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On Error Bounds of Inequalities in Asplund Spaces

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Dedicated to the memory of Hedy Attouch.

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Error bounds are central objects in optimization theory and its applications. They were for a long time restricted only to the theory before becoming over the course of time a field by itself. This paper is devoted to the study of error bounds of a general inequality defined by a proper lower semicontinuous function on an Asplund space. If one drops the convexity assumption, the dual characterization of error bounds for a general inequality may not be valid, but even in this case, several dual necessary conditions are still obtained in terms of Fréchet/Mordukhovich subdifferentials of the concerned function at points in the solution set. Moreover, for an inequality defined by a convex-composite function that is to say by a function which is the composition of a convex function with a smooth mapping, such dual conditions also turn out to be sufficient to have the error bound property. Our work is an extension of the results on dual characterizations of convex inequalities to the possibly non-convex case.

Keywords: Error bound, Fréchet subdifferential, Mordukhovich subdifferential, fuzzy calculus, Asplund space.

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1. Introduction

The main goal in this paper is to study error bounds of a general inequality in Asplund spaces. The starting point of theory of error bounds can be traced back to

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the pioneering work by Hoffman [19] for systems of affine functions in which this author proved that for a given matrix and a vector, the distance from the candidate point to the solution set is bounded from above by some scalar constant times the norm of the residual error. Hoffman's result was extensively and intensively studied by many authors. Robinson [43] extended Hoffman's work to a system of convex inequalities which defines a bounded feasible region with nonempty interior. Subsequently, Mangasarian [33] derived a global error bound for differentiable convex inequalities under the Slater condition and an asymptotic constraint qualification. Then, Auslender and Crouzeix [2] used closed proper convex functions satisfying an interior condition on their domains to improve Mangassarian's work for systems which are not necessary differentiable on the whole space. Pang [39] gave a comprehensive survey of an extensive theory and rich applications of error bounds for inequalities and optimization systems and solution sets to equilibrium problems. Lewis and Pang [30] gave a systematic and unified treatment for the existence of a global error bound for a convex inequality system. For the summary of the theory of error bounds and their various applications, the readers are invited to consult bibliographies [1, 4, 7, 13, 17, 20, 25, 28, 27, 29, 31, 36, 40, 48] and references therein for more details.

Error bounds have been well recognized in optimization and variational analysis and have played an important role in various its aspects such as sensitivity analysis of linear programs (cf. [42, 44]), the analysis of convergence of descent methods for linearly constrained minimization(cf. [18, 26, 32, 46]), the convex feasibility problem consisting of finding a point in the intersection of a finite collection of closed convex sets (cf. [5, 6, 7]) and the domain of image reconstruction (cf. [12]). Error bounds have also been shown to be closely related to weak and sharp minima of functions, metric regularity/subregularity, and the Aubin property/calmness of set-valued mappings (cf. [4, 9, 10, 11, 23, 24, 43, 52, 53] and references therein).

To study error bounds, an extensive literature is devoted to providing dual characterizations and criteria in terms of subdifferentials. We know that two directions have been followed to describe error bounds. One direction uses the subdifferentials of the given function at points outside the solution set. To the best of our knowledge, [22] was one of the first papers of this type to state sufficient conditions for the error bounds of a constraint system in terms of the Clarke subdifferential. In 2009, Ngai and Théra [38] discussed error bounds for systems of lower semi-continuous functions in Asplund spaces and provided sufficient conditions in terms of Fréchet subdifferentials of the functions in question at points outside the solution set.

A general classification scheme of necessary and sufficient conditions using several derivative-like objects from both primal space and dual space to characterize error bounds for lower semicontinuous extended-real-valued functions on metric, normed, Banach or Asplund spaces were given by Fabian, Henrion, Kruger and Outrata [16] and Cuong and Kruger [14]. The other direction focuses on subdifferentials of the given function at points inside the solution set. In this direction, Lewis and Pang [30] studied error bounds for systems of convex inequalities and provided necessary conditions via subdifferentials and normal cones. In 2003, Ngai and Théra [21] provided an error bound estimate and an implicit multifunction theorem in terms of smooth subdifferentials and abstract subdifferentials. In 2004, Zheng and Ng [50] proved dual characterizations of error bounds for convex inequalities in terms of subdifferentials and normal cones.

