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On Convex Cones for which Criticality Coincides with Antipodality

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Dedicated to the memory of Hedy Attouch.

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Let K be a proper cone in a Euclidean space E. An antipodal pair of K is a pair of unit vectors in K that achieves the maximum angle $\theta_{\max}(K)$ of the cone. A critical pair of K is a pair of unit vectors in K that satisfies the Karush-Kuhn-Tucker optimality conditions of the nonconvex optimization problem defining $\theta_{\max}(K)$. Antipodality implies criticality, but not conversely. This work studies the class of proper cones for which antipodality and criticality coincide.

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1. Introduction

Let E be a Euclidean vector space of dimension at least three, with inner product $\langle \cdot, \cdot \rangle$, associated norm $\| \cdot \|$, and unit sphere S_E. Let $\mathbb{P}(E)$ be the set of proper cones in E. By a proper cone we understand a closed convex cone that is pointed and solid. Recall that a closed convex cone is pointed if it contains no line and solid if it has nonempty interior. Throughout this work, K is a proper cone and K^* is its dual cone. Various issues concerning the angular structure of K can be formulated with the help of the *opening angle function* f_K given by

$$f_K(u) := \max_{v \in K \cap S_E} \arccos \langle u, v \rangle \tag{1}$$

for all $u \in S_E$. This function intervenes for instance in the definition

$$\theta_{\max}(K) := \max_{u \in K \cap S_E} f_K(u) \tag{2}$$

of the *maximum angle* of K. Problem (2) can be written in a symmetric form

$$\theta_{\max}(K) = \max_{u,\, v \in K \cap S_E} \arccos \langle u, v \rangle \tag{3}$$

in which the variables u and v are interchangeable.

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A pair $\{u, v\}$ of unit vectors in K that achieves the maximum (3) is called an *antipodal pair* of K. Critical pairs are obtained by working out the necessary optimality conditions for the nonconvex optimization problem (3), cf. Iusem and Seeger [7].

Definition 1.1. Let $K \in \mathbb{P}(E)$. A pair $\{u, v\}$ of distinct vectors in E is a *critical pair* of K if

$$u, v \in K \cap S_E\,, \quad v - \langle u, v\rangle u \in K^*\,, \quad u - \langle u, v\rangle v \in K^*. \tag{4}$$

The angle formed by a critical pair is called a *critical angle* of K. The set of critical angles, denoted by $\mathrm{AS}(K)$, is called the angular spectrum of K.

In contrast to [7, Definition 4.2], we do not consider a critical pair as a couple (u, v), but as an unordered pair $\{u, v\}$. Consistently, the case $u = v$ is ruled out and the trivial value 0 is not viewed as a critical angle. Every antipodal pair of K is a critical pair of K, but not conversely. In the next proposition, which is borrowed from [7], the acronym "bd" stands for boundary.

Proposition 1.2. *Let $K \in \mathbb{P}(E)$ and $\{u, v\}$ be a critical pair of K. Then:*

(a) *u and v are in* $\mathrm{bd}(K)$*.*

(b) *$v - \langle u, v\rangle u$ and $u - \langle u, v\rangle v$ are in* $\mathrm{bd}(K^*)$*.*

A proper cone K has at least one critical angle and $\theta_{\max}(K)$ is the largest one, i.e.,

$$\theta_{\max}(K) = \max(\mathrm{AS}(K)). \tag{5}$$

A vast majority of proper cones arising in practice have finitely many critical angles. This happens for instance with any polyhedral proper cone or any ellipsoidal cone. However, it is possible to construct fairly nasty proper cones admitting countably infinite or even uncountable many critical angles, cf. [9]. In this work we are interested in cones with just a few critical angles. In fact, we wish to explore in detail the following extreme case.

Definition 1.3. $K \in \mathbb{P}(E)$ is called *pure* if $\mathrm{AS}(K)$ is a singleton.

In case of purity, the unique element of $\mathrm{AS}(K)$ is of course $\theta_{\max}(K)$. In essence, that K is pure means that every critical pair of K is an antipodal pair of K. The notion of purity is mentioned in Iusem and Seeger [10, Definition 3], but just in passing. The motivation of these authors was to formulate in an elegant manner the next geometric result.

Proposition 1.4. *For all $K \in \mathbb{P}(E)$ one has*

$$\theta_{\max}(K) + \theta_{\max}(K^*) \geq \pi\,, \tag{6}$$

with equality in (6) *if and only if K is pure.*

Proposition 1.4 is a combination of [6, Theorem 7.2] and [10, Theorem 4]. This work aims at giving a full theoretical treatment of the notion of purity. The reader may wonder why we focus on such a seemingly narrow scope.

In fact, proper cones that are pure not only satisfy the relation

$$\theta_{\max}(K) + \theta_{\max}(K^*) = \pi, \tag{7}$$

which is an important property by itself, but they correspond to the conic version of the convex bodies of constant width. These are two good reasons for studying the notion of purity.

Example 1.5. Let $d \geq 2$. We equip the space Sym_d of symmetric matrices of order d with the Frobenius inner product $\langle X, Y\rangle := \mathrm{tr}(XY)$. In this Euclidean space there are plenty of proper cones of interest. Two of them widely used in linear algebra and optimization theory are

$$\begin{aligned} \mathrm{SDP}_d &:= \{X \in \mathrm{Sym}_d : X \text{ is positive semidefinite}\}, \\ \mathrm{DNN}_d &:= \{X \in \mathrm{Sym}_d : X \text{ is doubly nonnegative}\}. \end{aligned}$$

The first one, called the *Löwner cone*, is self-dual. Hence, SDP_d is pure by Proposition 1.4. That X is doubly nonnegative means that X is positive semidefinite and nonnegative componentwisely. The dual cone of DNN_d is equal to

$$\mathrm{Cop}_d := \{X \in \mathrm{Sym}_d : X \text{ is copositive}\}.$$

DNN_d is not pure because it does not satisfy the relation (7). Indeed, $\theta_{\max}(\mathrm{DNN}_d) = \pi/2$ and $\theta_{\max}(\mathrm{Cop}_d) \geq 3\pi/4$, cf. Hiriart-Urruty and Seeger [5]. For the same reason, Cop_d is not pure. □

Since $\mathrm{AS}(K)$ is a nonempty compact subset of the real line, we write

$$\theta_{\min}(K) := \min(\mathrm{AS}(K))$$

by analogy with (5). In other words, the *minimum angle* of K is defined as the smallest critical angle of K. The next definition corresponds to the "minimum" counterpart of an antipodal pair.

