that it is valid/amenable for subdifferentials in any Banach space, while the condition from [1, Theorem 3.1] was employed in Hilbert spaces but for any operator. One of the consequences of the (CTA) assumption is the maximal monotonicity of the limiting set of a sequence of subdifferentials in any Banach space, see Proposition 7.5. Moreover, the maximal monotonicity of the graphical limit of a sequence of subdifferentials of convex functions is a necessary condition to have the epigraphical convergence of their slopes, whenever the condition (CTA) is valid; this can be deduced by combining the results from Corollary 5.5 and Proposition 7.5. In the case of a reflexive Banach space with its norm having the Kadec-Klee property, it follows from Lemma 4.12 and Lemma 4.13 that the (CTA) condition is a necessary condition for the graphical convergence of subdifferentials of lower semicontiuous convex functions to the graph of the subdifferential of a lower semicontinuous proper convex function, see Proposition 7.3. Taking into account results from Sections 4, 5, 6 and 7, our paper answers a number of questions concerning convergences of subdifferentials and slopes, see also [11, Open problems].

At this point it is also worth recalling that there are several important definitions of convergence of sets, see [5]. Some aspects resulting from different approaches to the convergence of subdifferentials are discussed in Section 4. These convergences should be examined relative to natural topologies for these purposes, for example strong, or weak. This results in the need to consider several distance concepts of convergence of sets. The work managed to show that under the assumptions made, the limit sets are the same regardless of the topology utilized to test convergence, see for example Lemma 7.4. Let us also mention that renorming the space we change the values of slopes of the convex functions. This results in additional difficulties in the investigation on the epigraphical convergence of slopes. However, it turns out that in Asplund spaces, if the condition (CTA) holds for some equivalent norm and the epigraphical convergence of slopes is ensured for this norm, then the epigraphical convergence of slopes is ensured in every other equivalent norm, see [20, Theorems 3.1, 3.2, 4.5, 4.6 and 4.7].

The paper is organized as follows. Section 2 recalls diverse needed results on Painlevé-Kuratowski convergence of sets, epiconvergence of functions, monotone operators, and sudifferentials of convex functions. A slope determination theorem for convex functions in Banach spaces is also recalled in the same Section 2. Section 3 is devoted to the very short proof, in finite dimensions, of the theorem on the coincidence between the Painlevé-Kuratowski convergence of sudifferentials of lower semicontinuous convex functions and the epiconvergence of their slopes. For a Banach space E we introduce and study in Section 4 two specific *mixed* and *bounded mixed* convergences for subsets of $E \times E^*$ taking into account strong convergence on E and weak-star convergence on the topological dual E^*. Those convergences are utilized in the next sections for the investigation of extension in infinite dimension of the aforementioned theorem. Section 5 establishes various useful results for lower and upper epi-limits for slopes of convex functions. Section 6 provides a list of counter-examples, in certain classical infinite-dimensional Banach spaces, for the coincidence between the Painlevé-Kuratowski (resp. the bounded mixed) convergence of subdifferentials of lower semicontinuous convex functions with the epiconvergence of their slopes when no additional conditions are present. The condition (CTA) (see the first paragraph of this introduction) is introduced and analyzed in the final Section 7, and under that condition we prove in that section our main theorem, say Theorem 7.7, establishing on separable Banach spaces the coincidence between the three convergences: the Painlevé-Kuratowski convergence of subdifferentials of lower semicontinuous convex functions, their bounded mixed convergence, the epiconvergence of the slopes of the functions.

We emphasize that certain essential results of the present paper have been utilized in our study of similar features in Asplund spaces, see [20].

2. Notions and Preliminary results

Throughout the paper $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Q} \subset \mathbb{R}$ the subset of all rational numbers of $\mathbb{R}$ and $\mathbb{N}$ the set of positive integers (starting with 1), and E stands for a (real) Banach space with norm $\|\cdot\|$ and its (topological) dual E^* with the dual canonical norm $\|\cdot\|_*$, see e.g. [13, Definition 1.28]. For every real $r > 0$ and every $x \in E$ we denote by $\mathbb{B}_E(x, r)$ (resp. $\mathbb{B}_E[x, r]$) the open (resp. closed) ball centered at x and of radius r, the sphere is denoted by $\mathbb{S}_E[x, r] := \{y \in E \mid \|y - x\| = r\}$. As usual, "int" stands for the interior of a subset of E and "cl" for the strong closure, "cl^w" for the weak closure in E, and "cl^{w^*}" for the weak* closure in E^*, see [25] for more. For any function $g : E \to \mathbb{R} \cup \{+\infty\}$ and any operator $A : E \rightrightarrows E^*$ and for any set D in E, we denote as usual by $g_{|_D}$ the restriction of g to D and by $A_{|_D}$ the restriction of A to D.

2.1. P.K. convergence of sets

Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence of sets in a Hausdorff topological space (X, τ). One defines in (X, τ) the *sequential lower limit* $\mathrm{Li}_\tau\, S_k$ as the set of $x \in X$ for which there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X τ-converging to x such that $x_k \in S_k$ for all large k. When the set $S_k \neq \emptyset$ for every $k \in \mathbb{N}$, it is obviously equivalent above to require the existence of a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X τ-converging to x such that $x_k \in S_k$ for every $k \in \mathbb{N}$. Regarding the *sequential upper limit* $\operatorname{Ls}_\tau S_k$ in (X,τ), it is the set of $x \in X$ for which there is an increasing function $j : \mathbb{N} \to \mathbb{N}$ and a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X converging to x such that $x_k \in S_{j(k)}$ for all $k \in \mathbb{N}$. One says that a subset S of X is the Painlevé-Kuratowski (or P.K., for short) sequential limit or set limit of the sequence $\{S_k\}_{k\in\mathbb{N}}$ if the lower and upper sequential limits of the sequence coincide with S, which is equivalent to

$$\operatorname{Ls}_\tau S_k \subset S \subset \operatorname{Li}_\tau S_k. \tag{1}$$

When this holds, the limit S of the sequence is called the *Painlevé-Kuratowski* (or *P.K.*) *sequential limit* in (X,τ) of $\{S_k\}_{k\in\mathbb{N}}$, and one writes $S_k \xrightarrow{P.K.} S$.

If the sequence $\{S_k\}_{k\in\mathbb{N}}$ P.K. converges to a set S in (X,τ), for any subsequence $\{S_{j(k)}\}_{k\in\mathbb{N}}$ (i.e., $j : \mathbb{N} \to \mathbb{N}$ is increasing) we see that

$$\operatorname{Ls}_\tau S_{j(k)} \subset \operatorname{Ls}_\tau S_k \subset S \subset \operatorname{Li}_\tau S_k \subset \operatorname{Li}_\tau S_{j(k)}, \tag{2}$$

then the subsequence $\{S_{j(k)}\}_{k\in\mathbb{N}}$ also P.K. converges to the set S in (X,τ).

If the topology τ is metrizable by some distance d, one generally omits the index τ in just writing $\operatorname{Li} S_k$ and $\operatorname{Ls} S_k$ and one knows that

$$\operatorname{Li} S_k = \{x \in X \mid \limsup_{k\to\infty} d(x,S_k) = 0\},\ \operatorname{Ls} S_k = \{x \in X \mid \liminf_{k\to\infty} d(x,S_k) = 0\}. \tag{3}$$

In such a metric case, one just says that the limit S of the sequence (if it exists) is the Painlevé-Kuratowski (or P.K.) limit (omitting "*sequential*"). Let us also emphasize that the definitions preceding (1) ensure that the sets $\operatorname{Li} S_k$ and $\operatorname{Ls} S_k$ in a metrizable topological space do not depend on equivalent distance, i.e. for any equivalent metric to d the equalities in (3) hold true. Thus, in a metric space (X,d) the P.K. limit is the same for every equivalent metric to d. Moreover, if $S_k \xrightarrow{P.K.} S$ then for all $x \in E$ we have

$$\limsup_{k\to\infty} d(x,S_k) \le d(x,S). \tag{4}$$

In addition to (2), we recall a lemma on P.K. convergence established in the context of metric spaces in Kuratowski's book [18, Chapter II §25, IX, 1].

Lemma 2.1. *Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence of sets in a metric space (X,d). Then $\{S_k\}_{k\in\mathbb{N}}$ converges to a set S in (X,d) in the P.K. sense if and only if every subsequence has a further subsequence which converges to S in the P.K. sense.*

In 1927 K. Zarankiewicz obtained a result on limits of subsequences of sets, whenever the topology τ is metrizable, see [34]. We recall it following [4, Theorem 1.17] and [18, Chapter II, §25, VIII].

Theorem 2.2 (Zarankiewicz). *Every sequence of subsets $\{S_k\}_{k\in\mathbb{N}}$ of a separable metric space X contains a subsequence which has a (possibly empty) P.K. limit.*

We recall (see, e.g. [28, Proposition 1.7]) a known lemma (also easy to see via (3)).

Lemma 2.3. *If (X, d) is a metric space and $\{S_k\}_{k\in\mathbb{N}}$ is a sequence in X, then both sets $\operatorname{Ls} S_k$ and $\operatorname{Li} S_k$ are closed.*

2.2. Epiconvergence and semilimits

Recall that the *epigraph* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\operatorname{epi} f := \{(x, r) \in E \times \mathbb{R} \mid f(x) \le r\}.$$

Definition 2.4. A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ on the Banach space $(E, \|\cdot\|)$ epigraphically (for short, epi-) converges to f, denoted as $f_k \xrightarrow{e} f$, if $\operatorname{epi} f_k \xrightarrow{P.K.} \operatorname{epi} f$.

Since the P.K. limit does not depend on equivalent norm in E or in E^*, the epigraphical convergence of given functions does not depend on the choice of equivalent norms. Moreover, the epigraph of the limit function must be closed, see Lemma 2.3, so the function must be lower semicontinuous. It is known that the epi-convergence of functions $\{f_k\}_{k\in\mathbb{N}}$ is equivalent to requiring that the following two conditions hold: for every $x \in E$

$$\liminf_{k\to\infty} f_k(x_k) \ge f(x) \quad \text{for every sequence } \{x_k\}_{k\in\mathbb{N}} \text{ in } E \text{ converging to } x \tag{5}$$

$$\limsup_{k\to\infty} f_k(x_k) \le f(x) \quad \text{for some sequence } \{x_k\}_{k\in\mathbb{N}} \text{ in } E \text{ converging to } x. \tag{6}$$

The above is equivalent to saying that for every $x \in E$

$$\min\{\limsup_{k\to\infty} f_k(x_k) \mid x_k \to x\} \le f(x) \le \min\{\liminf_{k\to\infty} f_k(x_k) \mid x_k \to x\},$$

see e.g. [3, Proposition 12.1.1]. The function $\operatorname{li} f_k$ (resp. $\operatorname{ls} f_k$) whose epigraph coincides with $\operatorname{Ls}\operatorname{epi} f_k$ (resp. $\operatorname{Li}\operatorname{epi} f_k$) is called the *epi-limit inferior* (resp. *superior*) of $\{f_k\}_{k\in\mathbb{N}}$. One has $\operatorname{li} f_k \le \operatorname{ls} f_k$ and one knows (see, e.g. [2, Theorems 1.13 and 1.36]) that

$$(\operatorname{li} f_k)(x) = \min\{\liminf_{k\to\infty} f_k(x_k) \mid x_k \to x\},$$

and

$$(\operatorname{ls} f_k)(x) = \min\{\limsup_{k\to\infty} f_k(x_k) \mid x_k \to x\}. \tag{7}$$

2.3. Monotone operator and subdifferential

Given a (set-valued) operator $A : E \rightrightarrows E^*$, its *(effective) domain* and its *graph* are defined by

$$\operatorname{Dom} A := \{x \in E \,|\, A(x) \ne \emptyset\} \text{ and } \operatorname{gph} A := \{(x, x^*) \in E \times E^* \,|\, x^* \in A(x)\}, \tag{8}$$

respectively. We recall (see, e.g. [22]) that an operator $A : E \rightrightarrows E^*$ is *monotone* if

$$\langle x^* - y^*, x - y\rangle \ge 0 \text{ for all } (x, x^*) \text{ and } (y, y^*) \text{ in } \operatorname{gph} A;$$

it is *cyclically monotone* if for $x_i \in \operatorname{Dom} A$, $i = 0, \cdots, n$, $x_i^* \in A(x_i)$ and $x_{n+1} = x_0$

$$\sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \le 0.$$

A subset $S \subset E \times E^*$ is also said to be *monotone* (resp. *cyclically monotone)* if it is the graph of an operator $A : E \rightrightarrows E^*$ enjoying the same property. The operator A (similarly, the subset $S \subset E \times E^*$) is *maximal monotone* (resp. *maximal cyclically monotone*) whenever its graph is a maximal set for the inclusion (order) in the graphs of monotone (resp. cyclically monotone) operators from E into E^*. Below we provide a necessary condition for the P.K. convergence of monotone subsets of $E \times E^*$; we refer to [26] for more on monotone subsets.

A fundamental class of monotone operators is that of subdifferentials of convex functions. Let $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. In Variational Analysis, the *(effective) domain* of f is defined by

$$\operatorname{dom} f := \{x \in E : f(x) < +\infty\},$$

and f is said to be *proper* if it does not take on the value $-\infty$ and its domain $\operatorname{dom} f$ is nonempty. Assume now $f : E \to \mathbb{R} \cup \{+\infty\}$ is a convex function taking values in $\mathbb{R} \cup \{+\infty\}$. Its *subdifferential* at any $x \in \operatorname{dom} f$ is defined by (see for example [6, Chapter 4] or [28, Definition 3.2])

$$\partial f(x) := \{x^* \in E^* \mid f(y) \geq f(x) + \langle x^*, y - x\rangle, \ \forall y \in E\}.$$

One also defines $\partial f(x) = \emptyset$ when $x \in E \setminus \operatorname{dom} f$. When the convex function f on the Banach space E is proper and lower semicontinuous (lsc, for short), then $\operatorname{gph} \partial f \neq \emptyset$ by the Brøndsted-Rockafellar theorem, see e.g. [21, Theorem 3.17]. From the above definition, it is also readily seen for a convex function f that the operator ∂f is monotone. Further, it is also known that ∂f is a maximal monotone operator whenever the convex function f on the Banach space E is proper and lsc.

Regarding maximal cyclically monotone operators, they are characterized by the following Rockafellar theorem (see [23]).

Theorem 2.5. *Given a Banach space $(E, \|\cdot\|)$, an operator $A : E \rightrightarrows E^*$ is maximal cyclically monotone if and only if it coincides with the subdifferential ∂f of an lsc proper convex function $f : E \to \mathbb{R} \cup \{+\infty\}$.*

2.4. Slope determination

For the Banach space E and a lower semicontinuous proper convex function f, we recall that the *slope* of f at $x \in E$ can be defined by

$$s_f(x) := \inf \{\|x^*\|_* \mid x^* \in \partial f(x)\}, \tag{9}$$

with the usual convention that the latter infimum is $+\infty$ when $\partial f(x) = \emptyset$ (it is also known as a specific type of metric slope for f, see e.g. [28, page 266], see also [29] for the lower semicontinuity properties of slopes).

Given an operator $A : E \rightrightarrows E^*$, following (9) we will also put

$$s_A(x) := \inf\{\|x^*\|_* \mid x^* \in A(x)\} \quad \text{for all } x \in E, \tag{10}$$

with the usual convention (as above) that the latter infimum is $+\infty$ when $A(x) = \emptyset$. Further, if $A(x)$ is w^*-closed and nonempty, then

$$s_A(x) = \|\overline{x}^*\|_* \quad \text{for some } \overline{x}^* \in A(x). \tag{11}$$

This is due to the w^*-lower semicontinuity of $\|\cdot\|_*$ and the w^*-compactness of $A(x) \cap \mathbb{B}_{E^*}[0, r_x]$ with $r_x := 1 + s_A(x)$; keep in mind that $\mathbb{B}_{E^*}[0, r_x]$ is w^*-compact according to the Banach-Alaoglu Theorem (see, e.g., [25, Section 3.15]).

The above definition of $s_f(x)$ and $s_A(x)$ may also be rewritten as $s_f(x) = d(0, \partial f(x)$ and $s_A(x) = d(0, A(x))$.

Below it is recalled in Theorem 2.6 that slopes of lsc convex functions on Banach spaces can be used to establish an inequality between convex functions, see [30, Proposition 5.1 and Theorem 5.1]. As far as we know, the determination of convex functions by their slopes started in [7] with $\mathcal{C}^2$ convex functions on Hilbert spaces. We also refer, for the slope determination Theorem 2.6, to [19] for Hilbert spaces and to [16] for Banach spaces. Other related works can be found in [10, 8, 9, 31].

Theorem 2.6. *Let $(E, \|\cdot\|)$ be a Banach space and let $f, g : E \to \mathbb{R} \cup \{+\infty\}$ be two lsc proper convex functions.*

(a) *If g is bounded from below and $d(0, \partial f(x)) \le d(0, \partial g(x))$ for all $x \in E$, then f is also bounded from below and*
$$f(x) - \inf_E f \le g(x) - \inf_E g \quad \text{for all } x \in E.$$

(b) *Assume that either f or g is bounded from below. Then*
$$d(0, \partial f(x)) = d(0, \partial g(x)) \quad \text{for all } x \in E$$
if and only if there exists a real constant c such that
$$f(x) = g(x) + c \quad \text{for all } x \in E.$$

3. The finite dimensional setting

The coincidence of the convergence of subdifferentials with the epiconvergence of slopes in a finite-dimensional Euclidean space was first established in [11] for slopes taken with respect to the Euclidean norm. We furnish below a short proof of that result. In Section 7, see Theorem 7.7, the coincidence of these two convergences is established in general separable Banach spaces under a compactness-like condition, covering in this way the above coincidence theorem in any finite-dimensional vector space for slopes taken with respect to any norm of the space.