Inspired by [16, 38, 50, 14], we aim to study error bounds of a general inequality defined by a proper semi-continuous function on an Asplund space and also to give dual criteria guaranteeing error bounds. If one drops the convexity assumption, the dual characterization of error bounds for a general inequality may not be valid, but even in this case several necessary conditions for the error bounds may be still obtained in terms of Fréchet/Mordukhovich subdifferentials of the function given at points of the solution set. Moreover, such conditions also turn out to be sufficient for the error bounds of the convex-composite inequality (defined by a composition of a convex function with a smooth mapping). Our work is an extension of dual characterizations of error bounds via subdifferentials to the possibly non-convex case.

The paper is organized as follows. Several preliminaries and known results will be given in Section 2. Section 3 is devoted to error bounds of a general inequality in an Asplund space. Several dual necessary conditions for error bounds are obtained in terms of Fréchet/Mordukhovich subdifferentials of the concerned function at points in the solution set (see Theorems 3.1 to 3.3). Then we further discuss error bounds of a convex-composite inequality (defined by a composition of a convex function with a smooth mapping) and prove that such dual conditions are also sufficient for error bounds in this case (see Theorems 3.5 and 3.7). Conclusions of this paper are presented in Section 4.

2. Preliminaries

Let $\mathbb{X}$ be a Banach space whose norm is denoted by $\|\cdot\|$. $\mathbb{X}^*$ stands for the (continuous) dual of $\mathbb{X}$ equipped with the dual norm $\|\cdot\|_*$, while $\mathbb{B}_{\mathbb{X}}$ and $\mathbb{B}_{\mathbb{X}^*}$ are respectively the closed unit ball in $\mathbb{X}$ and $\mathbb{X}^*$ and $\mathbb{B}(x,r) = \{y \in \mathbb{X} : \|x-y\| < r\}$ is the open ball ball centered at x with radius equal to $r > 0$. Given a multifunction $\Phi : \mathbb{X} \rightrightarrows \mathbb{X}^*$, the symbol

$$\operatorname*{Limsup}_{y\to x} \Phi(x) := \left\{ x^* \in \mathbb{X}^* : \begin{array}{l} \exists \text{ sequences } x_n \to x \text{ and } x_n^* \xrightarrow{w^*} x^* \\ \text{with } \ x_n^* \in \Phi(x_n) \ \text{ for all } n \in \mathbb{N} \end{array} \right\}$$

signifies the sequential Painlevé-Kuratowski outer/upper limit of $\Phi(x)$ as $y \to x$, where the notation $x_n^* \xrightarrow{w^*} x^*$ means that the sequence $\{x_n^*\}$ converges to x^* with respect to the weak*-topology.

For a subset Ω of $\mathbb{X}$, we denote by $\mathrm{cl}(\Omega)$ and $\mathrm{bd}(\Omega)$ the closure and the boundary of Ω, respectively. For any $x \in \mathbb{X}$, the distance from x to the subset Ω is given by

$$\mathbf{d}(x,\Omega) := \inf\{\|x-u\| : u \in \Omega\}.$$

Let A be a closed set and $a \in A$. For $\varepsilon \geq 0$, we denote

$$\hat{N}_\varepsilon(A,a) := \left\{ x^* \in \mathbb{X}^* : \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x-a\rangle}{\|x-a\|} \leq \varepsilon \right\}$$

the set of ε-*normals* to A at a, where $x \xrightarrow{A} a$ means that x tends to a while staying in A.

When $\varepsilon = 0$, we will write $\hat{N}(A, a)$ instead of $\hat{N}_0(A, a)$. $\hat{N}(A, x)$ is a closed convex cone called the *Fréchet normal cone* to A at a.

The *Mordukhovich (or limiting) normal cone of A at a is the set $N(A, a)$ which is defined by*

$$N(A, a) := \operatorname*{Limsup}_{x \xrightarrow{A} a, \varepsilon \downarrow 0} \hat{N}_\varepsilon(A, x).$$

It is known that $\hat{N}(A, a) \subset N(A, a)$.