Definition 1.6. Let $K \in \mathbb{P}(E)$. A *minor pair* of K is a critical pair $\{u, v\}$ of K such that $\arccos\langle u, v\rangle = \theta_{\min}(K)$.

The next proposition is obtained by combining Theorem 2 and Theorem 3 in [10]. If $\{u, v\}$ is a pair of noncollinear unit vectors, then

$$a := \frac{v - \langle u, v\rangle u}{\sqrt{1 - \langle u, v\rangle^2}}\ , \quad b := \frac{u - \langle u, v\rangle v}{\sqrt{1 - \langle u, v\rangle^2}}$$

is a new pair of noncollinear unit vectors. We say that $\{a, b\}$ is the *conjugate pair* of $\{u, v\}$. Since the transformation $\{u, v\} \mapsto \{a, b\}$ is an involution, $\{u, v\}$ is in turn the conjugate pair of $\{a, b\}$. Note that $\{a, b, u, v\}$ are unit vectors on a same plane and $\langle a, u\rangle = \langle b, v\rangle = 0$. For the sake of convenience, to any pair $\{x, y\}$ of noncollinear unit vectors we associate

$$x \ominus y := \frac{x - \langle x, y\rangle y}{\sqrt{1 - \langle x, y\rangle^2}}$$

and call this vector the *spherical difference* of x and y.

Proposition 1.7. *Let $K \in \mathbb{P}(E)$. Let $\{u, v\}$ be a pair of noncollinear unit vectors and $\{a, b\}$ be its conjugate pair. Then:*

(a) $\arccos\langle u, v\rangle + \arccos\langle a, b\rangle = \pi$.

(b) *$\{u, v\}$ is a critical pair of K if and only if $\{a, b\}$ is a critical pair of K^*.*

(c) *$\{u, v\}$ is a minor pair of K if and only if $\{a, b\}$ is an antipodal pair of K^*.*

In classical spectral analysis of symmetric matrices, the largest and smallest eigenvalues are usually the most important ones and the intermediate eigenvalues are somehow less relevant. The same remark applies to the intermediate critical angles. By obvious reasons, we say that

$$\mathrm{ag}(K) := \theta_{\max}(K) - \theta_{\min}(K)$$

is the *angular gap* of K. Iusem and Seeger [10, p. 24] call this number the angular width of K, but we prefer to keep the word "width" for another use. The extreme angles $\theta_{\max}(K)$ and $\theta_{\min}(K)$ of a proper cone K have a number of applications. For instance, they serve to build suitable indices

$$\begin{aligned} \mathfrak{f}_{\mathrm{ptd}}(K) &:= \cos\left((1/2)\theta_{\max}(K)\right) \\ \mathfrak{f}_{\mathrm{sol}}(K) &:= \sin\left((1/2)\theta_{\min}(K)\right) \end{aligned}$$

for measuring the degree of pointedness and solidity of K, respectively. See Iusem and Seeger [6] for an axiomatic theory of pointedness and solidity indices.

1.1. Scope of this work

The main goal of this work is to provide necessary and sufficient conditions for a proper cone to be pure. We summarize in Theorem 1.8 what we know insofar about this subject.

Theorem 1.8. *Let $K \in \mathbb{P}(E)$. Then the next conditions are equivalent:*

$\mathfrak{c}_0$. *K is pure.*

$\mathfrak{c}_1$. *K^* is pure.*

$\mathfrak{c}_2$. *K obeys the upper reflection law $\theta_{\max}(K) + \theta_{\max}(K^*) = \pi$.*

$\mathfrak{c}_3$. *K obeys the lower reflection law $\theta_{\min}(K) + \theta_{\min}(K^*) = \pi$.*

$\mathfrak{c}_4$. *K has zero angular gap.*

$\mathfrak{c}_5$. $[\mathfrak{f}_{\mathrm{ptd}}(K)]^2 + [\mathfrak{f}_{\mathrm{sol}}(K)]^2 = 1$.

Proof. The characterization $\mathfrak{c}_2$ of purity is stated in Proposition 1.4, but we are writing it again for the sake of completeness. Proposition 1.7 shows that

$$\mathrm{AS}(K^*) = \{\pi - \theta : \theta \in \mathrm{AS}(K)\} \tag{8}$$

$$\theta_{\max}(K^*) = \pi - \theta_{\min}(K). \tag{9}$$

One can exchange of course the roles of K and K^*. Formula (8) shows that $\mathrm{AS}(K^*)$ and $\mathrm{AS}(K)$ have the same cardinality. This proves that $\mathfrak{c}_0$ is equivalent to $\mathfrak{c}_1$. Formula (9) takes care of the chain $\mathfrak{c}_2 \Leftrightarrow \mathfrak{c}_3 \Leftrightarrow \mathfrak{c}_4$. The equivalence between $\mathfrak{c}_4$ and $\mathfrak{c}_5$ follows from the standard trigonometric identities for the cosine and sinus of a half-angle. $\square$

By using the characterization $\mathfrak{c}_2$ of purity we readily see that any self-dual cone is pure. As we saw already, this explains why the Löwner cone SDP_d is pure. Note that the cone DNN_d of doubly nonnegative matrices is right-angled, but not self-dual. A proper cone is called *right-angled* if its maximal angle is equal to $\pi/2$.