Lemma 3.1. *Let $(E, \|\cdot\|)$ be a finite-dimensional normed space and $A : E \rightrightarrows E^*$ be an operator. Then $\operatorname{dom} s_A = \operatorname{Dom} A$, where s_A is defined as in* (10).

In addition, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E. The following conditions hold:

(a) *If $\operatorname{Ls} \operatorname{gph} \partial f_k \subset \operatorname{gph} A$, then for every $x \in E$ and every sequence $\{x_k\}_{k\in\mathbb{N}}$ in E converging to x the following inequality holds true*
$$\liminf_{k\to\infty} s_{f_k}(x_k) \ge s_A(x).$$

(b) *If $\operatorname{gph} A \subset \operatorname{Li} \operatorname{gph} \partial f_k$, then for every $x \in E$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in E converging to x such that*
$$\limsup_{k\to\infty} s_{f_k}(x_k) \le s_A(x).$$

Proof. The equality $\operatorname{dom} s_A = \operatorname{Dom} A$ is readily observed. For (a) fix $x \in E$ and take any sequence $\{x_k\}_{k\in\mathbb{N}}$ in E converging to x. We will prove $\liminf_{k\to\infty} s_{f_k}(x_k) \geq s_A(x)$. We may suppose $\liminf s_{f_k}(x_k) < +\infty$. There exists an increasing function $j : \mathbb{N} \to \mathbb{N}$ such that $s_{f_{j(k)}}(x_{j(k)})$ is finite for every $k \in \mathbb{N}$ and

$$\liminf_{k\to\infty} s_{f_k}(x_k) = \lim_{k\to\infty} s_{f_{j(k)}}(x_{j(k)}).$$

By the closedness of $\partial f_{j(k)}(x_{j(k)})$ and by (11), we can choose $x^*_{j(k)} \in \partial f_{j(k)}(x_{j(k)})$ for each $k \in \mathbb{N}$ such that $\|x^*_{j(k)}\|_* = s_{f_{j(k)}}(x_{j(k)})$. Since $\{x^*_{j(k)}\}_{k\in\mathbb{N}}$ is bounded, it has some (non relabed) subsequence converging to some $x^* \in E^*$. Clearly,

$$\|x^*\|_* = \lim_{k\to\infty} s_{f_{j(k)}}(x_{j(k)}) = \liminf_{k\to\infty} s_{f_k}(x_k)$$

and we have the inclusions $(x, x^*) \in \operatorname{Ls} \operatorname{gph} \partial f_k \subset \operatorname{gph} A$. This confirms (a).

For (b) fix any $x \in E$. We may suppose $s_A(x) < +\infty$, so $x \in \operatorname{Dom} A$. Take any sequence $\{x^*_i\}_{i\in\mathbb{N}}$ in E^* with $\lim_{i\to\infty} \|x^*_i\|_* = s_A(x)$ and $(x, x^*_i) \in \operatorname{gph} A$ for all $i \in \mathbb{N}$. The inclusion assumption gives for each $i \in \mathbb{N}$ a sequence $\{(b_{i,k}, b^*_{i,k})\}_{k\in\mathbb{N}}$ such that $(b_{i,k}, b^*_{i,k}) \in \operatorname{gph} \partial f_k$ for all $k \in \mathbb{N}$ and such that

$$\lim_{k\to\infty} \|b^*_{i,k} - x^*_i\|_* + \|b_{i,k} - x\| = 0,$$

hence $\limsup_{k\to\infty} s_{f_k}(b_{i,k}) \leq \|x^*_i\|_*$. Choose an increasing function $j : \mathbb{N} \longrightarrow \mathbb{N}$ such that for each $i \in \mathbb{N}$ and all $k \geq j(i)$

$$\|b^*_{i,k} - x^*_i\|_* + \|b_{i,k} - x\| \leq \frac{1}{i} \text{ and } s_{f_k}(b_{i,k}) \leq \|x^*_i\|_* + \frac{1}{i}.$$

For each $k \in \mathbb{N}$ satisfying $k \geq j(1)$ denote $i_k \in \mathbb{N}$ the unique integer such that $j(i_k) \leq k < j(1 + i_k)$, and put $x_k := b_{i_k,k}$. Put also $x_k := b_{1,k}$ for each integer $k < j(1)$ (if any). Notice that, as $k \to \infty$, both $j(i_k) \to \infty$ and $i_k \to \infty$. The inequality $\|x_k - x\| \leq 1/i_k$ gives $\lim_{k\to\infty} x_k = x$. Since $(x_k, b^*_{i_k,k}) \in \operatorname{gph}(\partial f_k)$ for all $k \geq j(1)$, we obtain $\limsup\limits_{k\to\infty} s_{f_k}(x_k) \leq \lim\limits_{k\to\infty} \|x^*_{i_k}\|_* = s_A(x)$. This finishes the proof of the lemma. □

It is known that in finite-dimensional Euclidean spaces the *P.K. nonempty limit* of a sequence of graphs of maximal monotone operators is the graph of a maximal monotone operator (see, e.g. [24, Theorem 12.32][1]). This being recalled, our short proof of the aforementioned coincidence theorem in [11] is the following.

Theorem 3.2. *Let $(E, \|\cdot\|)$ be a (finite-dimensional) Euclidean space, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E. Let f be a lsc proper convex function on E bounded from below. Then, one has the following equivalence:*

$$\operatorname{gph} \partial f_k \xrightarrow{P.K.} \operatorname{gph} \partial f \quad \textit{if and only if} \quad s_{f_k} \xrightarrow{e} s_f.$$

Proof. The implication $\Rightarrow$ is a consequence of Lemma 3.1.

Let us now show the converse implication $\Leftarrow$ by proving that any subsequence of $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$ contains a subsequence P.K.-converging to $\operatorname{gph} \partial f$ (see Lemma 2.1). By Zarankiewicz Theorem (see Theorem 2.2), given any subsequence of $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$,

[1] Theorem 12.32 in [24] with $\operatorname{Dom} A \neq \emptyset$

say $\{\operatorname{gph}\partial f_{j(k)}\}_{k\in\mathbb{N}}$ where $j : \mathbb{N} \longrightarrow \mathbb{N}$ is increasing, there exist an operator $A : E \rightrightarrows E$ and an increasing function $\ell : \mathbb{N} \to \mathbb{N}$ such that for $\gamma := j \circ \ell$ one has $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{P.K.} \operatorname{gph} A$. The assumption $s_{f_k} \xrightarrow{e} s_f$ also ensures by (2) that $s_{f_{\gamma(k)}} \xrightarrow{e} s_f$. Since f is lsc, proper and convex, we have $\operatorname{Dom}\partial f \neq \emptyset$, and hence $s_f(\cdot)$ is proper. According to the above convergence $s_{f_{\gamma(k)}} \xrightarrow{e} s_f$ and the inclusion $\operatorname{Ls}\operatorname{gph}\partial f_{\gamma(k)} \subset \operatorname{gph} A$, Lemma 3.1(a) tells us that $s_f(\cdot) \geq s_A(\cdot)$, which ensures by properness of $s_f(\cdot)$ that $s_A(\cdot)$ is proper, and hence $\operatorname{Dom} A \neq \emptyset$. The property $\operatorname{Dom} A \neq \emptyset$ and the convergence $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{P.K.} \operatorname{gph} A$ entail that A is maximal monotone since E is finite dimensional and Euclidean (see, e.g. [24, Theorem 12.32] recalled above). Since A is also cyclically monotone (as directy seen from the definition and the latter P.K. convergence), the Rockafellar Theorem (see Theorem 2.5) says that there exists an lsc proper convex function ϕ on E such that $\operatorname{gph} A = \operatorname{gph}\partial\phi$, which combined with the above implication entails $s_{f_{\gamma(k)}} \xrightarrow{e} s_\phi$. From the convergences $s_{f_{\gamma(k)}} \xrightarrow{e} s_f$ and $s_{f_{\gamma(k)}} \xrightarrow{e} s_\phi$ we obtain $s_f(x) = s_\phi(x)$ for all $x \in E$. We deduce from Theorem 2.6 that f and ϕ coincide up to an additive constant, so $\operatorname{gph}\partial f = \operatorname{gph}\partial\phi = \operatorname{gph} A$. Therefore, any subsequence $\{\operatorname{gph}\partial f_{j(k)}\}_{k\in\mathbb{N}}$ contains a further subsequence P.K.-converging to $\operatorname{gph}\partial f$, which by Lemma 2.1 confirms the implication $\Leftarrow$. □

After the above theorem in finite dimensions, we turn in the sequel to the situation of infinite-dimensional spaces. The study in that context is much more involved. In addition to the P.K. convergence we will need certain other concepts of convergence for subsets of $E \times E^*$.

4. More on convergence of sets

Working in the product space $E \times E^*$, it will be convenient below to denote d and d^* the distances on the normed space $(E, \|\cdot\|)$ and on the topological dual $(E^*, \|\cdot\|_*)$ respectively, that is,

$$d(x, y) = \|x - y\| \text{ for } x, y \in E, \quad \text{and} \quad d^*(x^*, y^*) = \|x^* - y^*\|_* \text{ for } x^*, y^* \in E^*.$$

The distance $d \times d^*$ is the distance on $E \times E^*$ defined by

$$(d \times d^*)\big((x, x^*), (y, y^*)\big) := \big[(d(x, y))^2 + (d^*(x^*, y^*))^2\big]^{1/2} \tag{12}$$

and related to the product of strong topologies on E and E^*. The topology associated with the distance $d \times d^*$ is then the strong topology of $E \times E^*$.

Before introducing the other convergences that we have to use for subsets in $E \times E^*$, we will establish Lemma 4.1. Under a certain condition involving the Fitzpatrick function the lemma will provide a necessary condition for the P.K. convergence of monotone subsets of $E \times E^*$. Recall that, for a given nonempty subset S of $E \times E^*$, the associated Fitzpatrick function (see [14, Definition 3.1]) is given by

$$F_S(v, v^*) := \sup_{(s,s^*)\in S} \langle s^*, v\rangle + \langle v^*, s\rangle - \langle s^*, s\rangle \text{ for all } (v, v^*) \in E \times E^*.$$

In the lemma and in all the sequel, we use the notation $\operatorname{cl}^{w\times w^*}$ for the topological closure with respect to the product topology on $E \times E^*$ of the weak topology on E and the weak* topology on E^*.

Lemma 4.1. *Let $(E, \|\cdot\|)$ be a Banach space, S and $\{S_k\}_{k\in\mathbb{N}}$ be monotone sets in $E \times E^*$. Let $W \subset E \times E^*$ be such that*

$$\bigcap_{k\in\mathbb{N}} \operatorname{cl}^{w\times w^*}\big(\bigcup_{j\geq k} S_j\big) \subset W$$

and such that for all $(v, v^*) \in W$ the following inequality

$$F_S(v, v^*) \geq \langle v^*, v\rangle \tag{13}$$

is fulfilled. A necessary condition for the inclusion

$$S \subset \operatorname{Li}_{d\times d^*} S_k$$

to hold is that for any increasing function $j : \mathbb{N} \longrightarrow \mathbb{N}$, for any sequences $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$ with $(b_k, b_k^*) \in S_{j(k)}$ for large $k \in \mathbb{N}$, and for any pair

$$(b, b^*) \in \bigcap_{k\in\mathbb{N}} \operatorname{cl}^{w\times w^*}\{(b_k, b_k^*), (b_{k+1}, b_{k+1}^*), \ldots\}$$

one has

$$\limsup_{k\to\infty} \langle b_k^*, b_k\rangle \geq \langle b^*, b\rangle. \tag{14}$$

Proof. Assume that $S \subset \operatorname{Li}_{d\times d^*} S_k$ and $j : \mathbb{N} \longrightarrow \mathbb{N}$ is an increasing function. Let us consider two sequences $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$ such that $(b_k, b_k^*) \in S_{j(k)}$ for large $k \in \mathbb{N}$, and fix $(b, b^*) \in \bigcap_{k\in\mathbb{N}} \operatorname{cl}^{w\times w^*}\{(b_k, b_k^*), (b_{k+1}, b_{k+1}^*), \ldots\}$. For every $(s, s^*) \in S$ there exists a sequene $\{(s_k, s_k^*)\}_{k\in\mathbb{N}}$ such that $(s_k, s_k^*) \in S_{j(k)}$ for large $k \in \mathbb{N}$ and $\lim_{k\to\infty}(s_k, s_k^*) = (s, s^*)$. By the monotonicity of $S_{j(k)}$ for all $k \in \mathbb{N}$ we get for large $k \in \mathbb{N}$

$$\langle b_k^*, b_k\rangle = \langle b_k^* - s_k^*, b_k - s_k\rangle + \langle s_k^*, b_k - s_k\rangle + \langle b_k^*, s_k\rangle \geq \langle s_k^*, b_k - s_k\rangle + \langle b_k^*, s_k\rangle.$$

Since $(b, b^*) \in \bigcap_{k\in\mathbb{N}} \operatorname{cl}^{w\times w^*}\{(b_k, b_k^*), (b_{k+1}, b_{k+1}^*), \ldots\}$, there is an increasing function $l : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\lim_{k\to\infty} \langle s_{l(k)}^*, b_{l(k)} - s_{l(k)}\rangle = \langle s^*, b - s\rangle \text{ and } \lim_{k\to\infty} \langle b_{l(k)}^*, s_{l(k)}\rangle = \langle b^*, s\rangle.$$

It follows that $\limsup_{k\to\infty}\langle b_k^*, b_k\rangle \geq \langle s^*, b - s\rangle + \langle b^*, s\rangle$. Since the above inequality is valid for every $(s, s^*) \in S$ we obtain the inequality

$$\limsup_{k\to\infty} \langle b_k^*, b_k\rangle \geq F_S(b, b^*),$$

which by (13) furnishes the inequality in (14). $\square$

Remark 4.2. It is easy to observe that if S is maximal monotone, then (13) is satisfied for all $(v, v^*) \in E \times E^*$. Indeed, first note that either we have $(v, v^*) \in S$ and $0 = \inf_{(s,s^*)\in S}\langle v^* - s^*, v - s\rangle$, or $(v, v^*) \notin S$ and $0 > \inf_{(s,s^*)\in S}\langle v^* - s^*, v - s\rangle$. If $F_S(v, v^*) < \langle v^*, v\rangle$ for some $(v, v^*) \in E \times E^*$, then

$$0 < \inf_{(s,s^*)\in S} \langle v^* - s^*, v - s\rangle, \tag{15}$$

which is impossible, whenever S is maximally monotone.

Also it is easy to verify that each one of the equalities either the equality

$$E = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\{s \in E \mid (\{s\} \times \mathbb{B}_{E^*}[0,n]) \cap S \neq \emptyset\}$$

or the other one $E^* = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\{s^* \in E^* \mid (\mathbb{B}_E[0,n] \times \{s^*\}) \cap S \neq \emptyset\}$ yield the inequality in (13) for all $(v, v^*) \in E \times E^*$, since the inequality in (15) does not hold either in the case $E = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\{s \in E \mid (\{s\} \times \mathbb{B}_{E^*}[0,n]) \cap S \neq \emptyset\}$ or in the case $E^* = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\{s^* \in E^* \mid (\mathbb{B}_E[0,n] \times \{s^*\}) \cap S \neq \emptyset\}$. □

In most (reasonable) infinite dimensional cases the product topology on $E \times E^*$ of the strong topology on E and the weak* topology (for short, w^*-topology) on E^* is not metrizable. However, in view of (3) and Theorem 2.2 the metrizability appears to be a desired property when one has to deal with convergences of sets in such spaces. It is of interest that the separability of the strong topology on E allows us to introduce on $E \times E^*$ a topology which makes the space separable metric (Hausdorff) in such a way that this new topology is somehow consistent with the product topology on $E \times E^*$ of the strong topology on E and the weak* topology on E^*. This is clarified in the example below.