If A is convex, the Fréchet normal cone coincides with the Mordukhovich normal cone and reduces to the normal cone in the sense of convex analysis; that is,

$$\hat{N}(A, a) = N(A, a) = \{x^* \in \mathbb{X}^* : \langle x^*, x - a \rangle \leq 0 \ \ \forall x \in A\}. \tag{1}$$

Let $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and consider $x \in \operatorname{dom} f := \{y \in \mathbb{X} : f(y) < +\infty\}$. We denote by

$$\operatorname{epi} f := \{(x, r) \in \mathbb{X} \times \mathbb{R} : f(x) \leq r\}$$

the *epigraph* of f. We denote by

$$\hat{\partial} f(x) := \{x^* \in \mathbb{X}^* : (x^*, -1) \in \hat{N}(\operatorname{epi} f, (x, f(x)))\}$$

the *Fréchet subdifferential* of f at x and by

$$\partial f(x) := \{x^* \in \mathbb{X}^* : (x^*, -1) \in N(\operatorname{epi} f, (x, f(x)))\},$$

the *Mordukhovich subdifferential* of f at x, respectively. We denote by

$$\hat{\partial}^\infty f(x) := \{x^* \in \mathbb{X}^* : (x^*, 0) \in \hat{N}(\operatorname{epi} f, (x, f(x)))\}$$

the *Fréchet singular subdifferential* of f at x and by

$$\partial^\infty f(x) := \{x^* \in \mathbb{X}^* : (x^*, 0) \in N(\operatorname{epi} f, (x, f(x)))\}$$

the *Mordukhovich singular subdifferential* of f at x, respectively. When f is Lipschitzian around x, the Fréchet and Mordukhovich singular subdifferentials of f at x are trivial; that is, $\hat{\partial}^\infty f(x) = \partial^\infty f(x) = \{0\}$.

Well known are the facts that $\hat{\partial} f(x) \subseteq \partial f(x)$, and

$$\hat{\partial} f(x) = \left\{ x^* \in \mathbb{X}^* : \liminf_{u \to x} \frac{f(u) - f(x) - \langle x^*, u - x \rangle}{\|u - x\|} \geq 0 \right\}.$$

Recall that a Banach space $\mathbb{X}$ is said to be an Asplund space if every continuous convex real-valued function defined on a nonempty open convex subset $D \subseteq \mathbb{X}$ is Fréchet differentiable at each point of some dense G_δ subset of D. There is a plethora of properties equivalent to Asplundness and the readers are invited to consult [41] or [49] for the definition of Asplund spaces and their equivalences. It is well known that among all these equivalences, $\mathbb{X}$ is Asplund if and only if every separable subspace of $\mathbb{X}$ has a separable dual. In particular, every reflexive Banach space is an Asplund space.

Variational analysis in the framework of Asplund spaces has been deeply carried out by Mordukhovich and Shao [35] and developed in several books and papers, see [34] and the recent book by Thibault [45, Comment on p. 555]. Two of the most important results in [35] are these following formulae that will be a key ingredient in our study:

The first one:
$$N(A,a) = \operatorname*{Limsup}_{x \xrightarrow{A} a} \hat{N}(A,x). \tag{2}$$

Equivalently, $x^* \in N(A,a)$ if and only if there exist $x_n \xrightarrow{A} a$ and $x_n^* \xrightarrow{w^*} x^*$ such that $x_n^* \in \hat{N}(A,x_n)$ for all n.

The second one:
$$\partial f(x) = \operatorname*{Limsup}_{y \xrightarrow{f} x} \hat{\partial} f(y) \text{ and } \partial^\infty f(x) = \operatorname*{Limsup}_{y \xrightarrow{f} x, \lambda \downarrow 0} \lambda \hat{\partial} f(y), \tag{3}$$

where the notation $y \xrightarrow{f} x$ means that $y \to x$ with $f(y) \to f(x)$.

Equivalently, $x^* \in \partial f(x)$ if and only if there exist $x_n \xrightarrow{f} x$ and $x_n^* \xrightarrow{w^*} x^*$ such that $x_n^* \in \hat{\partial} f(x_n)$ for all n, and $x^* \in \partial^\infty f(x)$ if and only if there exist $x_n \xrightarrow{f} x$, $\lambda_n \downarrow 0$ and $x_n^* \in \hat{\partial} f(x_n)$ such that $\lambda_n x_n^* \xrightarrow{w^*} x^*$.