Corollary 1.9. *Let $K \in \mathbb{P}(E)$. Then K is self-dual if and only if K is pure and right-angled.*

Being pure and being right-angled are independent properties, i.e., a proper cone may enjoy one of these properties but not the other. We retain from Corollary 1.9 that purity can be viewed as a relaxed version of self-duality: every self-dual cone is pure but not conversely. For instance, any revolution cone is pure, but not necessarily self-dual.

In this work we wish to prove Theorem 1.10 and a series of results that gravitate around Theorem 1.10. For notation convenience, we introduce the set

$$\Sigma_K := \mathrm{bd}(K) \cap S_E$$

called the *core* of K. This set captures all the information that is needed to recover K. Although Σ_K is nonconvex, this set plays somehow a similar role as a base of K.

Theorem 1.10. *Let $K \in \mathbb{P}(E)$. Then the next conditions are equivalent:*

$\mathfrak{c}_0$. *K is pure.*

$\mathfrak{c}_6$. *The oriented distance function of K^* is constant on Σ_K.*

$\mathfrak{c}_7$. *K is of constant width.*

$\mathfrak{c}_8$. *K is of constant opening angle.*

$\mathfrak{c}_9$. *K enjoys the antipodal mate property.*

$\mathfrak{c}_{10}$. *K enjoys the resolvant property.*

$\mathfrak{c}_{11}$. *K is diametrically complete in the angular sense.*

$\mathfrak{c}_{12}$. *K is of full antipodality range.*

Each one of the above conditions has an ad hoc geometric interpretation. Theorem 1.10 involves a number of concepts that have not been defined yet. The formal definitions of such concepts will be given progressively section after section. We are announcing Theorem 1.10 at this early point in time just in order to briefly indicate the different themes that we are going to encounter ahead. Proving the entire Theorem 1.10 in a single work is space prohibitive. In Seeger and Torki [14] we have partially accomplished this task by showing that $\mathfrak{c}_8 \Leftrightarrow \mathfrak{c}_9 \Leftrightarrow \mathfrak{c}_{10}$. The paper [14] is not concerned with the concept of purity, but it focuses on the analysis of the resolvant property, which is a surprising relation between the antipodal mates of K and the Gauss map G_K associated to K. The paper [14] also mentions the implication $\mathfrak{c}_{11} \Rightarrow \mathfrak{c}_9$, but the reverse implication, which is more difficult to prove, is left unattended. We shall encounter in our way two other characterizations of purity, namely $\mathfrak{c}_{13}$ and $\mathfrak{c}_{14}$, that we are not including in Theorem 1.10. These alternative characterizations of purity are more technical and serve essentially as analytical tools.

2. On proper cones of constant opening angle

Recall that the minimum angle $\theta_{\min}(K)$ of a proper cone K is defined as the smallest element of $\mathrm{AS}(K)$. We now introduce the concept of minimum opening angle of K, which is slightly different in spirit. Changing max by min in the bivariate problem (3) is a useless idea, because such a minimum would always be 0. It seems more reasonable to change max by min in the original formulation (2) and to further restrict the decision variable u to the boundary of K.

Definition 2.1. Let $K \in \mathbb{P}(E)$. For $u \in \Sigma_K$, the term $f_K(u)$ is called the *opening angle* of K relative to u. The *minimum opening angle* of K is defined as

$$\beta(K) := \min_{u \in \Sigma_K} f_K(u)\,. \tag{10}$$

We say that K is of *constant opening angle* if f_K is constant on Σ_K. □

Being of constant opening angle is one of the many conditions displayed in Theorem 1.10. Some examples of proper cones of constant opening angle are presented below.

Example 2.2. (a) Any revolution cone is of constant opening angle. A revolution cone in E is a proper cone of the form

$$\mathrm{Rev}(y,\phi) := \{x \in E : \langle y, x\rangle \geq \|x\| \cos\phi\},$$

where $y \in E$ is a unit vector that determines the central axis and $0 < \phi < \pi/2$ corresponds to the half-aperture angle of the cone. Evaluating the opening angle function of a revolution cone offers no difficulty. A simple computation shows that $f_{\mathrm{Rev}(y,\phi)}(u) = 2\phi$ for all $u \in \Sigma_{\mathrm{Rev}(y,\phi)}$.

(b) Any self-dual cone is of constant opening angle. Indeed, by applying Moreau's orthogonal decomposition theorem to a self-dual cone K, one can prove that

$$\min_{v \in K \cap S_E} \langle v, u\rangle = 0 \quad \text{for all } u \in \Sigma_K,$$

which is equivalent to saying that $f_K(u) = \pi/2$ for all $u \in \Sigma_K$.

(c) It is possible to go beyond revolution cones and self-dual cones. For instance, Moreno and Seeger [11] show that any Reuleaux cone in $\mathbb{R}^3$ is of constant opening angle. These authors define a Reuleaux cone in $\mathbb{R}^3$ as an intersection of three revolution cones $\mathrm{Rev}(y_1,\phi)$, $\mathrm{Rev}(y_2,\phi)$, $\mathrm{Rev}(y_3,\phi)$ with central axes determined by unit vectors y_1, y_2, y_3 satisfying the equiangularity condition

$$\langle y_1, y_2\rangle = \langle y_2, y_3\rangle = \langle y_3, y_1\rangle.$$

In the same way that a Reuleaux triangle is a "rounded" version of an equilateral triangle in the plane, a Reuleaux cone in $\mathbb{R}^3$ can be viewed as a rounded version of a three-dimensional equiangular simplicial cone. This example is not needed for understanding our exposition, but we mention it because it has a link to the theory of convex bodies of constant width. □

The next proposition shows the equivalence between $\mathfrak{c}_6$ and $\mathfrak{c}_8$. If $\Omega \subseteq E$ is a nonempty set whose set complement $\Omega^c := E \backslash \Omega$ is nonempty, then the *oriented distance function* of Ω is the function $\Delta_\Omega : E \to \mathbb{R}$ given by

$$\Delta_\Omega(z) := \mathrm{dist}(z, \Omega) - \mathrm{dist}(z, \Omega^c) .$$

Up to a sign, $\Delta_\Omega(z)$ measures how far is z from the boundary of Ω.