Example 4.3. Let $(E, \|\cdot\|)$ be a separable Banach space and $\{x_i \mid i \in \mathbb{N}\}$ be a dense subset of the unit sphere in E, that is

$$\mathbb{S}_E[0,1] = \{x \in E \mid \|x\| = 1\} = \mathrm{cl}\{x_1, x_2, x_3, \ldots\}.$$

For all $x^* \in E^*$ and $y^* \in E^*$ put
$$\rho^*(x^*, y^*) := \sum_{i\in\mathbb{N}} \frac{|\langle x^* - y^*, x_i\rangle|}{2^i}. \tag{16}$$

It is known and readily seen that ρ^* is a distance on E^* and the ρ^*-topology (associated to ρ^*) is included in the w^*-topology. For each $n \in \mathbb{N}$, since $\mathbb{B}_{E^*}[0,n]$ is w^*-compact, the topologies induced on $\mathbb{B}_{E^*}[0,n]$ by the w^*-topology and the metric ρ^* coincide. Then, $\mathbb{B}_{E^*}[0,n]$ endowed with the metric induced by ρ^* is a compact metric space, so it is separable. It ensues that (E^*, ρ^*) is a separable metric space. The space $E \times E^*$ is also a separable metric space for the distance $d \times \rho^*$, where $d(x, y) := \|x - y\|$ and

$$(d \times \rho^*)\big((x, x^*), (y, y^*)\big) := \left[(d(x,y))^2 + (\rho^*(x^*, y^*))^2\right]^{1/2}.$$

In a finite dimensional Banach space the distance defined in (16) is equivalent to d^*, thus in such a space we have the following equalities

$$\mathrm{Ls}_{d\times\rho^*} S_k = \mathrm{Ls}_{d\times d^*} S_k \quad \text{and} \quad \mathrm{Li}_{d\times\rho^*} S_k = \mathrm{Li}_{d\times d^*} S_k.$$ □

We proceed now to the introduction of the new convergences that will be required for subsets of $E \times E^*$. Given two Hausdorff topological spaces (X, τ_X) and (Y, τ_Y) (with $\tau_X \otimes \tau_Y$ the product topology on $X \times Y$) and given any Hausdorff topologies θ_X and θ_Y coarser than τ_X and τ_Y (i.e., $\theta_Y \subset \tau_Y$), we will say that a sequence $\{S_k\}_{k\in\mathbb{N}}$ in $X \times Y$ *sequentially* $G(\tau_X \otimes \tau_Y, \tau_X \otimes \theta_Y)$*-converges* to a set S in $X \times Y$, and we will write $S_k \xrightarrow{G(\tau_X\otimes\tau_Y, \tau_X\otimes\theta_Y)} S$ provided, for the sequential semilimits, if one has the equalities

$$\mathrm{Li}_{\tau_X\otimes\tau_Y} S_k = S = \mathrm{Ls}_{\tau_X\otimes\theta_Y} S_k, \text{ equivalently } \mathrm{Ls}_{\tau_X\otimes\theta_Y} S_k \subset S \subset \mathrm{Li}_{\tau_X\otimes\tau_Y}. \tag{17}$$

Similar other convergences can also be defined. Under (17), it is clear that the inclusions below

$$\mathrm{Ls}_{\tau_X\otimes\tau_Y} S_k \subset \mathrm{Ls}_{\tau_X\otimes\theta_Y} S_k \subset S \subset \mathrm{Li}_{\tau_X\otimes\tau_Y} \subset \mathrm{Li}_{\tau_X\otimes\theta_Y},$$

yield the following convergences

$$S_k \xrightarrow{G(\tau_X\otimes\tau_Y,\tau_X\otimes\tau_Y)} S \text{ and } S_k \xrightarrow{G(\tau_X\otimes\theta_Y,\tau_X\otimes\theta_Y)} S. \tag{18}$$

When X is a Banach space $(E,\|\cdot\|)$ and Y is its topological dual$(E^*,\|\cdot\|_*)$, denoting as usual w^* the weak* topology on E^*, we will write $S_k \xrightarrow{G(s,w^*)} S$ to mean that $\{S_k\}_{k\in\mathbb{N}}$ sequentially $G(\tau_E\otimes\tau_{E^*},\tau_E\otimes w^*)$ converges to S, where τ_E and τ_{E^*} denote here the strong topologies of E and E^* associated with the norms $\|\cdot\|$ and $\|\cdot\|_*$ respectively. Then (18) gives

$$S_k \xrightarrow{G(s,w^*)} S \implies S_k \xrightarrow{P.K.} S. \tag{19}$$

When $S_k \subset E\times E^*$ for a separable Banach space E, we say that the sequence $\{S_k\}_k$ converges to a subset S of $E\times E^*$ in a *mixed way* and write $S_k \xrightarrow{\mu} S$ provided that $S_k \xrightarrow{G(d\times\rho^*,d\times\rho^*)}$ converges to S, otherwise stated

$$\mathrm{Ls}_{d\times\rho^*} S_k = S = \mathrm{Li}_{d\times\rho^*} S_k, \tag{20}$$

where ρ^* is the distance on E^* defined in (16) above. Keeping E as a separable Banach space, the convergence defined in (20) obviously does not depend on the choice of the sequence $\{x_i\}_{i\in\mathbb{N}}$, which defines the distance ρ^*, see (16), whenever all the sets $\{x^*\in E^* \mid (E\times\{x^*\})\cap S_k \neq \emptyset\}$ are included in a same ball of E^*. This is a direct consequence of the coincidence of ρ^*-topology (associated to ρ^*) and the w^*-topology on any w^*-compact subset of E^*. This is stated explicitly in the following form.

Remark 4.4. If $S_k \subset E\times E^*$ for a separable Banach space E and if all the sets $\{x^*\in E^* \mid (E\times\{x^*\})\cap S_k\neq\emptyset\}$ are included in a same ball of E^*, then one has the equivalence

$$S_k \xrightarrow{\mu} S \iff S_k \xrightarrow{G(d\otimes w^*,d\otimes w^*)} S,$$

where (by abuse of notation) $d\otimes w^*$ stands for the product topology of the strong topology on E and the w^*-topology on E^*. □

In general cases, the convergence defined in (20) obviously depends on the choice of the sequence $\{x_i\}_{i\in\mathbb{N}}$, which defines the distance ρ^*, see (16). Below we propose a definition of a convergence referring to the "mixed way" convergence, which does not have this disadvantage. Let us first observe the following lemma.

Lemma 4.5. *Let $(E,\|\cdot\|)$ be a Banach space, let $\{a_k^*\}_{k\in\mathbb{N}}$ be a sequence in its dual space E^* and let $a^*\in E^*$ be such that for every increasing function $j:\mathbb{N}\to\mathbb{N}$*

$$a^* \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}^{w^*}\{a^*_{j(k)}, a^*_{j(k+1)},\ldots\}.$$

Then the sequence $\{a_k^*\}_{k\in\mathbb{N}}$ w^*-converges to a^*.

Proof. Suppose that $\{a_k^*\}_{k\in\mathbb{N}}$ does not w^*-converge to a^*. There exist some real $\varepsilon > 0$, some $h \in E$ and some increasing function $j : \mathbb{N} \to \mathbb{N}$ such that

$$|\langle a_{j(k)}^* - a^*, h\rangle| \geq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{21}$$

By w^*-continuity of $\langle \cdot, h\rangle$ and by the assumption of the lemma

$$\langle a^*, h\rangle \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}\{\langle a_{j(k)}^*, h\rangle, \langle a_{j(k+1)}^*, h\rangle, \cdots\},$$

so $\langle a^*, h\rangle$ is the limit of a subsequence of $\{\langle a_{j(k)}^*, h\rangle\}_{k\in\mathbb{N}}$, contradicting (21). □

Repeating the above reasoning for the w-topology in E yields the following

Lemma 4.6. *Let $(E, \|\cdot\|)$ be a Banach space, let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence in E and let $a \in E$ be such that for every increasing function $j : \mathbb{N} \to \mathbb{N}$*

$$a \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}^w \{a_{j(k)}, a_{j(k+1)}, \ldots\}.$$

Then the sequence $\{a_k\}_{k\in\mathbb{N}}$ w-converges to a.

Taking Lemma 4.5 into account, we propose another definition of mixed convergence as follows. Given any Banach space $(E, \|\cdot\|)$ and $S_k \subset E \times E^*$, our proposal to adjust the above definition of convergence to more general setting is:

Definition 4.7. We say that the sequence $\{S_k\}_{k\in\mathbb{N}}$ converges to a subset S of $E\times E^*$ in a *bounded mixed way* and write $S_k \xrightarrow{b\mu} S$ provided that

$$\mathcal{A}_i(\{S_k\}_{k\in\mathbb{N}}) = S = \mathcal{A}_s(\{S_k\}_{k\in\mathbb{N}}), \tag{22}$$

where

$$\mathcal{A}_s(\{S_k\}_{k\in\mathbb{N}}) := \left\{ \begin{array}{l} (a, a^*) \in E \times E^* \mid \text{ there is an increasing function} \\ j : \mathbb{N} \to \mathbb{N} \text{ and a sequence } \{(a_k, a_k^*)\}_{k\in\mathbb{N}} \text{ in } E \times E^* \text{ such} \\ \text{that } \{a_k^*\}_{k\in\mathbb{N}} \text{ is bounded, } \lim\limits_{k\to\infty} a_k = a \text{ and } (a_k, a_k^*) \in S_{j(k)} \\ \text{for all } k \in \mathbb{N}, \text{ and } a^* \in \bigcap\limits_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{a_k^*, a_{k+1}^*, \ldots\} \end{array} \right\}, \tag{23}$$

$$\mathcal{A}_i(\{S_k\}_{k\in\mathbb{N}}) := \left\{ \begin{array}{l} (a, a^*) \in E \times E^* \mid \text{ there is a sequence } \{(a_k, a_k^*)\}_{k\in\mathbb{N}} \\ \text{such that } a_k \to a \text{ and } (a_k, a_k^*) \in S_k \text{ for large } k \in \mathbb{N} \\ \text{and } \{a^*\} = \bigcap\limits_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{a_{j(k)}^*, a_{j(k+1)}^*, \ldots\} \text{ for any} \\ \text{increasing function } j : \mathbb{N} \to \mathbb{N} \end{array} \right\}. \tag{24}$$

When $S_k \neq \emptyset$ for every $k \in \mathbb{N}$ it is clearly equivalent in the definition of $\mathcal{A}_i(\{S_k\}_{k\in\mathbb{N}})$ to require $(a_k, a_k^*) \in S_k$ "for all $k \in \mathbb{N}$" instead of "for large $k \in \mathbb{N}$".

Let us also point out that the sets defined in (23) and (24) are not depending on the choice of equivalent norm in E or in E^*. Thus the bounded mixed way limit does not depend on the choice of equivalent norms. It is also easy to note that in finite dimensions the P.K. convergence and the bounded mixed way convergence coincide.

Remark 4.8. When the Banach space E is separable and the sets

$$\{s^* \in E^* \mid E \times \{s^*\} \cap S_k \neq \emptyset\}$$

lie in a same ball of E^*, it is important to notice as a consequence of Lemma 4.5 that $\mathcal{A}_i(\{S_k\}_{k\in\mathbb{N}})$ and $\mathcal{A}_s(\{S_k\}_{k\in\mathbb{N}})$ coincide with the *sequential* lower limit $\mathrm{Li}_{d\otimes w^*} S_k$and upper limit $\mathrm{Ls}_{d\otimes w^*} S_k$, where as in Remark 4.4 $d\otimes w^*$ is the product topology of the norm topology on E and the w^*-topology on E^*. $\square$

A natural question is whether under separability we can ensure the bounded mixed way convergence of a subsequence of a family of sets. Based on Zarankiewicz's result, see Theorem 2.2, such a property of the set family is ensured in the following proposition.

Proposition 4.9. *Let $(E, \|\cdot\|)$ be a separable Banach space. Every sequence $\{S_k\}_{k\in\mathbb{N}}$ of subsets of $E \times E^*$ contains a subsequence which has a (possibly empty) bounded mixed way limit, that is, there is an increasing function $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ and a subset S of $E \times E^*$ such that*

$$\mathcal{A}_i(\{S_{\gamma(k)}\}_{k\in\mathbb{N}}) = S = \mathcal{A}_s(\{S_{\gamma(k)}\}_{k\in\mathbb{N}}). \tag{25}$$

Proof. We use the notation $f \circ g$ for the composition of functions. Therefore we have $f \circ g(x) = f(g(x))$. For all $k, m \in \mathbb{N}$ put $S_{k,m} := S_k \cap (E \times \mathbb{B}_{E^*}[0, m])$.

It follows from the Zarankiewicz Theorem that there is an increasing function $\gamma_1 : \mathbb{N} \longrightarrow \mathbb{N}$ and a set C_1 in $E\times E^*$ such that $S_{\gamma_1(k),1} \xrightarrow{\mu} C_1$. Again, there is an increasing function $\gamma_2 : \mathbb{N} \longrightarrow \mathbb{N}$ and a set C_2 in $E \times E^*$ such that $S_{\gamma_1\circ\gamma_2(k),2} \xrightarrow{\mu} C_2$. Inductively, we are able to choose a sequence of increasing functions $\{\gamma_m\}_{m\in\mathbb{N}}$ from $\mathbb{N}$ to $\mathbb{N}$ and a sequence of sets $\{C_m\}_{m\in\mathbb{N}}$ in $E \times E^*$ such that

$$S_{\gamma_1\circ\ldots\circ\gamma_m(k),m} \xrightarrow{\mu} C_m, \tag{26}$$

for all $m \in \mathbb{N}$. Let us define $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ by $\gamma(k) := \gamma_1 \circ \ldots \circ \gamma_k(k)$ for all $k \in \mathbb{N}$, and let us note that the function γ is easily seen to be increasing.

Let any $(c, c^*) \in \mathcal{A}_s(\{S_{\gamma(k)}\}_{k\in\mathbb{N}})$. It follows from the definition of $\mathcal{A}_s(\{S_{\gamma(k)}\}_{k\in\mathbb{N}})$ that $(c, c^*) \in \mathcal{A}_s(\{S_{\gamma(k)} \cap (E \times \mathbb{B}_{E^*}[0, m])\}_{k\in\mathbb{N}})$ for some $m \in \mathbb{N}$, so by Remark 4.8 we have $(c, c^*) \in \mathrm{Ls}_{d\otimes w^*} \big(S_{\gamma(k)} \cap (E \times \mathbb{B}_{E^*}[0, m])\big)$. Thus, there is some $m \in \mathbb{N}$, some increasing function $j : \mathbb{N} \to \mathbb{N}$ and some sequence $\{(c_k, c_k^*)\}_{k\in\mathbb{N}}$ such that $c_k \to c$, $c_k^* \xrightarrow{w^*} c^*$ with $(c_k, c_k^*) \in S_{\gamma\circ j(k)} \cap (E \times B_{E^*}[0, m])$ for all $k \in \mathbb{N}$. For any integer $k > m$ we observe that

$$S_{\gamma\circ j(k)} = S_{\gamma_1\circ\cdots\circ\gamma_m\circ\cdots\circ\gamma_{j(k)}(j(k))} = S_{\gamma_1\circ\cdots\circ\gamma_m\circ\nu_m(k)}, \tag{27}$$

where $\nu_m(k) := \gamma_{m+1} \circ \cdots \circ \gamma_{j(k)}(j(k))$ for all $k > m$. It is not difficult to see that the function ν_m is increasing on $\mathbb{N}\backslash\{1, \ldots, m\}$, hence $(c, c^*) \in Ls_{d\otimes w^*} S_{\gamma_1\circ\ldots\circ\gamma_m(k),m}$. Since $S_{\gamma_1\circ\ldots\circ\gamma_m(k),m} \xrightarrow{G(d\otimes w^*,d\otimes w^*)} C_m$ by (26), see Remark 4.4, it follows that $(c, c^*) \in C_m$. This yields $\mathcal{A}_s(\{S_{\gamma(k)}\}_{k\in\mathbb{N}}) \subset \bigcup_{m\in\mathbb{N}} C_m$.

On the other hand, we claim that $\bigcup_{m\in\mathbb{N}} C_m \subset \mathcal{A}_i(\{S_{\gamma(k)}\}_{k\in\mathbb{N}})$. Indeed, fix $m \in \mathbb{N}$. If $C_m = \emptyset$, then $C_m \subset \mathcal{A}_i(\{S_{\gamma(k)}\}_{k\in\mathbb{N}})$. Suppose $C_m \neq \emptyset$ and take any $(s, s^*) \in C_m$.

By (26) there is a sequence $\{(s_k, s_k^*)\}_{k\in\mathbb{N}}$ in $E \times E^*$ such that

$$\lim_{k\to\infty} \|s - s_k\| + \rho^*(s^*, s_k^*) = 0$$

and $(s_k, s_k^*) \in S_{\gamma_1\circ\ldots\circ\gamma_m(k),m}$ for large k, see (20) and Remark 4.4. With $k > m$ and the increasing function ℓ_m defined by $\ell_m(k) := \gamma_{m+1} \circ \cdots \circ \gamma_k(k)$, like in (27) we have $S_{\gamma(k)} = S_{\gamma_1\circ\cdots\circ\gamma_m(\ell_m(k))}$, so putting $u_k := s_{\ell_m(k)}$ and $u_k^* := s_{\ell_m(k)}^*$ we see that $(u_k, u_k^*) \in S_{\gamma(k)}$ for large $k > m$. Further, since the sequence $\{s_k^*\}_{k\in\mathbb{N}}$ is bounded (say, $\|s_k^*\|_* \leq m$ for large k), we infer that $\{s_k^*\}_{k\in\mathbb{N}}$ and also $\{u_k^*\}_{k>m}$ must w^*-converge to s^*. This implies that $(s, s^*) \in \mathcal{A}_i(\{S_{\gamma(k)}\}_{k\in\mathbb{N}})$. This establishes the claim. Putting all together furnishes that (25) holds true. $\square$

Given a (set-valued) operator $A : E \rightrightarrows E^*$, its (effective) domain $\operatorname{Dom} A$ and its graph $\operatorname{gph} A$ are recalled in (8). When there is no risk of ambiguity, it is convenient to identify the graph $\operatorname{gph} A$ with A itself; so for a sequence of operators $\{A_k\}_{k\in\mathbb{N}}$ with $A_k : E \rightrightarrows E^*$, we are led to write $\mathcal{A}_s(\{A_k\}_{k\in\mathbb{N}})$ and $\mathcal{A}_i(\{A_k\}_{k\in\mathbb{N}})$ instead of $\mathcal{A}_s(\{\operatorname{gph} A_k\}_{k\in\mathbb{N}})$ and $\mathcal{A}_i(\operatorname{gph}\{A_k\}_{k\in\mathbb{N}})$ respectively. Similarly, by convenience we will say sometimes that $\{A_k\}_{k\in\mathbb{N}}$ $b\mu$-converges (resp. *P.K.* converges) to A instead of $\{\operatorname{gph} A_k\}_{k\in\mathbb{N}}$ $b\mu$-converges (resp. *P.K.* converges) to $\operatorname{gph} A$, and we will write $A_k \xrightarrow{b\mu} A$ (resp. $A_k \xrightarrow{P.K.} A$) instead of stating the graphs $\operatorname{gph} A_k$ and $\operatorname{gph} A$.