Proposition 2.3. *Let $K \in \mathbb{P}(E)$. Then K is of constant opening angle if and only if the oriented distance function of K^* is constant on Σ_K.*

Proof. It is shown in Seeger and Torki [14, Proposition 2.5] that

$$\cos[f_K(u)] + \Delta_{K^*}(u) \,=\, 0 \tag{11}$$

for all $u \in S_E$. This relation takes care of the proposition. □

Remark 2.4. Proposition 2.3 concerns the behavior on Σ_K of the oriented distance function of K^* and not that of K. Since purity is preserved after passage to dual cones, with the help of Theorem 1.10 one can deduce that condition $\mathfrak{c}_6$ is equivalent to saying that the oriented distance function of K is constant on Σ_{K^*}. But first one has to make sure that Theorem 1.10 is true! Incidentally, the general identity (11) confirms that any self-dual cone is of constant opening angle. □

Let us come back to the definition of $\beta(K)$. It is important to observe that the minimum opening angle of K may be different from the minimum angle of K. For establishing a link between both angles, we need first to prove a technical lemma that relates the functions f_K and f_{K^*}.

Lemma 2.5. *Let $K \in \mathbb{P}(E)$ and $x, y \in S_E$. Then*

$$K \ni x \perp y \in K^* \quad \Rightarrow \quad f_K(x) + f_{K^*}(y) \geq \pi . \tag{12}$$

Proof. Let $x, y \in S_E$ be as in the left-hand side of (12), i.e., $x \in K$, $y \in K^*$, and $\langle x, y \rangle = 0$. Hence, $x \in \Sigma_K$ and $y \in \Sigma_{K^*}$. It is also clear that $f_K(x) > 0$ and $f_{K^*}(y) > 0$. We may assume that $\min\{f_K(x), f_{K^*}(y)\} < \pi/2$, otherwise we are done. By exchanging the roles of K and K^* if necessary, we may consider now the case $f_K(x) < \pi/2$. Since $\langle x, v \rangle \geq \|v\| \cos f_K(x)$ for all $v \in K$, we have $K \subseteq \mathrm{Rev}(x, f_K(x))$. If we let

$$z := (\sin f_K(x))\, x - (\cos f_K(x))\, y \,,$$

then $\|z\| = 1$ and $\langle x, z \rangle = \sin f_K(x) = \cos[(\pi/2) - f_K(x)]$ and, a posteriori,

$$z \in \mathrm{Rev}(x, (\pi/2) - f_K(x)) = [\mathrm{Rev}(x, f_K(x))]^* \subseteq K^*.$$

From here we get

$$f_{K^*}(y) \geq \arccos \langle y, z \rangle = \arccos[-\cos f_K(x)] = \pi - f_K(x),$$

which proves the right-hand side of (12). □

The next proposition establishes a link between $\theta_{\min}(K)$ and $\beta(K)$. Such a result shows in particular that any proper cone with zero angular gap is of constant opening angle.

Proposition 2.6. *Let $K \in \mathbb{P}(E)$ and $u \in \Sigma_K$. Then*

$$\theta_{\min}(K) \;\leq\; \beta(K) \;\leq\; f_K(u) \;\leq\; \theta_{\max}(K). \tag{13}$$

Proof. The last two inequalities in (13) are tautological, so we focus on the first one. Let u_0 be a solution to problem (10). Moreau's orthogonal decomposition theorem implies the existence of a point $y \in K^* \cap S_E$ such that $\langle y, u_0 \rangle = 0$. We obtain

$$\beta(K) = f_K(u_0) \geq \pi - f_{K^*}(y) \;\geq \pi - \theta_{\max}(K^*) = \theta_{\min}(K)\,,$$

where the first inequality is due to Lemma 2.5. □

Remark 2.7. The first inequality in (13) may be strict. The next example involving an ellipsoidal cone shows that the difference $\beta(K) - \theta_{\min}(K)$ could be almost $\pi/2$. For a small positive ε, let

$$K_\varepsilon = \{x \in \mathbb{R}^3 : \left[(x_1/\varepsilon)^2 + (\varepsilon x_2)^2\right]^{1/2} \leq x_3\}.$$

Computing the exact value of $\beta(K_\varepsilon)$ is time consuming, but clearly $\beta(K_\varepsilon)$ is near the value $\pi/2$. On the other hand, by using [13, Proposition 9], we directly see that $\theta_{\min}(K_\varepsilon) = 2\arctan\varepsilon$ is near 0.

3. On proper cones of constant width

It turns out that a proper cone K is of constant width if and only if K^* is of constant opening angle. We consider this equivalence as a mathematical result, cf. Proposition 3.5, and not as a definition. For introducing the concept of proper cone of constant width in a natural and geometric way, we recall first the concept of width of a convex body Ω in E. Our exposition will be long, but it will pay to spent some time clarifying this matter. The width of Ω in a direction $a \in S_E$, denoted by $w_\Omega(a)$, is the distance between a pair

$$\{x \in E : \langle a, x \rangle = \gamma_i\} \qquad i = 1, 2$$

of parallel supporting hyperplanes to Ω. Equivalently,

$$\omega_\Omega(a) \;=\; \min_{\substack{\gamma_1,\gamma_2 \in \mathbb{R} \\ \Omega \subseteq S_a(\gamma_1,\gamma_2)}} \gamma_2 - \gamma_1\,,$$

where $S_a(\gamma_1, \gamma_2) := \{x \in E : \gamma_1 \leq \langle a, x \rangle \leq \gamma_2\}$ is a *slab* perpendicular to a. The convex body Ω is said to be of constant width if the function $w_\Omega : S_E \to \mathbb{R}$ is constant. The idea of working with parallel supporting hyperplanes does not carry over to proper cones.

Proposition 3.1. *A proper cone in E cannot be enclosed by a slab.*

Proof. Let $K \in \mathbb{P}(E)$. Suppose the existence of $a \in S_E$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 \leq \langle a, x\rangle \leq \gamma_2$ for all $x \in K$. Pick any $z \in K$. By taking $x = tz$ and letting t go to infinity, we get $\langle a, z\rangle = 0$. Hence, K is contained in the hyperplane

$$a^{\perp} := \{z \in E : \langle a, z\rangle = 0\} ,$$

contradicting the fact that K is solid. □

Since a proper cone cannot be placed within a pair parallel hyperplanes, we suggest to change slabs by V-shaped cantilevers. For the sake of brevity, such mathematical objects will be called just cantilevers.

Definition 3.2. A set C in E is called a *cantilever* if it is a polyhedral cone of the special form

$$H_{a,b} := \{x \in E : \langle a, x\rangle \geq 0 , \langle b, x\rangle \geq 0\}$$

for some pair $\{a, b\}$ of noncollinear unit vectors in E. The pair $\{a, b\}$ used to represent C is (unique and) called the normal pair of C. The conjugate pair

$$u = b \ominus a , \ v = a \ominus b$$

is called the tangent pair of C. Finally, the width of C is defined as the angle $\omega(C) := \arccos\langle u, v\rangle$ formed by the tangent pair. □

Cantilevers are unpointed cones and, therefore, they are not proper. Cantilevers have been studied from different points of view and used in a variety of applications, cf. [1, 4, 12]. Any proper cone K can be enclosed by a cantilever. Hence, it makes sense to consider the optimization problem

$$e(K) := \min_{\substack{C \text{ cantilever} \\ K \subseteq C}} \omega(C) \tag{14}$$

which consists in finding a cantilever of smallest width enclosing K. By obvious reasons, we call $e(K)$ the *width* of K. The next proposition explores in detail such an optimization problem. It comes as a surprise that $\theta_{\min}(K)$ is just the width of K. Such an observation provides a new interpretation for the minimum angle of a proper cone.

Proposition 3.3. *Let $K \in \mathbb{P}(E)$. Then*

(a) *The minimum in* (14) *is attained.*

(b) $e(K) = \theta_{\min}(K)$.

(c) *Let C_0 be a cantilever having $\{u_0, v_0\}$ as tangent pair. Then C_0 is a solution to* (14) *if and only if $\{u_0, v_0\}$ is a minor pair of K.*

Proof. In view of Proposition 1.7 (a) and the fact that a cantilever can be identified with a pair of noncollinear unit vectors, problem (14) is about minimizing

$$\vartheta(a, b) := \pi - \arccos\langle a, b\rangle \tag{15}$$

with respect to $a, b \in E$ such that

$$\begin{cases} \text{normalization}: \ \|a\| = 1, \|b\| = 1 \\ \text{noncollinearity}: \ |\langle a, b\rangle| < 1 \\ \text{cantilever containment}: \ K \subseteq H_{a,b}\,. \end{cases} \tag{16}$$

The term $\vartheta(a, b)$ corresponds to the width of the cantilever $H_{a,b}$. Note that the right-hand side of (15) is well defined even if a and b are collinear. So, we may see ϑ as a continuous function on the bi-sphere $S_E \times S_E$. The inclusion $K \subseteq H_{a,b}$ can be rewritten as cone$\{a, b\} \subseteq K^*$ and this is the same thing as saying that $a, b \in K^*$. Hence, (14) can be written as a minimization problem

$$e(K) \ = \min_{a,b\in K^*\cap S_E} \{\pi - \arccos\langle a, b\rangle\} \tag{17}$$

involving the variables $a, b \in E$. The noncollinearity constraint mentioned in (16) has not been incorporated into (17) and there is a good reason for that: the case $a = b$ is clearly non-optimal and the case $a = -b$ is impossible because K^* is pointed. We are dealing then the minimization of a continuous function on a compact set. This completes the proof of (a). Note that (17) yields

$$e(K) \ = \ \pi - \max_{a,b\in K^*\cap S_E} \arccos\langle a, b\rangle \ = \ \pi - \theta_{\max}(K^*) \ = \ \theta_{\min}(K)\,.$$

Observe also that solving the width minimization problem (14) boils down to finding an antipodal pair of K^*. More precisely, a cantilever H_{a_0,b_0} with normal pair $\{a_0, b_0\}$ is a smallest width cantilever enclosing K if and only if $\{a_0, b_0\}$ is an antipodal pair of K^*. As we saw in Proposition 1.7 (c), the later condition is equivalent to saying that the conjugate pair of $\{a_0, b_0\}$ is a minor pair of K. □

We now explain what does it means that a proper cone is of constant width. Motivated by the unfolding of formula (2) into the bivariate formulation (3), we write

$$\begin{aligned} e(K) \ &= \ \min_{a\in K^*\cap S_E} g_K(a) \\ g_K(a) \ &:= \ \min_{b\in K^*\cap S_E} \{\pi - \arccos\langle a, b\rangle\} \end{aligned}$$

and call $g_K : S_E \to \mathbb{R}$ the width function of K. The "width" counterpart of Definition 2.1 reads as follows.

Definition 3.4. Let $K \in \mathbb{P}(E)$. If $a \in \Sigma_{K^*}$, then $g_K(a)$ is called the *width* of K relative to a. We say that K is of constant width if g_K is constant on Σ_{K^*}.

Without further ado we state the result announced at the beginning of the section.