Similarly, given a sequence of convex functions $\{f_k\}_{k\in\mathbb{N}}$ from E into $\mathbb{R} \cup \{+\infty\}$, for their subdifferential operators $\partial f_k : E \rightrightarrows E^*$ it will be natural by our above convention to write $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}})$ in place of $\mathcal{A}_s(\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}})$ and also $\mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$ in place of $\mathcal{A}_i(\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}})$.

The analysis of the above convergences for subdifferentials will utilize the next two propositions. The first states a direct consequence of [26, Theorem 10.3] (or Theorem 29.2 in the 2nd edition [27]) in regards of the maximal monotonicity of subdifferentials of convex functions.

Proposition 4.10. *Let $(E, \|\cdot\|)$ be a reflexive Banach space and assume that $f : E \to \mathbb{R} \cup \{+\infty\}$ is an lsc proper convex function. Then for every $(z, z^*) \in E \times E^*$ there is $(b, b^*) \in \operatorname{gph}(\partial f)$ such that*

$$\|z^* - b^*\|_*^2 + \|z - b\|^2 + 2\langle z^* - b^*, z - b\rangle = 0.$$

The second proposition is related to nonreflexive Banach spaces. In such spaces an evaluation of the type in Proposition 4.10 has been also presented in [33, Corollary 3.4]. It is stated explicitly below, as a direct consequence of Property 2.3 and Corollary 3.4 from [33], see also [6, Proposition 9.7.2].

Proposition 4.11. *Let $(E, \|\cdot\|)$ be a Banach space and let $f : E \to \mathbb{R} \cup \{+\infty\}$ an lsc proper convex function. Then for every $(z, z^*) \in E \times E^*$ and every $\epsilon > 0$ there is $(b, b^*) \in \operatorname{gph}(\partial f)$ such that*

$$\|z^* - b^*\|_*^2 + \|z - b\|^2 + 2\langle z^* - b^*, z - b\rangle \leq \epsilon. \tag{28}$$

Moreover, there are $\epsilon_0 > 0$ and $R > 0$ such that if $\epsilon \in]0, \epsilon_0[$ and $(s_\epsilon, s_\epsilon^*) \in \operatorname{gph} \partial f$ fulfill the inequality in (28) then $\|s_\epsilon^*\|_* + \|s_\epsilon\| \leq R$.

Condition (28) can be used to distinguish some sequences $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$ with the property that its $w \times w^*$ convergence implies the strong convergence of the sequence $\{b_k\}_{k\in\mathbb{N}}$, whenever the norm has the Kadec-Klee property, see e.g. [2, R property (3.52)] or [13, Exercise 8.45], and also the strong convergence of the sequence $\{b_k^*\}_{k\in\mathbb{N}}$, whenever the norm has the w^*-Kadec-Klee property, see e.g. [13, Exercise 8.45]. Let us also recall that from a renorming theorem of S. Trojanski, see e.g. [12, Corollary 3, p. 167], any reflexive Banach space can be renormed so as to have the Kadec-Klee property for both primal and dual norms. It is also well known that both norm and dual norm of any Hilbert space possess the Kadec-Klee property.

Lemma 4.12. *Let $(E, \|\cdot\|)$ be a Banach space and $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc convex functions from E into $\mathbb{R}\cup\{+\infty\}$. Let $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$ be sequences w-converging to b in E and w^*-converging to some b^* in E^*, respectively, and such that $(b_k, b_k^*) \in \operatorname{gph}\partial f_k$ for all $k\in\mathbb{N}$ and the inequality*

$$\liminf_{k\to\infty}\langle b_k^*, b_k\rangle \geq \langle b^*, b\rangle \tag{29}$$

holds true. Let $\{\epsilon_k\}_{k\in N}$ be a sequence of positive reals converging to 0 and let $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ be a sequence converging strongly to (x, x^*), i.e. in the topology given by the distance $d\times d^*$ on $E\times E^*$ defined in (12), and such that the following inequalities

$$\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + 2\langle x_k^* - b_k^*, x_k - b_k\rangle \leq \epsilon_k$$

hold true for all $k\in\mathbb{N}$. Then the equalities

$$\lim_{k\to\infty}\|x_k - b_k\| = \|x - b\| \text{ and } \lim_{k\to\infty}\|x_k^* - b_k^*\|_* = \|x^* - b^*\|_*,$$

hold true along with $\|x - b\| = \|x^* - b^*\|_*$. Moreover, if the norm $\|\cdot\|$ (resp. the dual norm $\|\cdot\|_*$) has the Kadec-Klee property (resp. the w^*-Kadec-Klee property), then $\lim_{k\to\infty}\|b_k - b\| = 0$ (resp. $\lim_{k\to\infty}\|b_k^* - b^*\|_* = 0$).

Proof. Let $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$ be sequences w-converging to b in E and w^*-converging to some b^* in E^*, respectively, with $(b_k, b_k^*) \in \operatorname{gph}\partial f_k$ for all $k\in\mathbb{N}$ and such that the inequality in (29) is valid. Additionally, let $\{\epsilon_k\}_{k\in N}$ be a sequence of positive reals converging to 0 and let $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ be a sequence converging strongly to (x, x^*) (i.e. in the topology given by the distance defined in (12)) and such that the inequality assumption is satisfied, so for each $k\in\mathbb{N}$

$$0 \leq (\|x_k^* - b_k^*\|_* - \|x_k - b_k\|)^2 \leq \|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + 2\langle x_k^* - b_k^*, x_k - b_k\rangle \leq \epsilon_k,$$

hence, by the boundedness of $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$, we have

$$\lim_{k\to\infty}(\|x_k^* - b_k^*\|_* - \|x_k - b_k\|) = 0 \text{ and } \lim_{k\to\infty}(\|x_k^* - b_k^*\|_*^2 - \|x_k - b_k\|^2) = 0, \tag{30}$$

and
$$\lim_{k\to\infty}(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + 2\langle x_k^* - b_k^*, x_k - b_k\rangle) = 0. \tag{31}$$

We also have

$$\begin{aligned} 0 &= \lim_{k\to\infty}\left(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + 2\langle x_k^* - b_k^*, x_k - b_k\rangle\right) \\ &\geq \limsup_{k\to\infty}\left(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2\right) + 2\liminf_{k\to\infty}\langle x_k^* - b_k^*, x_k - b_k\rangle \end{aligned}$$

$$\geq \|x^* - b^*\|_*^2 + \|x - b\|^2 + 2\liminf_{k\to\infty}\langle x_k^* - b_k^*, x_k - b_k\rangle$$
$$\geq \|x^* - b^*\|_*^2 + \|x - b\|^2 + 2\langle x^* - b^*, x - b\rangle \geq (\|x^* - b^*\|_* - \|x - b\|)^2 \geq 0,$$

where (29) is applied in the last line. The three latter inequalities entail

$$\liminf_{k\to\infty}\langle x_k^* - b_k^*, x_k - b_k\rangle = \langle x^* - b^*, x - b\rangle = -\|x^* - b^*\|_* \cdot \|x - b\|$$

and
$$\|x^* - b^*\|_* = \|x - b\|.$$

Taking the latter equalities, (30) and the equality

$$0 = \limsup_{k\to\infty}\left(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2\right) + 2\liminf_{k\to\infty}\langle x_k^* - b_k^*, x_k - b_k\rangle$$

into account, we can write

$$\begin{aligned} 2\limsup_{k\to\infty}\|x_k^* - b_k^*\|_*^2 &= \limsup_{k\to\infty}\left(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + \|x_k^* - b_k^*\|_*^2 - \|x_k - b_k\|^2\right) \\ &= \limsup_{k\to\infty}(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2) \\ &= -2\liminf_{k\to\infty}\langle x_k^* - b_k^*, x_k - b_k\rangle \\ &= 2\|x^* - b^*\|_* \cdot \|x - b\| = 2\|x^* - b^*\|_*^2. \end{aligned}$$

Thus $\limsup_{k\to\infty}\|x_k^* - b_k^*\|_* \leq \|x^* - b^*\|_*$, hence by w^*-lsc of $\|\cdot\|_*$

$$\|x^* - b^*\|_* \geq \limsup_{k\to\infty}\|x_k^* - b_k^*\|_* \geq \liminf_{k\to\infty}\|x_k^* - b_k^*\|_* \geq \|x^* - b^*\|_*,$$

so $\lim_{k\to\infty}\|x_k^* - b_k^*\|_* = \|x^* - b^*\|_*$. Putting together this, the left equality in (30) and the equality $\|x^* - b^*\|_* = \|x - b\|$, we conclude $\lim_{k\to\infty}\|x_k - b_k\| = \|x - b\|$. Let us observe that the sequence $\{x_k - b_k\}_{k\in\mathbb{N}}$ is w-converging to $x - b$ and

$$\lim_{k\to\infty}\|x_k - b_k\| = \|x - b\|.$$

Thus if the norm has the Kadec-Klee property, then

$$\lim_{k\to\infty}\|b_k - b\| = \lim_{k\to\infty}\|(x_k - b_k) - (x - b)\| = 0,$$

since $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ is a sequence converging to (x, x^*). Similarly we get the strong convergence of the sequence $\{b_k^*\}_{k\in\mathbb{N}}$ to b, whenever the norm of dual space has the w^*-Kadec-Klee property. □

The next lemma furnishes certain sufficient conditions for (29) to be valid.

Lemma 4.13. *Let $(E, \|\cdot\|)$ be a Banach space and $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc convex functions from E into $\mathbb{R}\cup\{+\infty\}$. Let $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$ be sequences w-converging to b in E and w^*-converging to some b^* in E^*, respectively, and such that $(b_k, b_k^*) \in \operatorname{gph}\partial f_k$ for all $k\in\mathbb{N}$. Assume that at least one of the following conditions is satisfied:*

(a) *for some real $M > 0$*

$$b \in \operatorname{cl}\left(\operatorname{Dom}\operatorname{Li}_{d\times d^*}(\operatorname{gph}\partial f_k \cap (E \times \mathbb{B}_{E^*}[0, M]))\right); \tag{32}$$

(b) *for some real $M > 0$*

$$b^* \in \operatorname{cl}\big(\operatorname{Rge} \operatorname{Li}_{d\times d^*} (\operatorname{gph}\partial f_k \cap (\mathbb{B}_E[0, M] \times E^*))\big); \tag{33}$$

(c) *for all $(v, v^*) \in E \times E^*$ the inequality $F_S(v, v^*) \geq \langle v^*, v\rangle$ holds true, where $S := \operatorname{Li}_{d\times d^*} \operatorname{gph}\partial f_k$, see* (13) *and the comments following Lemma* 4.1.

Then the following inequality holds true

$$\liminf_{k\to\infty} \langle b_k^*, b_k\rangle \geq \langle b^*, b\rangle. \tag{34}$$

Proof. Let $b_k \xrightarrow{w} b$ in E and $b_k^* \xrightarrow{w^*} b^*$ in E^*, with $(b_k, b_k^*) \in \operatorname{gph}\partial f_k$ for all $k \in \mathbb{N}$ and such that $\lim_{k\to\infty}\langle b_k^*, b_k\rangle$ exists.

(a) Fix any $\epsilon > 0$. The inclusion in (32) ensures the existence of $c \in \mathbb{B}_E[0, \epsilon]$ and $a^* \in \mathbb{B}_{E^*}[0, M]$ such that $(b + c, a^*) \in \operatorname{Li}_{d\times d^*} (\operatorname{gph}\partial f_k \cap (E \times \mathbb{B}_{E^*}[0, M]))$. By the definition of $\operatorname{Li}_{d\times d^*} (\operatorname{gph}\partial f_k \cap (E \times \mathbb{B}_{E^*}[0, M]))$ there is a sequence $\{(a_k, a_k^*)\}_{k\in\mathbb{N}}$ with $(a_k, a_k^*) \in \operatorname{gph}\partial f_k$ for large $k \in \mathbb{N}$ andsuch that $\lim_{k\to\infty} \|a_k - (b + c)\| = 0$ and $\lim_{k\to\infty} \|a_k^* - a^*\|_* = 0$. The monotonicity of subdifferentials ensures that for large $k \in \mathbb{N}$

$$\begin{aligned}
&\langle b_k^* - b^*, b_k - b - c\rangle = \langle b_k^* - a_k^*, b_k - b - c\rangle + \langle a_k^* - b^*, b_k - b - c\rangle \\
&= \langle b_k^* - a_k^*, b_k - a_k\rangle + \langle b_k^* - a_k^*, a_k - b - c\rangle + \langle a_k^* - b^*, b_k - b - c\rangle \\
&\geq \langle b_k^* - a_k^*, a_k - b - c\rangle + \langle a_k^* - b^*, b_k - b - c\rangle
\end{aligned}$$

and passing to the limit inferior gives

$$\liminf_{k\to\infty} \langle b_k^* - b^*, b_k - b - c\rangle \geq \langle a^* - b^*, -c\rangle \geq -(M + \|b^*\|_*)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we deduce $\liminf_{k\to\infty}\langle b_k^* - b^*, b_k - b\rangle \geq 0$, which by the w-convergence of $\{b_k\}_{k\in\mathbb{N}}$ and w^*-convergence of $\{b_k^*\}_{k\in\mathbb{N}}$ yields the inequality in (34).

(b) Fix any $\epsilon > 0$. The inclusion in (33) ensures the existence of $c^* \in \mathbb{B}_{E^*}[0, \epsilon]$ and $a \in \mathbb{B}_E[0, M]$ such that $(a, b^* + c^*) \in \operatorname{Li}_{d\times d^*} (\operatorname{gph}\partial f_k \cap (\mathbb{B}_E[0, M] \times E^*))$. Then, there is a sequence $\{(a_k, a_k^*)\}_{k\in\mathbb{N}}$ with $(a_k, a_k^*) \in \operatorname{gph}\partial f_k$ for large $k \in \mathbb{N}$ and such that $\lim_{k\to\infty} \|a_k - a\| = 0$ and $\lim_{k\to\infty} \|a_k^* - (b^* + c^*)\|_* = 0$. Therefore, for large $k \in \mathbb{N}$

$$\begin{aligned}
&\langle b_k^* - b^* - c^*, b_k - b\rangle = \langle b_k^* - b^* - c^*, b_k - a_k\rangle + \langle b_k^* - b^* - c^*, a_k - b\rangle \\
&= \langle b_k^* - a_k^*, b_k - a_k\rangle + \langle a_k^* - b^* - c^*, b_k - a_k\rangle + \langle b_k^* - b^* - c^*, a_k - b\rangle \\
&\geq \langle a_k^* - b^* - c^*, b_k - a_k\rangle + \langle b_k^* - b^* - c^*, a_k - b\rangle.
\end{aligned}$$

Analogously to the previous case, we get that (34) holds.

(c) Let $\gamma : \mathbb{N} \to \mathbb{N}$ be increasing with $\liminf_{k\to\infty}\langle b_k^*, b_k\rangle = \lim_{k\to\infty}\langle b_{\gamma(k)}^*, b_{\gamma(k)}\rangle$. Putting $S' := \operatorname{Li}_{d\times d^*} \operatorname{gph}\partial f_{\gamma(k)}$ we have $S \subset S'$, hence $F_{S'} \geq F_S$ according to the definition of F_S. This and the assumption in (c) give $F_{S'}(v, v^*) \geq \langle v^*, v\rangle$ for all $(v, v^*) \in E \times E^*$. Therefore, the inequality $\lim_{k\to\infty}\langle b_{\gamma(k)}^*, b_{\gamma(k)}\rangle \geq \langle b^*, b\rangle$ is a consequence of Lemma 4.1, thus the inequality in (34) is justified. □

The next lemma is related to operators A with $\operatorname{gph} A \subset \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$.

Lemma 4.14. *Let $(E, \|\cdot\|)$ be a Banach space, $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc convex functions from E into $\mathbb{R} \cup \{+\infty\}$, and $A : E \rightrightarrows E^*$ be an operator such that*

$$\operatorname{gph} A \subset \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}). \tag{35}$$

Then the operator A is cyclically monotone.