Proposition 3.5. *Let $K \in \mathbb{P}(E)$. Then K is of constant width if and only if K^* is of constant opening angle.*

Proof. The work leading to the introduction and interpretation of g_K has been done in the proof of Proposition 3.3. Here we just need to observe that $g_K(a) = \pi - f_{K^*}(a)$ for all $a \in S_E$. Hence, g_K is constant on Σ_{K^*} if and only is $\mathfrak{f}_{K^*}$ is constant on Σ_{K^*}. □

4. The antipodal mate property and the resolvant property

Let $K \in \mathbb{P}(E)$. For each $u \in S_E$, the maximization problem (1) has the same solution set as the problem which consists in minimizing the linear form $\langle u, \cdot\, \rangle$ on $K \cap S_E$. In other words,

$$\begin{aligned} A_K(u) \;&:=\; \operatorname{argmax}_{v \in K \cap S_E} \arccos\langle u, v\rangle \\ &=\; \operatorname{argmin}_{v \in K \cap S_E} \langle u, v\rangle\,. \end{aligned}$$

The set $A_K(u)$ is always nonempty and contains the possibly empty set

$$M_K(u) \;:=\; \{v \in K \cap S_E : \{u, v\} \text{ is an antipodal pair of } K\}.$$

Each element of $M_K(u)$ is called an *antipodal mate* of u (relative to K). The recent work [14] introduces and gives several characterizations of the so-called antipodal mate property.

Definition 4.1. $K \in \mathbb{P}(E)$ enjoys the *antipodal mate property* if every point in Σ_K has an antipodal mate. □

The resolvant property mentioned in Proposition 4.2 is less easy to visualize. It establishes a surprising link between M_K and the Gauss map G_K of K. Recall that the Gauss map of a general closed convex set Ω assigns to $u \in \mathrm{bd}(\Omega)$ the nonempty set

$$G_\Omega(u) := N_\Omega(u) \cap S_E\,,$$

where N_Ω is the normal cone map of Ω in the sense of convex analysis, i.e.,

$$N_\Omega(u) := \{\xi \in E : \langle \xi, x - u\rangle \leq 0 \text{ for all } x \in \Omega\}.$$

We say that K enjoys the *resolvant property* if

$$M_K(u) = c_K u - s_K G_K(u) \tag{18}$$

for all $u \in \Sigma_K$, where c_K and s_K are trigonometric coefficients given by

$$c_K := \cos[\theta_{\max}(K)]\ ,\ s_K := \sin[\theta_{\max}(K)]. \tag{19}$$

The resolvant identity (18) allows to compute antipodal mates with the help of the Gauss map. Instead of indulging on theoretical considerations concerning the resolvant property, we simply exploit (18) as a technical mathematical tool.

Proposition 4.2. (Seeger and Torki [14]) *Let $K \in \mathbb{P}(E)$. Then the next conditions are equivalent:*

$\mathfrak{c}_8$. *K is of constant opening angle.*

$\mathfrak{c}_9$. *K enjoys the antipodal mate property.*

$\mathfrak{c}_{10}$. *K enjoys the resolvant property.*

$\mathfrak{c}_{13}$. *$A_K(u) = M_K(u)$ for all $u \in \Sigma_K$.*

Proving the equivalence $\mathfrak{c}_9 \Leftrightarrow \mathfrak{c}_{10}$ was an important achievement of [14]. The next proposition shows that the resolvant property suffices to ensure purity.

Proposition 4.3. *If $K \in \mathbb{P}(E)$ enjoys the resolvant property, then K is pure.*

Proof. Let $\mathfrak{c}_{10}$ be true. In addition to (19), we consider the trigonometric coefficients

$$d_K := \cos[\theta_{\min}(K)]\ ,\ t_K := \sin[\theta_{\min}(K)]$$

corresponding to the minimum angle of K. We shall prove that K has zero angular gap or, equivalently, that $d_K = c_K$. Let $\{u, y\}$ be any minor pair of K. Proposition 1.2 implies that $u, y \in \Sigma_K$. Furthermore, the second and third condition in (4) take here the form

$$y - d_K u \in K^*,\ u - d_K y \in K^*\,, \tag{20}$$

respectively. From (20) we deduce that

$$\begin{aligned} \xi &:= -(1/t_K)(y - d_K u) \in G_K(u) \\ z &:= -(u - d_K y) \in N_K(y)\,. \end{aligned}$$

The resolvant identity (18) implies that $v := c_K\, u - s_K \xi$ is an antipodal mate of u. Clearly,

$$L := \operatorname{span}\{u, v\} = \operatorname{span}\{u, \xi\} \ni y.$$

Since $0 < \theta_{\min}(K) \leq \theta_{\max}(K)$, we have in fact

$$y \in P := \operatorname{cone}\{u, v\}\ ,\ y \neq u.$$

We claim that $y = v$. Suppose to the contrary that $y \neq v$. In such a case, y belongs to the relative interior of P and $N_P(y) \subseteq L^\perp$. Since $P \subseteq K$, we also have $N_K(y) \subseteq N_P(y)$ and, therefore,

$$z \in N_P(y) \subseteq L^\perp.$$

On the other hand, $z \in \operatorname{span}\{u, y\} = L$. Hence, $z \in L \cap L^\perp = \{0\}$ and, a posteriori, $u = d_K y$. But this is impossible because $\{u, y\}$ is a pair of noncollinear unit vectors. This confirms that $y = v$, from where we get the desired equality $d_K = c_K$. □

5. Diametric completeness in the angular sense

A convex body Ω in E called *diametrically complete* if

$$\operatorname{diam}(\Omega \cup \{x\}) > \operatorname{diam}(\Omega) \quad \text{for all} \quad x \in E \setminus \Omega\,. \tag{21}$$

A celebrated result of convex geometry, known as Eggleston's theorem, asserts that Ω is diametrically complete if and only if Ω is of constant width, cf. [2, 3]. We present below a conic version of this classical result. The first thing to do is to introduce an angular version of inequality (21).

Definition 5.1. $K \in \mathbb{P}(E)$ is *diametrically complete* in the angular sense if

$$\mathfrak{f}_K(x) > \theta_{\max}(K) \quad \text{for all} \quad x \in S_E \setminus K. \tag{22}$$

The next proposition equates diametric completeness in the angular sense, constancy of the opening angle, and a certain condition involving the 0-sublevel set of the continuous positively homogeneous function $\varphi_K : E \to \mathbb{R}$ given by

$$\varphi_K(x) := c_K\|x\| - \min_{v\in K\cap S_E} \langle x, v\rangle \,.$$

Note that φ_K is defined on the entire space E and not just on the unit sphere as the opening angle function of K. Although φ_K has some advantages over f_K as an analytical tool (tangential convexity, subdifferential calculus, et cetera), its geometric interpretation is less transparent.

Proposition 5.2. *Let $K \in \mathbb{P}(E)$. Then the next conditions are equivalent:*

$\mathfrak{c}_8$. *K is of constant opening angle.*

$\mathfrak{c}_{11}$. *K is diametrically complete in the angular sense.*

$\mathfrak{c}_{14}$. *$K = \{x \in E : \varphi_K(x) \leq 0\}$.*

Proof. The equivalence $\mathfrak{c}_{11} \Leftrightarrow \mathfrak{c}_{14}$ is proven [14, Proposition 3.12], so we focus on $\mathfrak{c}_8 \Leftrightarrow \mathfrak{c}_{11}$. Suppose that $\mathfrak{c}_8$ is false. Then $\mathfrak{f}_K(u) < \theta_{\max}(K)$ for some $u \in \Sigma_K$. Since $\mathfrak{f}_K$ is continuous and u belongs to the boundary of K, there exists a point $x \in S_E\backslash K$ near u such that $\mathfrak{f}_K(x) \leq \theta_{\max}(K)$. Hence, $\mathfrak{c}_{11}$ is false. Suppose now that $\mathfrak{c}_8$ is true and pick any $x \in S_E\backslash K$. We must prove the inequality (22). There is no loss of generality in assuming that x is close to $\mathrm{bd}(K)$. In such a case, $x \notin -K^*$ and the metric projection $P_K(x)$ of x onto K is a nonzero vector. Hence $z := \|P_K(x)\|^{-1}x$ is well defined and

$$\begin{aligned} u := P_K(z) = \|P_K(x)\|^{-1}P_K(x) \quad &\in \quad \Sigma_K \\ \xi := \|z-u\|^{-1}(z-u) \quad &\in \quad G_K(u). \end{aligned}$$

Since K enjoys the resolvant property following Proposition 4.2, the unit vector $v := c_K u - s_K \xi$ is an antipodal mate of u. Given the definition of ξ, we get

$$u = (1/\gamma)\left(v + \|z-u\|^{-1} s_K\, z\right)$$

with $\gamma := c_K + \|z-u\|^{-1}s_K$. If $c_K \geq 0$, then γ is positive. If $c_K < 0$, then we take x near $\mathrm{bd}(K)$, so that $\|z-u\|$ is small enough and γ is positive again. Hence, $u \in \mathrm{cone}\{v, z\} = \mathrm{cone}\{v, x\}$ and

$$\mathfrak{f}_K(x) \geq \arccos\langle x, v\rangle = \arccos\langle x, u\rangle + \arccos\langle u, v\rangle > \arccos\langle u, v\rangle = \theta_{\max}(K).$$

This proves (22) and implies that $\mathfrak{c}_{11}$ is true. □

The equivalence $\mathfrak{c}_8 \Leftrightarrow \mathfrak{c}_{11}$ has a striking resemblance with Eggleston's theorem, except for the fact that we are comparing diametric completeness in the angular sense with the constancy of f_K and not with the constancy of g_K. However, as well shall see later, both constancy conditions are equivalent.

6. On proper cones of full antipodality range

Let $\mathrm{AP}(K)$ denote the set of antipodal pairs of a proper cone K. Depending on the choice of K, the set $\mathrm{AP}(K)$ could be finite or infinite. The *antipodality range* of K is defined as

$$\Upsilon(K) := \bigcup_{\{u,v\}\in \mathrm{AP}(K)} \mathrm{span}\{u,v\}\,.$$

This set is a union of finitely many or infinitely many two-dimensional linear subspaces of E. Although $\Upsilon(K)$ is stable under scalar multiplication, it is not stable under addition. In general, $\Upsilon(K)$ is not a linear subspace.

Definition 6.1. Let $K \in \mathbb{P}(E)$. K is of *full antipodality range* if $\Upsilon(K) = E$, that is to say,

$$\begin{cases} \text{for all } \ x \in E, \ \text{ there is an antipodal pair } \{u,v\} \text{ of } K \\ \text{such that } \ x \text{ belongs to the plane spanned by } u \text{ and } v. \end{cases}$$

The concept of antipodality range is of interest also beyond the context of this work. The next example illustrates how to compute the antipodality range of an ellipsoidal cone.

Example 6.2. Consider an ellipsoidal cone in $\mathbb{R}^n$ written in the format

$$E(s) := \left\{x \in \mathbb{R}^n : \left[(x_1/s_1)^2 + \ldots + (x_{n-1}/s_{n-1})^2\right]^{1/2} \leq x_n\right\},$$

where the entries of $s := (s_1,\ldots,s_{n-1})^\top$ are positive reals arranged in nonincreasing order. Let r be the multiplicity of s_1, i.e., $s_1 = \ldots = s_r > s_{r+1} \geq \ldots \geq s_{n-1}$. Theorem 1 in [8] implies that $\{u,v\}$ is an antipodal pair of $E(s)$ if and only if

$$\begin{aligned} u &= (z_1,\ldots,z_r,0,\ldots,0,\gamma)^\top \\ v &= (-z_1,\ldots,-z_r,0,\ldots,0,\gamma)^\top \end{aligned}$$

with $\gamma := (s_1^2+1)^{-1/2}$ and $z_1^2 + \ldots + z_r^2 = (s_1^2+1)^{-1}s_1^2$. Hence,

$$\Upsilon(E(s)) = \{x \in \mathbb{R}^n : x_{r+1} = 0,\ldots,x_{n-1} = 0\}$$

is a linear subspace of dimension $r+1$. Note that $\Upsilon(E(s)) = \mathbb{R}^n$ if and only if $r = n-1$. In conclusion, an ellipsoidal cone is of full antipodality range if and only if it is a revolution cone.