Proof. Let $(a_i, a_i^*) \in \operatorname{gph} A$ where $i \in \{0, \dots n\}$, $n \in \mathbb{N} \cup \{0\}$. Due to relation (35) and Lemma 4.5, there are sequences $\{a_{i,k}\}_{k\in\mathbb{N}}$ and $\{a_{i,k}^*\}_{k\in\mathbb{N}}$, $i = 0, 1, \cdots, n$, with $a_{i,k} \to a_i$ and $a_{i,k}^* \xrightarrow{w^*} a_i^*$ and such that $(a_{i,k}, a_{i,k}^*) \in \operatorname{gph} \partial f_k$ for large $k \in \mathbb{N}$. Note that for every $i = 0, 1, \cdots, n$ with $a_{n+1,k} = a_{0,k}$ and $a_{n+1} = a_0$ we get the equality $\langle a_i^*, a_{i+1} - a_i\rangle = \lim_{k\to\infty}\langle a_{i,k}^*, a_{i+1,k} - a_{i,k}\rangle$. By cyclic monotonicity of ∂f_k we have for large $k \in \mathbb{N}$ the inequality $0 \geq \langle a_{0,k}^*, a_{1,k} - a_{0,k)}\rangle + \ldots + \langle a_{n,k}^*, a_{n+1,k} - a_{n,k}\rangle$, hence making $k \to \infty$ gives $0 \geq \langle a_0^*, a_1 - a_0\rangle + \ldots + \langle a_n^*, a_{n+1} - a_n\rangle$, which yields the cyclic monotonicity of the operator A. □

Before passing to the next section, we need a result linking sequential limits with bounded mixed convergence for maximal monotone operators.

Lemma 4.15. *Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of monotone operators from a Banach space $(E, \|\cdot\|)$ into its topological dual E^* and let $A : E \rightrightarrows E^*$ be an operator such that* $\operatorname{gph} A \subset \operatorname{Li}_{d\times d^*} \operatorname{gph} A_k$. *If A is maximal monotone, then $\mathcal{A}_s(\{A_k\}_{k\in\mathbb{N}}) \subset \operatorname{gph} A$.*

Proof. Consider any point $(a, a^*) \in \mathcal{A}_s(\{A_k\}_{k\in\mathbb{N}})$. We are going to prove that

$$\langle x^* - a^*, x - a\rangle \geq 0, \text{ for all } (x, x^*) \in \operatorname{gph} A. \tag{36}$$

Indeed, fix $(x, x^*) \in \operatorname{gph} A$. Since $(a, a^*) \in \mathcal{A}_s(\{A_k\}_{k\in\mathbb{N}})$ we have that there exists an increasing function $j : \mathbb{N} \to \mathbb{N}$ and a bounded sequence $\{(a_k, a_k^*)\}_{k\in\mathbb{N}}$ in $E \times E^*$ such that $\lim_{k\to\infty} a_k = a$ and $(a_k, a_k^*) \in \operatorname{gph} A_{j(k)}$ for all $k \in \mathbb{N}$, and such that

$$a^* \in \bigcap_{k\in\mathbb{N}} \operatorname{cl}^{w^*} \{a_k^*, a_{k+1}^*, \ldots\}. \tag{37}$$

Now, for the point (x, x^*) there exists, by the inclusion $\operatorname{gph} A \subset \operatorname{Li}_{d\times d^*} \operatorname{gph} A_k$, a sequence $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ converging strongly to (x, x^*) with $(x_k, x_k^*) \in \operatorname{gph} A_k$ for large $k \in \mathbb{N}$. Putting $u_k := x_{j(k)}$ and $u_k^* := x_{j(k)}^*$ the monotonicity of $A_{j(k)}$ ensures that

$$\langle u_k^* - a_k^*, u_k - a_k\rangle \geq 0, \text{ for large } k \in \mathbb{N}. \tag{38}$$

On the other hand, according to (37) the real $\langle a^*, x - a\rangle$ is a cluster point in $\mathbb{R}$ of the sequence $\{\langle a_k^*, x - a\rangle\}_{k\in\mathbb{N}}$, so there is a subsequence $\{\langle a_{\nu(k)}^*, x - a\rangle\}_{k\in\mathbb{N}}$ tending to $\langle a^*, x - a\rangle$. Using this along with the boundedness of $\{a_{\nu(k)}^*\}_{k\in\mathbb{N}}$ and the strong convergence of $\{(a_{\nu(k)}, u_{\nu(k)}, u_{\nu(k)}^*)\}_{k\in\mathbb{N}}$ to (a, x, x^*), it results from (38) that

$$\langle x^* - a^*, x - a\rangle = \lim_{k\to\infty} \langle u_{\nu(k)}^* - a_{\nu(k)}^*, u_{\nu(k)} - a_{\nu(k)}\rangle \geq 0.$$

From the arbitrariness of (x, x^*) in $\operatorname{gph} A$ we deduce that (36) holds, and by the maximal monotonicity of A we obtain that $(a, a^*) \in \operatorname{gph} A$. This ends the proof. □

5. Lower and upper epi-limits of slopes

First, it is worth noticing that imposing restrictions on the slope of a function f, we are able in the following proposition to ensure that f is bounded below. The proposition will be obtained as a consequence of [30, Lemma 4.2].

Proposition 5.1. *Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be an lsc proper convex function. For any sequence $\{x_k\}_{k\in\mathbb{N}}$ in E one has the implication*

$$\Big(\lim_{k\to\infty} s_f(x_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty}(s_f(x_k) + s_f(x_{k+1}))\|x_k - x_{k+1}\|\Big)$$
$$\Longrightarrow \liminf_{k\to\infty} f(x_k) = \inf_E f > -\infty. \tag{39}$$

Moreover, if $\inf_E f > -\infty$, then there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that the sequence $\{s_f(x_k)\}_{k\in\mathbb{N}}$ is nonincreasing and

$$\lim_{k\to\infty} s_f(x_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty} s_f(x_k)\|x_k - x_{k+1}\|. \tag{40}$$

Proof. The implication

$$\Big(\lim_{k\to\infty} s_f(x_k) = 0 \text{ and}, +\infty > \sum_{k=1}^{\infty} s_f(x_{k+1})\|x_k - x_{k+1}\|\Big) \Longrightarrow \liminf_{k\to\infty} f(x_k) = \inf_E f$$

was obtained for every sequence $\{x_k\}_{k\in\mathbb{N}}$ in [30, Lemma 4.2], see also [30, Lemma 3.2]. Let us consider any sequence $\{x_k\}_{k\in\mathbb{N}}$ of E such that

$$+\infty > \sum_{k=1}^{\infty} s_f(x_k)\|x_k - x_{k+1}\|.$$

By definition of slope we have $f(x_{k+1}) + \sum_{i=1}^{k} s_f(x_i)\|x_i - x_{i+1}\| \geq f(x_1)$ for every $k \in \mathbb{N}$, which by the convergence of the series implies $\liminf_{k\to\infty} f(x_k) > -\infty$, thus the implication in (39) is translated.

Now consider the case when $\inf_E f > -\infty$. If $s_f(\bar{x}) = 0$ for some $\bar{x} \in E$, then it suffices to put $x_k := \bar{x}$ for all $k \in \mathbb{N}$. Suppose that $s_f(x) > 0$ for every $x \in E$. Fix any $x_1 \in E$ such that $f(x_1) < 1 + \inf_E f$ and $s_f(x_1) < +\infty$, see e.g. the Brøndsted-Rockafellar Theorem [21, Theorem 3.17]. By the Ekeland Variational Principle, we use the version presented in [28, Corollary 4.40], there is $x_2 \in E$ such that

(a_1) $f(x_2) + \frac{s_f(x_1)}{2}\|x_1 - x_2\| \leq f(x_1)$;

(b_1) the point x_2 is a strong minimizer over E of $f(\cdot) + \frac{s_f(x_1)}{2}\|\cdot - x_2\|$.

By induction we define the sequence $\{x_k\}_{k\in\mathbb{N}}$, satisfying

(a_k) $f(x_{k+1}) + \frac{s_f(x_k)}{2}\|x_k - x_{k+1}\| \leq f(x_k)$;

(b_k) the point x_{k+1} is a strong minimizer over E of $f(\cdot) + \frac{s_f(x_k)}{2}\|\cdot - x_{k+1}\|$.

For each $k \in \mathbb{N}$ we have by (b_k) that $0 \in \partial f(x_{k+1}) + \frac{s_f(x_k)}{2}\mathbb{B}_E[0, 1]$, so

$$s_f(x_{k+1}) \leq \frac{s_f(x_k)}{2} \quad \text{for every } k \geq 1.$$

By (a_k) and by the above inequality $f(x_1) < 1 + \inf_E f$ we also easily see that $2 \geq \sum_{k=1}^{\infty} s_f(x_k)\|x_k - x_{k+1}\|$. Therefore, the defined sequence $\{x_k\}_{k\in\mathbb{N}}$ satisfies requirements from (40) and it is clear that the sequence$\{s_f(x_k)\}_{k\in\mathbb{N}}$ is nonincreasing, which terminates the proof. $\square$

In the next lemma we point out inequality connections between epi-limits superior of slopes related to equivalent norms.

Lemma 5.2. *Let E be a Banach space with two equivalent norms $\|\cdot\|$ and $\|\!|\cdot|\!\|$, so there are reals $\eta_1 > 0$ and $\eta_2 > 0$ such that for all $x^* \in E^*$*

$$\|x\|_* \leq \eta_1 |\!\|x|\!\|_* \leq \eta_2 \|x\|_*,$$

where $\|\cdot\|_*$ and $|\!\|\cdot|\!\|_*$ stand for the canonical dual norms, respectively. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E with slopes denoted by

$$s_{f_k}(x) := \inf\{\|x^*\|_* \mid x^* \in \partial f_k(x)\} \quad \text{and} \quad t_{f_k}(x) := \inf\{|\!\|x^*|\!\|_* \mid x^* \in \partial f_k(x)\}.$$

Then for every $x \in E$ one has

$$\eta_2(\operatorname{ls} s_{f_k})(x) \geq \eta_1(\operatorname{ls} t_{f_k})(x) \geq (\operatorname{ls} s_{f_k})(x).$$

Moreover, for every sequence $\{c_k\}_{k\in\mathbb{N}}$ we have the following equivalence:

$$\lim_{k\to\infty} s_{f_k}(c_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty}(s_{f_k}(c_k) + s_{f_k}(c_{k+1}))\|c_k - c_{k+1}\|$$

if and only if $\lim_{k\to\infty} t_{f_k}(c_k) = 0$ and $+\infty > \sum_{k=1}^{\infty}(t_{f_k}(c_k) + t_{f_k}(c_{k+1}))\|c_k - c_{k+1}\|$.

Proof. Let us put $\psi(x) := \operatorname{ls} s_{f_k}(x)$. It is easy to notice that if $\psi(x) = +\infty$, then $\operatorname{ls} t_{f_k}(x) = +\infty$. If $+\infty > \psi(x) \geq 0$, then it is easily seen that

$$\frac{\eta_2}{\eta_1}\psi(x) \geq \min\{\limsup_{k\to\infty} t_{f_k}(x_k) \mid x_k \to x\} \geq \frac{1}{\eta_1}\psi(x),$$

which justifies the first statement. The second part of the statement (the equivalence) is also easy to observe. $\square$

The next lemma in infinite dimensions can be seen as a brother of Lemma 3.1.

Lemma 5.3. *Let $(E, \|\cdot\|)$ be a Banach space and $A : E \rightrightarrows E^*$ be an operator. Then $\operatorname{dom} s_A = \operatorname{Dom} A$, where s_A is defined as in (10).*

In addition, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E. The following hold:

(a) *If $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \operatorname{gph} A$, then for every $x \in E$ and every sequence $\{x_k\}_{k\in\mathbb{N}}$ in E converging to x the following inequality holds true*

$$\liminf_{k\to\infty} s_{f_k}(x_k) \geq s_A(x).$$

(b) *If $\operatorname{gph} A \subset \operatorname{Li}_{d\times d^*} \operatorname{gph} \partial f_k$, then for every $x \in E$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in E converging to x such that*

$$\limsup_{k\to\infty} s_{f_k}(x_k) \leq s_A(x).$$

Proof. The equality $\operatorname{dom} s_A = \operatorname{Dom} A$ is easy to observe. For (a) let us fix an arbitrary $x \in E$ and any sequence $\{x_k\}_{k\in\mathbb{N}}$ in E converging to x. We claim that $\liminf_{k\to\infty} s_{f_k}(x_k) \geq s_A(x)$. We may assume that $\liminf s_{f_k}(x_k) < +\infty$, otherwise the statement trivially holds. Then there exists an increasing function $j : \mathbb{N} \to \mathbb{N}$ such that $s_{f_{j(k)}}(x_{j(k)})$ is finite for every $k \in \mathbb{N}$ and $\liminf_{k\to\infty} s_{f_k}(x_k) = \lim_{k\to\infty} s_{f_{j(k)}}(x_{j(k)})$. For each $k \in \mathbb{N}$ by w^*-closedness of $\partial f_{j(k)}(x_{j(k)})$ and by (11) we can choose $x^*_{j(k)} \in \partial f_{j(k)}(x_{j(k)})$ such that $\|x^*_{j(k)}\|_* = s_{f_{j(k)}}(x_{j(k)})$. Since $\{x^*_{j(k)}\}_{k\in\mathbb{N}}$ is bounded, by the Banach-Alaoglu Theorem [25, Section 3.15] there is some $x^* \in E^*$ such that

$$\forall \epsilon > 0, x^* \in \bigcap_{k\in\mathbb{N}} \operatorname{cl}^{w^*} \{x^*_{j(k)}, x^*_{j(k+1)}, \ldots\} \subset \mathbb{B}_{E^*}[0, \epsilon + \liminf_{k\to\infty} s_{f_k}(x_k)].$$

It is easy to observe that this implies $\|x^*\|_* \leq \lim s_{f_{j(k)}}(x_{j(k)}) = \liminf_{k\to\infty} s_{f_k}(x_k)$ and $(x, x^*) \in \mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \operatorname{gph} A$, which terminates the proof of (a).

Regarding (b) let any $x \in E$. We may suppose $s_A(x) < +\infty$, so $x \in \operatorname{Dom} A$. Take any sequence $\{x^*_i\}_{i\in\mathbb{N}}$ in E^* with $\lim_{i\to\infty} \|x^*_i\|_* = s_A(x)$ and $(x, x^*_i) \in \operatorname{gph} A$ for all $i \in \mathbb{N}$. By the inclusion assumption for each $i \in \mathbb{N}$ there is a sequence $\{(b_{i,k}, b^*_{i,k})\}_{k\in\mathbb{N}}$ such that $(b_{i,k}, b^*_{i,k}) \in \operatorname{gph}(\partial f_k)$ for all $k \in \mathbb{N}$ and such that

$$\lim_{k\to\infty} \|b^*_{i,k} - x^*_i\|_* + \|b_{i,k} - x\| = 0.$$

Then $\limsup_{k\to\infty} s_{f_k}(b_{i,k}) \leq \|x^*_i\|_*$. As in the proof of Lemma 3.1 choose an increasing function $j : \mathbb{N} \longrightarrow \mathbb{N}$ such that for each $i \in \mathbb{N}$ and all $k \geq j(i)$

$$\|b^*_{i,k} - x^*_i\|_* + \|b_{i,k} - x\| \leq \frac{1}{i} \quad \text{and} \quad s_{f_k}(b_{i,k}) \leq \|x^*_i\|_* + \frac{1}{i}.$$

For each $k \in \mathbb{N}$ satisfying $k \geq j(1)$ denote $i_k \in \mathbb{N}$ the unique integer such that $j(i_k) \leq k < j(1 + i_k)$, and set $x_k := b_{i_k,k}$. Set also $x_k := b_{1,k}$ for every integer $k < j(1)$ (if any). We note that both $j(i_k) \to \infty$ and $i_k \to \infty$ as $k \to \infty$. The inequality $\|x_k - x\| \leq 1/i_k$ yields $\lim_{k\to\infty} x_k = x$. Since $(x_k, b^*_{i_k,k}) \in \operatorname{gph}(\partial f_k)$ for all $k \geq j(1)$ it follows that $\limsup_{k\to\infty} s_{f_k}(x_k) \leq \lim_{k\to\infty} \|x^*_{i_k}\|_* = s_A(x)$. This translates the desired assertion (b). $\square$

Let $(E, \|\cdot\|)$ be an Asplund space, see [21, Definition 1.22], that is, a Banach space for which every continuous convex function f defined on a nonempty open convex set $U \subset E$ is Fréchet-differentiable at each point of some dense G_δ subset of U. We use this fact to provide a sufficient condition for the inclusion in (32).

Lemma 5.4. *Let $(E, \|\cdot\|)$ be an Asplund space such that its canonical dual norm has the w^*-Kadec-Klee property, $U \subset E$ be a given nonempty open convex subset of E and $A : E \rightrightarrows E^*$ be an operator. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc convex functions from E into $\mathbb{R} \cup \{+\infty\}$ such that $\mathcal{A}_s(\{\partial f_{k|_U}\}_{k\in\mathbb{N}}) = \mathcal{A}_i(\{\partial f_{k|_U}\}_{k\in\mathbb{N}}) = \operatorname{gph} A_{|_U}$ and such that for some dense G_δ subset U' of U and all $x \in U'$*

$$(\operatorname{ls} s_{f_k})(x) \leq s_A(x) < +\infty.$$

Then for every $b \in U$ there is $M \geq 0$ such that

$$b \in \operatorname{cl}\left(\operatorname{Dom}\left(\operatorname{Li}_{d\times d^*}\left(\operatorname{gph}\partial f_k \cap (E \times \mathbb{B}_{E^*}[0, M])\right)\right)\right).$$

Proof. It follows from Lemma 4.14 that $\operatorname{gph} A_{|U}$ is cyclically monotone, so there is a maximal cyclically monotone subset B of $E \times E^*$ such that $\operatorname{gph} A_{|U} \subset B$. By Theorem 2.5 there is an lsc convex function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{gph} \partial f = B$. The inclusion $\operatorname{gph} A_{|U} \subset B$ yields the following inequality

$$s_f(x) \leq s_{A_{|U}}(x) \quad \text{for all } x \in U. \tag{41}$$

For every $k \in \mathbb{N}$ let us define the following closed set $E_k := \{x \in E \mid f(x) \leq k\}$. Since $A_{|U}(x) = A(x)$ and $s_A(x) < +\infty$ for all $x \in U'$, by (41) we have $U' \subset \bigcup_{k\in\mathbb{N}} E_k$. Suppose that $\operatorname{int} E_k = \emptyset$ for every $k \in \mathbb{N}$. By the Baire Category Theorem, see for example [15], it follows that the set $\bigcap_{k\in\mathbb{N}}(E \setminus E_k)$ is a dense G_δ subset of E, thus $U' \cap \bigcap_{k\in\mathbb{N}}(E \setminus E_k) \neq \emptyset$, which is impossible. Thus $\operatorname{int} E_k \neq \emptyset$ for some $k \in \mathbb{N}$, which implies $\operatorname{int} \operatorname{dom} f \neq \emptyset$. The inclusions $U' \subset \operatorname{dom} f$ and $U \subset \operatorname{cl} U' \subset \operatorname{cl} \operatorname{dom} f$ imply $U \subset \operatorname{int} \operatorname{dom} f$ (see, e.g. [13, Exercise 1.57]), so in particular the lsc convex function f is locally Lipschitz on U (see, e.g. [28, Theorem 3.19]). Since E is an Asplund space, there is a dense G_δ subset D' of U such that f is Fréchet-differentiable at each point of D'. Again, applying the inclusion $\operatorname{gph} A_{|U} \subset B$ we get the equality $s_f(d) = s_{A_{|U}}(d)$ at each element $d \in D' \cap U'$. Let us fix any $b \in U$. By the local Lipschitz property of f on U we easily derive (or see, e.g. [28, Proposition 3.1(b)]) that there are reals $\epsilon > 0$ and $M > 0$ such that $\mathbb{B}_E[b, \epsilon] \subset U$ and $\partial f(x) \subset \mathbb{B}(0, M)$, and as a consequence $s_f(x) < M$ for all $x \in \mathbb{B}_E[b, \epsilon]$. Hence, for all $d \in \mathbb{B}_E[b, \epsilon] \cap D' \cap U'$ we get $s_{A_{|U}}(d) < M$. Let us fix any $d \in \mathbb{B}_E[b, \epsilon] \cap D' \cap U'$. Using (11) and the fact that

$$(\operatorname{ls} s_{f_k})(d) := \min\{\limsup_{k\to\infty} s_{f_k}(x_k) \mid x_k \to d\} \leq s_{A_{|U}}(d) < M,$$

there are sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{x_k^*\}_{k\in\mathbb{N}}$ such that $\limsup_{k\to\infty} \|x_k^*\|_* \leq s_{A_{|U}}(d)$ with $\|x^*\|_* < M$ and $(x_k, x_k^*) \in \operatorname{gph} \partial f_k$ for large $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|x_k - d\| = 0$. Any w^*-cluster point of the sequence $\{x_k^*\}_{k\in\mathbb{N}}$ must be in $A_{|U}(d)$. Since $d \in D' \cap U'$, $\mathcal{A}_s(\{\partial f_{k_{|U}}\}_{k\in\mathbb{N}}) = \mathcal{A}_i(\{\partial f_{k_{|U}}\}_{k\in\mathbb{N}}) = \operatorname{gph} A_{|U}$ and $\operatorname{gph} A_{|U} \subset B$, the set $A_{|U}(d)$ must be a singleton, hence the sequence $\{x_k^*\}_{k\in\mathbb{N}}$ w^*-converges to the element $x^* \in E^*$ such that $\{x^*\} = A_{|U}(d)$. Taking into account the equality $\|x^*\|_* = s_{A_{|U}}(d)$ and the lower w^*-semicontinuity of the dual norm, the inequality $\limsup_{k\to\infty} \|x_k^*\|_* \leq s_{A_{|U}}(d)$ ensures the following equality $\lim_{k\to\infty} \|x_k^*\|_* = \|x^*\|_*$, thus by the w^*-Kadec-Klee property of the norm $\|\cdot\|_*$ we obtain the strong convergence of the sequence $\{x_k^*\}_{k\in\mathbb{N}}$, which implies that

$$d \in \Delta := \operatorname{Dom}\left(\operatorname{Li}_{d\times d^*}\left(\operatorname{gph} \partial f_k \cap (E \times \mathbb{B}_{E^*}[0, M])\right)\right).$$

Since $d \in \mathbb{B}_E[b, \epsilon] \cap D' \cap U'$ is arbitrary, the proof is finished with $\mathbb{B}_E[b, \epsilon] \subset \operatorname{cl} \Delta$. $\square$

It is known that any reflexive Banach space is Asplund, see for example [13, Corollary 11.10]. Thus we have the following corollary.

Corollary 5.5. *Let $(E, \|\cdot\|)$ be a reflexive Banach space such that both norms, the primal one and the dual canonical one, have the Kadec-Klee property and let $A : E \rightrightarrows E^*$ be an operator. Let $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ be an lsc convex function such that $U := \operatorname{int} \operatorname{dom} f \neq \emptyset$, and such that if $\{c_k\}_{k\in\mathbb{N}}$ is a sequence weakly convergent to some element of* $\operatorname{cl} \operatorname{dom} f \setminus \operatorname{int} \operatorname{dom} f$*, then it converges in the norm topology.*

Let $\{f_k\}_{k\in\mathbb{N}}$ *be a sequence of lsc convex functions from* E *into* $\mathbb{R}\cup\{+\infty\}$ *such that*

$$\bigcap_{i\in\mathbb{N}} \mathrm{cl}^w\Big(\bigcup_{k\geq i}\mathrm{dom}\, f_k\Big) \subset \mathrm{cl}\,\mathrm{dom}\, f,$$

$\mathcal{A}_s(\{\partial f_{k_{|U}}\}_{k\in\mathbb{N}}) = \mathcal{A}_i(\{\partial f_{k_{|U}}\}_{k\in\mathbb{N}}) = \mathrm{gph}\, A_{|_U}$ *and for some dense* G_δ *subset* U' *of* $U = \mathrm{int}\,\mathrm{dom}\, f$ *and all* $x\in U'$

$$(\mathrm{ls}\, s_{f_k})(x) \leq s_A(x) < +\infty.$$

Let $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$ *be a sequence with* $(b_k, b_k^*)\in \mathrm{gph}\,\partial f_k$ *for all* $k\in\mathbb{N}$ *and such that* $\{b_k\}_{k\in\mathbb{N}}$ *and* $\{b_k^*\}_{k\in\mathbb{N}}$ w*-converges to* $b\in E$ *and* w^**-converges to* $b^*\in E^*$*, respectively. Assume that there are a sequence of positive reals* $\{\epsilon_k\}_{k\in N}$ *converging to* 0 *and a sequence* $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$ *converging strongly to* (x, x^*) *such that*

$$\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + 2\langle x_k^* - b_k^*, x_k - b_k\rangle \leq \epsilon_k \quad \textit{for all } k\in\mathbb{N}. \tag{42}$$

Then one has: $\|b_k - b\| \to 0$ *and* $\|b_k^* - b^*\|_* \to 0.$

Proof. First, we have $b\in \mathrm{cl}\,\mathrm{dom}\, f$ according to the inclusion

$$\bigcap_{i\in\mathbb{N}} \mathrm{cl}^w\Big(\bigcup_{k\geq i}\mathrm{dom}\, f_k\Big) \subset \mathrm{cl}\,\mathrm{dom}\, f.$$

If $b\in \mathrm{int}\;\mathrm{dom}\, f$, then the strong convergence of both sequences $\{b_k\}_{k\in\mathbb{N}}$ and $\{b_k^*\}_{k\in\mathbb{N}}$ is a simple consequence of Lemma 5.4, Lemma 4.13(a) and Lemma 4.12. Consider the case $b\in \mathrm{cl}\,\mathrm{dom}\, f\setminus \mathrm{int}\;\mathrm{dom}\, f$. Since the sequence $\{b_k\}_{k\in\mathbb{N}}$ w-converges to $b\in E$, so by the assumption the sequence converges in the norm topology. Thus in both cases the sequence $\{b_k\}_{k\in\mathbb{N}}$ is strongly convergent to b. On the other hand, since the left-hand side of (42) is not less than $(\|x_k^* - b_k^*\|_* - \|x_k - b_k\|)^2 \geq 0$, the same inequality (42) and the lower w^*-semicontinuity of the dual norm furnish

$$\begin{aligned} 0 &= \lim_{k\to\infty}\big(\|x_k^* - b_k^*\|_*^2 + \|x_k - b_k\|^2 + 2\langle x_k^* - b_k^*, x_k - b_k\rangle\big) \\ &\geq \|x^* - b^*\|_*^2 + \|x - b\|^2 + 2\langle x^* - b^*, x - b\rangle \geq 0. \end{aligned}$$

This entails, like in the proof of Lemma 4.12, $\lim_{k\to\infty}\|x_k^* - b_k^*\|_* = \|x^* - b^*\|_*$, hence $\{b_k^*\}_{k\in\mathbb{N}}$ converges strongly to b^* by Kadec-Klee property of the dual norm. □

6. Graph convergence of subdifferentials in Banach spaces, counterexamples

The first example shows the convergence of slopes of convex differentiable functions on $\mathbb{R}$ does not entail the P.K. convergence of graphs of their derivatives.

Example 6.1. Take the convex differentiable functions f, f_k on $\mathbb{R}$, $k\in\mathbb{N}$ with

$$f_k(t) := (-1)^k t \text{ and } f(t) := t \quad \text{for all } t\in\mathbb{R}.$$

We have $s_f \equiv s_{f_k} \equiv 1$ for all $k\in\mathbb{N}$, so we have the epi-convergence of slopes. However, we observe that $Df(t) = 1$ and $Df_k(t) = (-1)^k$, so $(0,-1)\notin \mathrm{gph}\, Df$ while $(0,-1)\in \mathrm{Ls}\,\mathrm{gph}\, Df_k$ since $(0,-1)\in \mathrm{gph}\, Df_{2k+1}$ for all $k\in\mathbb{N}$.

Thus the convergence of graphs of derivatives does not hold. □

The function f above is not bounded below. The next example is related to bounded below functions. For classical spaces c_0, ℓ_p we use the usual notation. We thank D. Salas for a useful remark on a previous version of the example.

Example 6.2. We consider in $E = \ell_2$ the sets $C_k := \operatorname{co}\{0, e_1 + e_k\}$ and $C := \{0\}$, where $\{e_k\}_{k\in\mathbb{N}}$ is the canonical basis of E. Clearly, the sequence $\{C_k\}_{k\in\mathbb{N}}$ P.K. converges to C, so for $f_k := \psi_{C_k}$ and $f := \psi_C$ the sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ epi-converges to f, where ψ_C is the indicator function of C, i.e. $\psi_C(u) = 0$ if $u \in C$ and $\psi_C(u) = +\infty$ if $u \notin C$. On the other hand, $s_{f_k} = \psi_{C_k}$ and $s_f = \psi_C$, hence the sequence of slopes $\{s_{f_k}\}_{k\in\mathbb{N}}$ epi-converges to the slope s_f. Moreover, the sequence $\{f_k\}_k$ does not converge in the Mosco sense. Indeed, consider $x_k := e_1 + e_k$, which converges weakly to $x = e_1$. However, we have

$$\lim_{k\to\infty} f_k(x_k) = 0 < f(x) = +\infty.$$

In addition, the normalization condition (namely, the existence of $(a_k, a_k^*) \in \operatorname{gph}\partial f_k$ and $(a, a^*) \in \operatorname{gph}\partial f$ such that $(a_k, a_k^*, f_k(a_k)) \to (a, a^*, f(a))$) is trivially satisfied, since all the functions attain their global minimum at zero.

Therefore, by the classical Attouch's theorem, the subdifferentials ∂f_k do not converge to ∂f in the Painlevé-Kuratowski sense. □

Below we recall an example, see [32, Example 1.1], where f is the norm in ℓ_1 (hence bounded below) and the f_k are Lipschitz with constant 1. The example illustrates that, under those assumptions, the bounded mixed convergence of $\operatorname{gph}\partial f_k$ to $\operatorname{gph}\partial f$ and the epi-convergence of f_k to f (even the Mosco convergence) can hold while one has neither the epi-convergence of $\{s_{f_k}\}_k$ to s_f nor the P.K. convergence of $\{\operatorname{gph}\partial f_k\}_k$ to $\operatorname{gph}\partial f$.

Example 6.3. Let ℓ_1 be endowed with its usual norm $\|\cdot\|$ given by $\|x\| = \sum_{i=1}^{\infty} |x_i|$ for all $x \in \ell_1$. Let $f, f_k : \ell_1 \to \mathbb{R}$ with

$$f(x) := \sum_{i=1}^{\infty} |x_i| \;\text{ for }\; x \in \ell_1 \;\text{ and }\; f_k(x) := \sum_{i=1}^{k} |x_i| + \sum_{i=k+1}^{\infty} x_i \;\text{ for }\; x \in \ell_1.$$

It is obvious that $0 \in \partial f(0)$ and for every $(x, x^*) \in \operatorname{gph}\partial f_k$ we have $\|x^*\|_* = 1$. Further, $f_k(x) \le f_{k+1}(x)... \longrightarrow f(x)$ (hence $\{f_k\}_k$ epigraphically converges to f) but there is no sequence $\{(x_k, x_k^*)\}_{n\in\mathbb{N}}$ in $\ell_1 \times \ell_\infty$ such that $(x_k, x_k^*) \in \operatorname{gph}\partial f_k$ and $(x_k, x_k^*) \longrightarrow 0$, where the convergence is in the norm topology. Then, $\{\operatorname{gph}\partial f_k\}_k$ does not P.K. converge to $\operatorname{gph}\partial f$.

It is easy to notice that $s_f(0) = 0$, however, if $z_k \longrightarrow 0$ in ℓ_1 with $z_k \neq 0$, then we see that $s_{f_k}(z_k) \longrightarrow 1$, so it is impossible that $s_{f_k} \xrightarrow{e} s_f$.

On the other hand, it ensues from Proposition 4.9 that for every subsequence of $\{\operatorname{gph}\partial f_k\}_k$, say $\{\operatorname{gph}\partial f_{j(k)}\}_k$, there is an increasing function $\ell : \mathbb{N} \to \mathbb{N}$ such that for $\gamma := j \circ \ell$ one has $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{b\mu} \operatorname{gph} A$ for some operator $A : \ell_1 \rightrightarrows \ell_\infty$.

Let us put $y_k := (0, 0, ...)$ and $y_k^* = (0, \dots, 0, 1, 1, \dots)$ (where 0 is taken k-times). Then $(y_k, y_k^*) \in \operatorname{gph}\partial f_k$, $y_k \to 0$ and $\lim_{k\to\infty} \langle y_k^*, h\rangle = 0$ for every $h \in \ell_1$, so $\{y_k^*\}_{k\in\mathbb{N}}$ is weak* convergent to 0.

It is obvious that $(0,0) \in \operatorname{gph} A$. In fact, it follows from [32, Theorem 3.1] that $\partial f \subset \operatorname{gph} A$. Since A is cyclically monotone, see Lemma 4.14 above, and ∂f is maximal monotone, so $\partial f = A$. Thus every subsequence of $\{\operatorname{gph} \partial f_k\}_k$ has a subsequence converging in a bounded mixed way to $\operatorname{gph} \partial f$. Now it is not difficult to verify that $\operatorname{gph} \partial f_k \xrightarrow{b\mu} \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$, keep in mind that [32, Theorem 3.1] ensures the inclusion $\operatorname{gph} \partial f \subset \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$. □

We pass now to an example where f, f_k are (uniformly) bounded below, Lipschitz and $\mathcal{C}^1$ functions with $\partial f_k \xrightarrow{b\mu} \partial f$ while one has neither $s_{f_k} \xrightarrow{e} s_f$ nor $\partial f_k \xrightarrow{P.K.} \partial f$.

Example 6.4. Let $E = \ell_2$ (with canonical basis $\{e_k\}_k$) and denote $x_k := \langle x, e_k\rangle$ for any $x \in E$. Define $\mathcal{C}^1$ functions $f, f_k : \ell_2 \to \mathbb{R}$, $k \in \mathbb{N}$, by

$$f_k(x) := \sqrt{x_k^2 + 1} + x_k \quad \text{and} \quad f(x) := 0 \quad \text{for all } x \in E, k \in \mathbb{N}.$$

Then, f_k, f are bounded below ($f, f_k \geq 0$) and uniformly Lipschitz and

$$\nabla f_k(x) = \Big(\frac{x_k}{\sqrt{x_k^2+1}} + 1\Big) e_k \quad \text{and} \quad \nabla f(x) = 0.$$

In this case we have that $\operatorname{gph} \partial f_k \xrightarrow{b\mu} \operatorname{gph} \partial f$. Nevertheless, for any $z \in \ell_2$ and any sequence $z^k \to z$ in ℓ^2 (recall $\langle e_k, z^k\rangle \to \langle 0_{\ell_2}, z\rangle = 0$)

$$\lim_{k\to\infty} s_{f_k}(z^k) = \lim_{k\to\infty} \left| \frac{\langle e_k, z^k\rangle}{\sqrt{\langle e_k, z^k\rangle^2 + 1}} + 1 \right| = 1 \text{ and } s_f(z) = 0,$$

which shows that the sequence of slopes $\{s_{f_k}\}_{k\in\mathbb{N}}$ cannot epi-converge to s_f. Observe also that $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$ does not converge in the P.K sense.