Proposition 6.3. *Let $K \in \mathbb{P}(E)$. Then K is of full antipodality range if and only if K enjoys the antipodal mate property.*

Proof. Suppose that K enjoys the antipodal mate property. By Proposition 4.2, K enjoys the resolvant property and we also have $A_K = M_K$ on Σ_K. Let $x \in S_E$. We must prove that $x \in \Upsilon(K)$. We consider first the case $x \notin -K$. Pick any

$$y^* \in A_K(x) = \mathrm{argmin}_{y\in K\cap S_E}\langle x,y\rangle\,. \tag{23}$$

Such an optimal solution y^* is necessarily on the boundary of K. Hence, $y^* \in \Sigma_K$ and

$$A_K(y^*) = M_K(y^*)\,. \tag{24}$$

By writing down the optimality conditions for problem (23), we get $\langle x, y^*\rangle y^* - x \in N_K(y^*)$. But the hypothesis $x \notin -K$ implies that $|\langle x, y^*\rangle| \neq 1$. Hence,

$$\xi := \frac{\langle x, y^*\rangle y^* - x}{\sqrt{1-\langle x, y^*\rangle^2}}$$

is well defined and belongs to $G_K(y^*)$. By using (24) and the resolvant identity (18) at y^*, we deduce that $z := c_K y^* - s_K \xi$ is an antipodal mate of y^*. Clearly, $x \in \operatorname{span}\{y^*, \xi\}$ and $\xi \in \operatorname{span}\{y^*, z\}$. Hence,

$$x \in \operatorname{span}\{y^*, z\} \subseteq \Upsilon(K).$$

We now consider the case $x \in -K$. Note that $-x \notin -K$ because K is pointed. By using the same argument as before, we deduce that $-x \in \Upsilon(K)$. Hence, $x \in \Upsilon(K)$.
Conversely, suppose that K is of full antipodality range. Pick any $x \in S_E \backslash K$. Since $\Upsilon(K) = E$, there exists an antipodal pair $\{u, v\}$ of K such that $x \in L := \operatorname{span}\{u, v\}$. We view $P := \operatorname{conv}\{u, v\}$ as a proper cone in the two-dimensional space L. Since

$$\theta(x, K) \,\geq\, \max_{w \in P \cap S_L} \arccos\langle x, w\rangle \,>\theta(P) \,=\, \theta(K)\,,$$

we deduce that K is DCA. A posteriori, K enjoys the antipodal mate property. $\square$

7. By way of conclusion

For a proper cone K the property $\mathfrak{c}_0$ means that K is pure, i.e., every critical pair of the cone is an antipodal pair. All in all, we have 14 characterizations of purity at our disposal:

$$\mathfrak{c}_0 \;\Leftrightarrow\; \mathfrak{c}_1 \Leftrightarrow \ldots \Leftrightarrow \mathfrak{c}_5 \quad \text{by Theorem 1.8} \tag{25}$$

$$\mathfrak{c}_0 \;\Leftrightarrow\; \mathfrak{c}_6 \Leftrightarrow \ldots \Leftrightarrow \mathfrak{c}_{13} \quad \text{by Theorem 1.10} \tag{26}$$

$$\mathfrak{c}_0 \;\Leftrightarrow\; \mathfrak{c}_{13} \Leftrightarrow \mathfrak{c}_{14} \quad \text{by auxiliary results.} \tag{27}$$

Some of these equivalences are almost tautological. For instance, it is clear that K is pure if and only if K has zero angular gap. However, several of these characterizations have an entirely different geometric interpretation. When we started our research on this subject, we were not aware that purity has something to do with the antipodal mate property, with the resolvant property, or with diametric completeness in the angular sense, just to mention a few options. It took us a good deal of intellectual effort to present these geometric properties in an harmonious way and to prove that all these properties are just different faces of the same mathematical concept. We now explain in a few words why the properties listed in (26)-(27) are all equivalent.

- The chain $\mathfrak{c}_8 \Leftrightarrow \mathfrak{c}_9 \Leftrightarrow \mathfrak{c}_{10} \Leftrightarrow \mathfrak{c}_{13}$ is taken care in [14], a paper of ours that forms a diptych with the present one. Despite its technical character, the resolvant property $\mathfrak{c}_9$ is quite interesting because it explains how construct antipodal mates by using the Gauss map G_K of the cone.

- $\mathfrak{c}_6 \Leftrightarrow \mathfrak{c}_8$ is proven in Proposition 2.3. The oriented distance function Δ_{K^*} enters into the picture because it serves to represent the opening angle function f_K.
- Proposition 2.6 shows that $\mathfrak{c}_4 \Rightarrow \mathfrak{c}_8$. Hence, $\mathfrak{c}_0 \Rightarrow \mathfrak{c}_8$.
- $\mathfrak{c}_8 \Leftrightarrow \mathfrak{c}_{11}$ is shown in Proposition 5.2 and $\mathfrak{c}_{11} \Leftrightarrow \mathfrak{c}_{14}$ is proven in [14].
- $\mathfrak{c}_9 \Leftrightarrow \mathfrak{c}_{12}$ is shown in Proposition 6.3.
- Proposition 4.3 shows that $\mathfrak{c}_{10} \Rightarrow \mathfrak{c}_0$. This completes the proof of (26)-(27), except that the property $\mathfrak{c}_7$ is missing. For incorporating $\mathfrak{c}_7$ into the list, we combine Proposition 3.5 and the fact that purity is preserved after passing to dual cones.

Proposition 2.3 to Proposition 6.3 were stated with the purpose of proving Theorem 1.10. Such auxiliary propositions contain however more information than strictly needed. For instance, Proposition 3.3 can be considered as a cornerstone result for building smallest width cantilevers that enclose a given proper cone. This theme deserves to be explored with greater attention, but this is not the place to indulge on such a matter.

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