7. Convergence of subdifferentials and slopes

In this section it is shown that under a compactness type assumption, see (CTA) below, the bounded mixed convergence of subdifferentials is equivalent to the P.K.-convergence. We formulate the condition (CTA) as follows.

Compactness Type Assumption. Let $(E, \|\cdot\|)$ be a Banach space and $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E.

(CTA): For any $a \in E$, any $a^* \in E^*$, any increasing function $j : \mathbb{N} \to \mathbb{N}$, and any bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$, such that $(b_k, b_k^*) \in \operatorname{gph} \partial f_{j(k)}$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \big(\|b_k^* - a^*\|_*^2 + \|b_k - a\|^2 + 2\langle b_k^* - a^*, b_k - a\rangle\big) = 0, \tag{43}$$

the sequence $\{b_k\}_{k\in\mathbb{N}}$ admits a subsequence converging strongly to some $b \in E$.

Below an equivalent form of (CTA) is pointed out.

Lemma 7.1. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on a Banach space $(E, \|\cdot\|)$. Then* (CTA) *holds if and only if* (CTA') *holds, where*

(CTA'): *For any $a^* \in E^*$, any convergent sequence $\{a_k\}_{k\in\mathbb{N}}$ in E, any increasing function $j : \mathbb{N} \to \mathbb{N}$, and any bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$, such that $(b_k, b_k^*) \in \operatorname{gph} \partial f_{j(k)}$ for all $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} \left(\|b_k^* - a^*\|_*^2 + \|b_k - a_k\|^2 + 2\langle b_k^* - a^*, b_k - a_k\rangle\right) = 0, \tag{44}$$

the sequence $\{b_k\}_{k\in\mathbb{N}}$ admits a subsequence converging strongly to some $b \in E$.

Proof. It is easy to see that (CTA') implies (CTA), just considering $a_k = a$. Now, let us assume that (CTA) holds. Let us consider $a^* \in E^*$, a convergent sequence $\{a_k\}_{k\in\mathbb{N}}$ in E, an increasing function $j : \mathbb{N} \to \mathbb{N}$, and a bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$, such that $(b_k, b_k^*) \in \operatorname{gph} \partial f_{j(k)}$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \left(\|b_k^* - a^*\|_*^2 + \|b_k - a_k\|^2 + 2\langle b_k^* - a^*, b_k - a_k\rangle\right) = 0. \tag{45}$$

Then, let us denote a the limit of $\{a_k\}_{k\in\mathbb{N}}$. On the one hand, we have that

$$0 \le (\|b_k^* - a^*\|_* - \|b_k - a\|)^2 \le \|b_k^* - a^*\|_*^2 + \|b_k - a\|^2 + 2\langle b_k^* - a^*, b_k - a\rangle. \tag{46}$$

Notice that $\langle b_k^* - a^*, b_k - a\rangle = \langle b_k^* - a^*, b_k - a_k\rangle + \langle b_k^* - a^*, a_k - a\rangle$ and

$$\begin{aligned}\|b_k - a\|^2 &\le (\|b_k - a_k\| + \|a_k - a\|)^2 \\ &= \|b_k - a_k\|^2 + 2\|b_k - a_k\|\|a_k - a\| + \|a_k - a\|^2.\end{aligned}$$

Since $\{b_k^*\}_{k\in\mathbb{N}}$ is bounded, we see that there exist positive reals $\epsilon_k \downarrow 0$ such that $\langle b_k^* - a^*, b_k - a\rangle \le \langle b_k^* - a^*, b_k - a_k\rangle + \epsilon_k$ and $\|b_k - a\|^2 \le \|b_k - a_k\|^2 + \epsilon_k$. Consequently, we can write

$$\|b_k - a\|^2 + 2\langle b_k^* - a^*, b_k - a\rangle \le \|b_k - a_k\|^2 + 2\langle b_k^* - a^*, b_k - a_k\rangle + 3\epsilon_k.$$

This inequality, (45) and (46) entail that

$$\lim_{k\to\infty} \left(\|b_k^* - a^*\|_*^2 + \|b_k - a\|^2 + 2\langle b_k^* - a^*, b_k - a\rangle\right) = 0,$$

so (CTA) implies that a subsequence of $\{b_k\}_{k\in\mathbb{N}}$ converges strongly to some $b \in E$. □

Remark 7.2. Let $(E, \|\cdot\|)$ be a Banach space and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E.

Observe that, if the inequality $\limsup_{k\to\infty} (\|b_k\| + s_{f_k}(b_k)) < +\infty$ ensures that the sequence $\{b_k\}_{k\in\mathbb{N}}$ is contained in a compact subset of E, then (CTA) holds true; compare to the compactness assumption A_c from [1, Theorem 1]. In order to verify that, let us assume that $(b_k, b_k^*) \in \operatorname{gph} \partial f_k$ for all $k \in \mathbb{N}$ are such that the sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$ is bounded.

Then
$$\limsup_{k\to\infty}\left(\|b_k\| + s_{f_k}(b_k)\right) \le \limsup_{k\to\infty}\left(\|b_k\| + \|b_k^*\|_*\right) < +\infty$$

and hence the sequence $\{b_k\}_{k\in\mathbb{N}}$ is contained in a compact subset, so it contains a convergent subsequence.

As examples of when condition (CTA) is satisfied in infinite dimension, we can refer to [1, Remark 3.2(c)] for a concrete instance. Additionally, the ideas in [1, Remark 3.2(b)] can be applied to obtain sufficient conditions in Banach spaces. Another case where sufficient conditions can be provided is in the space $E = \ell_1$. Precisely, if instead of (CTA) we assume that for any increasing function $j : \mathbb{N} \to \mathbb{N}$ and any bounded sequence $\{b_k\}_{k\in\mathbb{N}}$ such that $(b_k, b_k^*) \in \operatorname{gph}\partial f_{j(k)}$ for all $k \in \mathbb{N}$, the sequence $\{b_k\}_{k\in\mathbb{N}}$ admits a w-convergent subsequence, then Schur's Lemma (see e.g. [15, p. 149]) implies that this subsequence is actually strongly convergent. A similar reasoning applies whenever the norm of the space has the Kadec-Klee property. Specifically, knowing that a bounded sequence $\{b_k\}_{k\in\mathbb{N}}$ with $(b_k, b_k^*) \in \operatorname{gph}\partial f_{j(k)}$ for all $k \in \mathbb{N}$ has a subsequence $\{b_{\ell(k)}\}_{k\in\mathbb{N}}$ w-converging to b along with

$$\lim_{k\to\infty} \|b_{\ell(k)} - x\| = \|b - x\|, \text{ for some } x \in E,$$

we can conclude that this subsequence must be strongly convergent.

Assume that E is a reflexive Banach space with its norm having the Kadec-Klee property. The Eberlein-Šmulian Theorem, see [13, Theorem 3.109], gives by reflexivity that any bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$ satisfying (43) and $(b_k, b_k^*) \in \operatorname{gph}\partial f_{j(k)}$ for all $k \in \mathbb{N}$ has a w-convergent subsequence. It follows from Lemmas 4.1, 4.12 and 4.13, see also Remark 4.2, that any of the two equalities

$$E = \operatorname{cl}\operatorname{Dom}(\operatorname{Li}_{d\times d^*}\operatorname{gph}\partial f_k),\ E^* = \operatorname{cl}\operatorname{Rge}(\operatorname{Li}_{d\times d^*}\operatorname{gph}\partial f_k)$$

makes this subsequence strongly convergent. Thus, in any reflexive Banach space with its norm having the Kadec-Klee property, the condition (CTA) holds true whenever anyone of the two above equalities is satisfied.

Of course, in finite dimensions the condition (CTA) is satisfied. □

Below it is pointed out that the (CTA) condition is a necessary condition for the graphical convergence of subdifferentials of lower semicontiuous convex functions to the graph of the subdifferential of a lower semicontinuous proper convex function, whenever E is a reflexive Banach space with its norm having the Kadec-Klee property.

Proposition 7.3. *Assume that $(E, \|\cdot\|)$ is a reflexive Banach space with the norm having the Kadec-Klee property and $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of lsc proper convex functions on E. If f is an lsc proper convex function on E such that*

$$\operatorname{gph} f = \operatorname{Li}_{d\times d^*}\operatorname{gph}\partial f_k,$$

then (CTA) *is valid.*

Proof. Let us fix any $a \in E$, any $a^* \in E^*$, any increasing function $j : \mathbb{N} \to \mathbb{N}$, and any bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$, such that $(b_k, b_k^*) \in \operatorname{gph}\partial f_{j(k)}$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty}\left(\|b_k^* - a^*\|_*^2 + \|b_k - a\|^2 + 2\langle b_k^* - a^*, b_k - a\rangle\right) = 0. \tag{47}$$

There is an increasing function $l : \mathbb{N} \to \mathbb{N}$such that the sequences $\{b_{l(k)}\}_{k\in\mathbb{N}}$ and $\{b^*_{l(k)}\}_{k\in\mathbb{N}}$ are weakly convergent (w-convergent) to b and weakly* convergent (w^*-convergent) to b^*, respectively. It follows from Lemma 4.1 and Remark 4.2 that

$$\lim_{k\to\infty} \langle b^*_{l(k)}, b_{l(k)}\rangle \geq \langle b^*, b\rangle,$$

thus by Lemma 4.12 we get the strong convergence of the sequence $\{b_{l(k)}\}_{k\in\mathbb{N}}$ to b, which translates the condition (CTA). □

Lemma 7.4. *Let $(E, \|\cdot\|)$ be a Banach space, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E. Assuming that* (CTA) *is valid, then the following equality*

$$\mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) = \mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k$$

holds true.

Proof. The inclusion $\mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k \subset \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$ is easy to notice.

In particular, the equality is trivial if $\mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) = \emptyset$.

Take any $(a, a^*) \in \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$. The latter inclusion and Lemma 4.5 furnish sequences $\{a_k\}_{k\in\mathbb{N}}$ in E and $\{a^*_k\}_{k\in\mathbb{N}}$ in E^* such that $(a_k, a^*_k) \in \operatorname{gph} \partial f_k$ for all $k \in \mathbb{N}$ and such that

$$\lim_{k\to\infty} a_k = a \quad \text{and} \quad a^*_k \xrightarrow{w^*} a^*, \tag{48}$$

so we observe in particular that the sequence $\{(a_k, a^*_k)\}_{k\in\mathbb{N}}$ is bounded. By Proposition 4.11 for each $k \in \mathbb{N}$ there is $(b_k, b^*_k) \in \operatorname{gph} \partial f_k$ such that

$$\|b^*_k - a^*\|^2_* + \|b_k - a_k\|^2 + 2\langle b^*_k - a^*, b_k - a_k\rangle \leq \frac{1}{k}. \tag{49}$$

By the monotonicity of the subdifferential ∂f_k we have for every $k \in \mathbb{N}$

$$\langle b^*_k - a^*, b_k - a_k\rangle \geq \langle b^*_k - a^*_k, b_k - a_k\rangle + \langle a^*_k - a^*, b_k - a_k\rangle \geq \langle a^*_k - a^*, b_k - a_k\rangle.$$

This and (49) ensure that for every $k \in \mathbb{N}$

$$\|b^*_k - a^*\|^2_* + \|b_k - a_k\|^2 + 2\langle a^*_k - a^*, b_k - a_k\rangle \leq \frac{1}{k},$$

which for $\alpha := \sup_{k\in\mathbb{N}} \|a^*_k - a^*\|_*$ gives

$$\|b_k - a_k\| \left(\|b_k - a_k\| - 2\|a^*_k - a^*\|_*\right) \leq 1 \text{ and } \|b^*_k - a^*\|^2_* \leq 1 + 2\alpha\|b_k - a_k\|.$$

We then see that the sequences $\{b_k\}_{k\in\mathbb{N}}$ and $\{b^*_k\}_{k\in\mathbb{N}}$ are bounded.

Assume that (CTA) is valid, so (CTA') is also valid by Lemma 7.1. There is an increasing function $j : \mathbb{N} \to \mathbb{N}$ such that $\{b_{j(k)}\}_{k\in\mathbb{N}}$ converges strongly to some $b \in E$. Using the convergence $a^*_k \xrightarrow{w^*} a^*$ and the monotonicity of the subdifferential $\partial f_{j(k)}$ we can write via (49):

$$
\begin{aligned}
0 &\geq \liminf_{k\to\infty} \Big(\|b^*_{j(k)} - a^*\|_*^2 + \|b_{j(k)} - a_{j(k)}\|^2 + 2\langle b^*_{j(k)} - a^*, b_{j(k)} - a_{j(k)}\rangle \Big) \\
&= \liminf_{k\to\infty} \Big(\|b^*_{j(k)} - a^*\|_*^2 + \|b_{j(k)} - a_{j(k)}\|^2 \\
&\quad + 2\langle b^*_{j(k)} - a^*_{j(k)}, b_{j(k)} - a_{j(k)}\rangle + 2\langle a^*_{j(k)} - a^*, b_{j(k)} - a_{j(k)}\rangle \Big) \\
&\geq \liminf_{k\to\infty} \left(\|b^*_{j(k)} - a^*\|_*^2 + \|b_{j(k)} - a_{j(k)}\|^2 \right) \\
&= \liminf_{k\to\infty} \left(\|b^*_{j(k)} - a^*\|_*^2 + \|b - a\|^2 \right) \geq 0.
\end{aligned}
$$

The above development can be proceeded with every subsequence of $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$ according to the definition of (CTA'). Therefore, $(a, a^*) \in \mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k$, hence we conclude the proof with the inclusion

$$\mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k. \qquad \square \tag{50}$$

The next proposition complements Lemma 7.4 above.

Proposition 7.5. *Let $(E, \|\cdot\|)$ be a Banach space, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E and let $A : E \rightrightarrows E^*$ be an operator.*

(a) *If* (CTA) *is valid, then the following equivalence*

$$\operatorname{gph} A = \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) \Longleftrightarrow \operatorname{gph} A = \mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k \tag{51}$$

holds true. Further, one has

$$(\operatorname{gph} \partial f_k \xrightarrow{b\mu} \operatorname{gph} A) \Longrightarrow (\operatorname{gph} \partial f_k \xrightarrow{P.K.} \operatorname{gph} A).$$

The latter implication is an equivalence whenever A is maximal monotone.

(b) *If* (CTA) *is valid and $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$ converges to $\operatorname{gph} A$ in a bounded mixed way (i.e. (22) holds true for $S_k := \operatorname{gph} \partial f_k$ and $S := \operatorname{gph} A$), then there exists an lsc convex function $f : E \to \mathbb{R} \cup \{+\infty\}$ such that $A = \operatorname{gph} \partial f$.*

Proof. It follows from Lemma 4.14 that the operator A is cyclically monotone, whenever either $\operatorname{gph} A = \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}})$ or $\operatorname{gph} A = \mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k$.

(a) The equivalence in (51) is justified by Lemma 7.4.

Assume that $\operatorname{gph} \partial f_k \xrightarrow{b\mu} \operatorname{gph} A$. Then by (51) we can write

$$\mathrm{Ls}_{d\times d^*} \operatorname{gph} \partial f_k \subset \mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) = \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) = \operatorname{gph} A = \mathrm{Li}_{d\times d^*} \operatorname{gph} \partial f_k,$$

which justifies the implication in (a).

Conversely, assume A is maximal monotone and $\operatorname{gph} \partial f_k \xrightarrow{P.K.} \operatorname{gph} A$. It follows from Lemma 4.15 that $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \operatorname{gph} A$, and this finishes the proof of (a).

(b) Assume now that (CTA) is valid and $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$ converges to $\operatorname{gph} A$ in a bounded mixed way.

First, if $\mathrm{Dom}\, A = \emptyset$, we have that $f \equiv +\infty$ satisfies the requirement. Then, in the rest of the proof we assume that $\mathrm{Dom}\, A \neq \emptyset$.

As noticed at the beginning of the proof, the operator A must be monotone.

Consider any $(x, x^*) \in E \times E^*$ such that

$$\langle x^* - y^*, x - y \rangle \geq 0, \quad \text{for all } (y, y^*) \in \mathrm{gph}\, A. \tag{52}$$

By Proposition 4.11, for each $k \in \mathbb{N}$, there is $(b_k, b_k^*) \in \mathrm{gph}\, \partial f_k$ such that

$$\forall k \in \mathbb{N},\ \|b_k^* - x^*\|_*^2 + \|b_k - x\|^2 + 2\langle b_k^* - x^*, b_k - x \rangle \leq \frac{1}{k}, \tag{53}$$

which gives in particular

$$0 \leq (\|b_k^* - x^*\|_* - \|b_k - x\|)^2 = \|b_k^* - x^*\|_*^2 + \|b_k - x\|^2 - 2\|b_k^* - x^*\|_* \cdot \|b_k - x\| \leq 1,$$

hence

$$\|b_k^* - x^*\|_* \leq \|b_k - x\| + 1. \tag{54}$$

Now, let us consider a pair $(a, a^*) \in \mathrm{gph}\, A$, and a sequence $(a_k, a_k^*) \to (a, a^*)$ with $a_k^* \in \partial f_k(a_k)$ according to the equivalence (51). Observe that by the monotonicity of subdifferentials of convex functions we have for every $k \in \mathbb{N}$

$$\langle b_k^* - x^*, b_k - x \rangle \geq \langle b_k^* - a_k^*, a_k - x \rangle + \langle a_k^* - x^*, b_k - x \rangle,$$

which allows us to obtain from (53) that

$$\|b_k^* - x^*\|_*^2 + \|b_k - x\|^2 + 2\langle b_k^* - a_k^*, a_k - x \rangle + 2\langle a_k^* - x^*, b_k - x \rangle \leq 1.$$

We derive from this and (54) that for every $k \in \mathbb{N}$

$$\|b_k - x\|^2 \leq 2\|b_k - x\|\,\|a_k - x\| + 2\|a_k - x\| + 2\|a_k^* - x^*\|_*\,\|a_k - x\| + 2\|b_k - x\| \cdot \|a_k^* - x^*\|_* + 1,$$

and hence

$$\|b_k - x\|(\|b_k - x\| - 2\|a_k^* - x^*\|_* - 2\|a_k - x\|) \leq 2\|a_k - x\| + 2\|a_k^* - x^*\|_*\,\|a_k - x\| + 1.$$

The latter inequality entails that $\{b_k\}_{k\in\mathbb{N}}$ is bounded, so $\{b_k^*\}_{k\in\mathbb{N}}$ is also bounded by (54). By (CTA) the sequence $\{b_k\}_{k\in\mathbb{N}}$ contains a strongly convergent subsequence, i.e. there is some increasing function $j : \mathbb{N} \to \mathbb{N}$ such that $\{b_{j(k)}\}_{k\in\mathbb{N}}$ converges strongly to some $b \in E$. Further, by boundedness of the sequence $\{b_k^*\}_{k\in\mathbb{N}}$ in E^* we can choose some

$$b^* \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{b_{j(k)}^*, b_{j(k+1)}^*, \ldots\}, \tag{55}$$

so $(b, b^*) \in \mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}})$. By (22) we deduce that

$$(b, b^*) \in \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) = \mathrm{gph}\, A.$$

Thus, using (52) we get that $\langle b^* - x^*, b - x \rangle \geq 0$.

Further, the inclusion (55) entails that $\langle b^*, b - x \rangle$ is a cluster point in $\mathbb{R}$ of the sequence $\{\langle b_{j(k)}^*, b - x \rangle\}_{k\in\mathbb{N}}$, hence there is an increasing function $\ell : \mathbb{N} \to \mathbb{N}$ such that for $\gamma := j \circ \ell$ one has $\{\langle b^*_{\gamma(k)}, b - x\rangle\}_{k\in\mathbb{N}}$ tends to $\langle b^*, b - x\rangle$. Because $\{b_{j(k)}\}_{k\in\mathbb{N}}$ converges strongly to $b \in E$, this and the boundedness of $\{b^*_{\gamma(k)}\}_{k\in\mathbb{N}}$ ensure that

$$\lim_{k\to\infty} \langle b^*_{\gamma(k)} - x^*, b_{\gamma(k)} - x\rangle = \langle b^* - x^*, b - x\rangle \geq 0.$$

Passing to the lim sup in (53) we get

$$\begin{aligned} 0 &\geq \limsup_{k\to\infty} \left(\|b^*_{\gamma(k)} - x^*\|_*^2 + \|b_{\gamma(k)} - x\|^2\right) + 2 \lim_{k\to\infty} \langle b^*_{\gamma(k)} - x^*, b_{\gamma(k)} - x\rangle \\ &= \limsup_{k\to\infty} \left(\|b^*_{\gamma(k)} - x^*\|_*^2 + \|b_{\gamma(k)} - x\|^2\right) + 2\langle b^* - x^*, b - x\rangle \\ &\geq \limsup_{k\to\infty} \left(\|b^*_{\gamma(k)} - x^*\|_*^2 + \|b_{\gamma(k)} - x\|^2\right) \geq 0, \end{aligned}$$

which entails $x = \lim_{k\to\infty} b_{\gamma(k)}$ and $x^* = \lim_{k\to\infty} b^*_{\gamma(k)}$. We conclude that (x, x^*) belongs to $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}})$. Since by assumption $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \operatorname{gph} A$ we deduce $(x, x^*) \in \operatorname{gph} A$, hence the operator A is maximal cyclically monotone and Theorem 2.5 ensures the existence of the function f fulfilling the statement (b). □

If E is a finite dimensional Banach space, the condition (CTA) is satisfied, and the bounded way convergence and the Painlevé-Kuratowski convergence of subdifferetials coincide. Thus it follows from Proposition 7.5(b) that the corresponding limits must be the same maximal monotone operator, whenever $\operatorname{Dom} A \neq \emptyset$.

Under (CTA) and through Proposition 7.5(a) we can show that the epigraphical convergence of slopes is implied by the bounded mixed convergence of subdifferentials.

Theorem 7.6. *Let $(E, \|\cdot\|)$ be a Banach space and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E and f be an lsc convex function on E.*

(a) *If f is proper, one has the implication*

$$\partial f_k \xrightarrow{P.K.} \partial f \Longrightarrow s_{f_k} \xrightarrow{e} s_f.$$

(b) *If* (CTA) *is valid, then the other implication*

$$\partial f_k \xrightarrow{b\mu} \partial f \Longrightarrow s_{f_k} \xrightarrow{e} s_f$$

also holds true (without imposing the properness of f).

Proof. Note that, under the assumption of either (a) or (b), we have the inclusion

$$\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \operatorname{gph} \partial f$$

(in case (a) by Lemma 4.15 since in (a) the operator ∂f is maximal monotone) and we have also

$$\operatorname{gph} \partial f \subset \operatorname{Li}_{d\times d^*} \operatorname{gph} \partial f_k$$

(in case (b) by Lemma 7.4). Then, Lemma 5.3 implies that $s_{f_k} \xrightarrow{e} s_f$, which concludes the proof. □

The above development in this section culminates in the following theorem.

Theorem 7.7. *Let* $(E, \|\cdot\|)$ *be a separable Banach space, let* $\{f_k\}_{k\in\mathbb{N}}$ *be a sequence of lsc proper convex functions on* E *such that* (CTA) *is valid. Let* f *be an lsc proper convex function on* E *bounded below. Then, the following statements are equivalent:*

(a) $\partial f_k \xrightarrow{b\mu} \partial f$; (b) $\partial f_k \xrightarrow{P.K.} \partial f$; (c) $s_{f_k} \xrightarrow{e} s_f$.

Proof. The equivalence (a)⇔(b) follows from Proposition 7.5(a) while the implication (a)⇒(c) is a consequence of Theorem 7.6(b).

In order to get the other implication (c)⇒(b) it suffices to prove that any subsequence of $\{\operatorname{gph}\partial f_k\}_{k\in\mathbb{N}}$ contains a subsequence converging in P.K. sense to $\operatorname{gph}\partial f$ (see Lemma 2.1). It follows from Proposition 4.9 that for every subsequence of $\{\operatorname{gph}\partial f_k\}_{k\in\mathbb{N}}$, say $\{\operatorname{gph}\partial f_{j(k)}\}_{k\in\mathbb{N}}$, where $j : \mathbb{N} \longrightarrow \mathbb{N}$ is an increasing function, there are an operator $A : E \rightrightarrows E^*$ and an increasing function $\ell : \mathbb{N} \to \mathbb{N}$ such that for $\gamma := j \circ \ell$ one has $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{b\mu} \operatorname{gph} A$.

Hence, by Proposition 7.5, $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{P.K.} \operatorname{gph} A$. Further, from the convergence $s_{f_k} \xrightarrow{e} s_f$ we infer by (2) (see also Definition 2.4) that

$$s_{f_{\gamma(k)}} \xrightarrow{e} s_f. \tag{56}$$

It is assumed that f is proper convex and lsc, so $\operatorname{Dom}\partial f \neq \emptyset$ and then as a result the function $s_f(\cdot)$ is proper. According to (56) and the inclusion

$$\mathcal{A}_s(\{\partial f_{\gamma(k)}\}_{k\in\mathbb{N}}) \subset \operatorname{gph} A$$

(see equalities in (22), Definition 4.7) Lemma 5.3(a) guarantees that $s_f(\cdot) \geq s_A(\cdot)$, which ensures $\operatorname{Dom} A \neq \emptyset$ since the function $s_A(\cdot)$ is also proper. The property $\operatorname{Dom} A \neq \emptyset$ and the convergence $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{b\mu} \operatorname{gph} A$ entail by Proposition 7.5(b) that there exists an lsc proper convex function ϕ on E such that $\operatorname{gph} A = \operatorname{gph}\partial\phi$, which combined with the above equivalence (a)⇔(b) entails $\operatorname{gph}\partial f_{\gamma(k)} \xrightarrow{P.K.} \operatorname{gph}\partial\phi$. This and Theorem 7.6(a) give $s_{f_{\gamma(k)}} \xrightarrow{e} s_\phi$, so (56) yields

$$\forall x \in E, s_f(x) = s_\phi(x).$$

It follows from Theorem 2.6 that f and ϕ coincide up to an additive constant, hence $\operatorname{gph}\partial f = \operatorname{gph}\partial\phi = \operatorname{gph} A$. We conclude by what precedes that any subsequence $\{\operatorname{gph}\partial f_{j(k)}\}_{k\in\mathbb{N}}$ contains a further subsequence converging in the P.K. sense to $\operatorname{gph}\partial f$, which by Lemma 2.1 translates (*b*) and finishes the proof. □

Remark 7.8. Let us notice that in Theorem 7.7 the equivalence (a)⇔(c) holds even when f is not proper, i.e. $f \equiv +\infty$, which is equivalent to $\operatorname{gph}\partial f = \emptyset$, see for example [6, Theorem 4.3.2 (Brøndsted-Rockafellar)] or [28, Theorem 3.142]. Indeed, the implication (a)⇒(c) follows from Theorem 7.6(b) even when $f \equiv +\infty$. Regarding the converse implication (c)⇒(a), assume that $s_{f_k} \xrightarrow{e} s_f \equiv +\infty$. It follows from Proposition 7.5(b) (see also Proposition 4.9) that every subsequence of $\{\operatorname{gph}\partial f_k\}_{k\in\mathbb{N}}$ has the empty set as bounded mixed way limit.

Indeed, in order to justify this statement let us observe that if

$$\operatorname{gph} \partial f_{j(k)} \xrightarrow{b\mu} \operatorname{gph} A$$

for some increasing function $j : \mathbb{N} \to \mathbb{N}$, then there exists an lsc convex function $g : E \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{gph} A = \operatorname{gph} \partial g$, see Proposition 7.5(b). If $\operatorname{gph} A \neq \emptyset$, then we get a contradiction with the convergences

$$s_{f_k} \xrightarrow{e} s_f \equiv +\infty \quad \text{and} \quad s_{f_{j(k)}} \xrightarrow{e} s_g \not\equiv +\infty,$$

for the latter one see Theorem 7.7. Hence $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) = \emptyset$, see Proposition 4.9, which translates the implication (c)⇒(a) in this case.

Finally, we mention that Theorem 3.2 is also a consequence of Theorem 7.7.

If (CTA) is valid and $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$ converges to $\operatorname{gph} A$ in a bounded mixed way (i.e. (22) holds true for $S_k := \operatorname{gph} \partial f_k$ and $S := \operatorname{gph} A$), then there exists an lsc convex function $f : E \to \mathbb{R} \cup \{+\infty\}$ such that $A = \operatorname{gph} \partial f$. □

The following theorem also deserves to be pointed out. It is a direct consequence of Theorem 7.7 and Proposition 7.3.

Theorem 7.9. *Let $(E, \|\cdot\|)$ be a separable reflexive Banach space with its norm having the Kadec-Klee property, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E, and let f be an lsc proper convex function on E bounded below. Then, the following statements are equivalent:*

(a) $\partial f_k \xrightarrow{P.K.} \partial f$;

(b) $s_{f_k} \xrightarrow{e} s_f$ *and* (CTA) *is satisfied.*

Acknowledgement. P. Pérez-Aros was partially supported by Centro de Modelamiento Matemático (CMM), ACE210010 and FB210005, BASAL funds for center of excellence and ANID-Chile grant: MATH-AMSUD 23-MATH-09 and MATH-AMSUD 23-MATH-17, ECOS-ANID ECOS320027, Fondecyt Regular 1220886, Fondecyt Regular 1240335, Fondecyt Regular 1240120 and Exploración 13220097.

References

[1] S. Adly, H. Attouch, R. T. Rockafellar: *Preservation or not of the maximally monotone property by graph-convergence*, J. Convex Analysis 30 (2023) 413–440.

[2] H. Attouch: *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman, London (1984).

[3] H. Attouch, G. Buttazzo, G. Michaille: *Variational Analysis in Sobolev an BV Spaces. Applications to PDEs and Optimization*, 2nd ed., Society for Industrial and Applied Mathematics, Philadelphia (2014).

[4] J.-P. Aubin, H. Frankowska: *Set-Valued Analysis*, Birkhäuser, Boston (1990).

[5] G. Beer: *Topologies on Closed and Closed Convex Sets*, Mathematics and its Applications Vol. 268, Kluwer Academic Publishers, Dordrecht (1993).

[6] J. M. Borwein, J. D. Vanderwerff: *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, New York (2010).

[7] T. Boulmezaoud, P. Cieutat, A. Daniilidis: *Gradient flows, second-order gradient systems and convexity*, SIAM J. Optim. 28 (2018) 2049–2066.

[8] A. Daniilidis, D. Drusvyatskiy: *The slope robustly determines convex functions*, Proc. Amer. Math. Soc. 151 (2023) 4751–4756.

[9] A. Daniilidis, L. Miclo, D. Salas: *Descent modulus and applications*, J. Funct. Analysis 287/11 (2024), art.no. 110626.

[10] A. Daniilidis, D. Salas: *A determination theorem in terms of the metric slope*, Proc. Amer. Math. Soc. 150/10 (2022) 4325–4333.

[11] A. Daniilidis, D. Salas, S. Tapia-García: *A slope generalization of Attouch theorem*, Math. Programming A 212/1-2 (2025) 319–348.

[12] J. Diestel: *Geometry of Banach Spaces – Selected Topics*, Springer, Berlin (1975).

[13] M. Fabian, P. Habal, P. Hajek, V. Montesinos, V. Zizler: *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, Springer, New York (2011).

[14] S. Fitzpatrick: *Representing monotone operators by convex functions*, in: *Functional Analysis and Optimization*, Workshop Canberra 1988, Proc. Cent. Math. Anal. Aust. Natl. Univ. 20 (1988) 59–65.

[15] R. B. Holmes: *Geometric Functional Analysis and its Applications*, Springer, New York (1975).

[16] M. Ivanov, N. Zlateva: *Slopes and Moreau-Rockafellar theorem*, J. Convex Analysis 32 (2025) 359–374.

[17] A. Jourani, L. Thibault, D. Zagrodny: *$C^{1,\omega(\cdot)}$-regularity and Lipschitz-like properties of subdifferential*, Proc. London Math. Soc. (3) 105 (2012) 189–223.

[18] K. Kuratowski: *Topologie*, Academic Press, New York (1966).

[19] P. Perez-Aros, D. Salas, E. Vilches: *Determination of convex functions via subgradients of minimal norms*, Math. Programming A 190/1-2 (2021) 561–583.

[20] P. Perez-Aros, L. Thibault, D. Zagrodny; *Convergence of slopes in Asplund spaces*, J. Convex Analysis 32 (2025) 661–680.

[21] R. R. Phelps: *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Springer, Berlin (1989).

[22] R. T. Rockafellar: *Characterization of the subdifferentials of convex functions*, Pacific J. Math. 17 (1966) 497–510.

[23] R. T. Rockafellar: *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. 33 (1970) 209–216.

[24] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften Vol. 317, Springer, Berlin (1998).

[25] W. Rudin: *Functional Analysis*, 2nd ed., McGraw-Hill, New York (1991).

[26] S. Simons: *Minimax and Monotonicity*, Lecture Notes in Mathematics Vol. 1693, Springer, Berlin (1998).

[27] S. Simons: *From Hahn-Banach to Monotonicity*, Lecture Notes in Mathematics Vol. 1693, Springer, Berlin (2008).

[28] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. I: General Theory*, World Scientific, Hackensack (2023).

[29] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. II: Special Classes of Functions and Sets*, World Scientific, Hackensack (2023).

[30] L. Thibault, D. Zagrodny: *Determining functions by slopes*, Comm. Contemp. Math. 25/7 (2023), art.no. 2250014.

[31] E. Vilches: *Proximal determination of convex functions*, J. Convex Analysis 28/4 (2021) 1187–1192.

[32] D. Zagrodny: *On the weak* convergence of subdifferentials of convex functions*, J. Convex Analysis 12/1 (2005) 213–219.

[33] D. Zagrodny: *The convexity of the closure of the domain and the range of a maximal monotone multifunction of type NI*, Set-Valued Analysis 16 (2008) 759–783.

[34] K. Zarankiewicz: *Sur les points de division dans les ensembles connexes*, Fund. Math. 9 (1927) 124–171.