The following important fuzzy calculus result was proved by Fabian [15]. Readers could also consult [15, Theorem 3] and [34, Theorem 2.33] for the details.

Lemma 2.1. *Let $\mathbb{X}$ be an Asplund space and $f_i : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ $(i = 1,2)$ be proper lower semicontinuous functions such that one of them is Lipschitz continuous around $x \in dom\, f_1 \cap dom\, f_2$. Then for any $x^* \in \hat{\partial}(f_1 + f_2)(x)$ and $\varepsilon > 0$, there exist $x_i \in \mathbb{B}(x,\varepsilon)$ with $|f_i(x_i) - f_i(x)| < \varepsilon$ $(i = 1,2)$ such that*

$$x^* \in \hat{\partial} f_1(x_1) + \hat{\partial} f_2(x_2) + \varepsilon \mathbb{B}_{\mathbb{X}^*}.$$

The following Lemma 2.2 (see [37, Lemma 3.6] and also [3, Lemma 3.7]) and Lemma 2.3 (see [51, Theorem 3.1]) will be used in our analysis later.

Lemma 2.2. *Let $\mathbb{X}$ be an Asplund space and A be a nonempty closed subset of $\mathbb{X}$. Let $x \in \mathbb{X}$ and $x^* \in \hat{\partial}\mathbf{d}(\cdot,A)(x)$. Then for any $\varepsilon > 0$ there exist $a \in A$ and $a^* \in \hat{N}(A,a)$ such that*

$$\|x - a\| < \mathbf{d}(x,A) + \varepsilon \quad \textit{and} \quad \|x^* - a^*\| < \varepsilon.$$

Lemma 2.3. *Let $\mathbb{X}$ be an Asplund space, A be a nonempty closed subset of $\mathbb{X}$ and $x \notin A$. Then for any $\beta \in (0,\ 1)$ there exist $z \in \mathrm{bd}(A)$ and $z^* \in \hat{N}(A,z)$ with $\|z^*\| = 1$ such that*

$$\beta\|x - z\| < \min\{\mathbf{d}(x,A),\ \langle z^*, x - z\rangle\}.$$

The following lemma is cited from [8, Theorem 3.5.8].

Lemma 2.4. *Let $\mathbb{X}$ be Fréchet smooth Banach space and $f_i : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ $(i = 1, \cdots, m)$ be proper lower semicontinuous functions and $f := \max\{f_1, \cdots, f_m\}$. Suppose that $x^* \in \hat{\partial} f(\bar{x})$. Then for any $\epsilon > 0$ and any weak* neighborhood U^* of 0 in $\mathbb{X}^*$, there exist $x_k \in \mathbb{B}(\bar{x}, \epsilon)$ with $|f_k(\bar{x}) - f_k(x_k)| < \epsilon$, $x_k^* \in \hat{\partial} f_k(x_k)$ and $\lambda_k \geq 0 (k = 1, \cdots, m)$ such that*

$$\Big|\sum_{k=1}^{m} \lambda_k - 1\Big| < \epsilon \quad and \quad x^* \in \sum_{k=1}^{m} \lambda_k x_k^* + U^*. \tag{4}$$

We conclude this section with the following lemma which is of independent interest.

Lemma 2.5. *Let $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and $(z, r) \in \operatorname{epi} f$.*

Then
$$\hat{N}(\operatorname{epi} f, (z, r)) \subseteq \hat{N}(\operatorname{epi} f, (z, f(z))). \tag{5}$$

Assume further that f is continuous at z.

Then
$$N(\operatorname{epi} f, (z, r)) \subseteq N(\operatorname{epi} f, (z, f(z))). \tag{6}$$

Proof. Let $(x^*, t) \in \hat{N}(\operatorname{epi} f, (z, r))$. Then

$$\limsup_{(x,\alpha) \xrightarrow{\operatorname{epi} f} (z,r)} \frac{\langle (x^*, t), (x - z, \alpha - r) \rangle}{\|(x - z, \alpha - r)\|} \leq 0.$$

Then for any $(y, \beta) \xrightarrow{\operatorname{epi} f} (z, f(z))$, one has $(y, \beta + r - f(z)) \xrightarrow{\operatorname{epi} f} (z, r)$ as $f(z) \leq r$ and thus

$$\limsup_{(y,\beta) \xrightarrow{\operatorname{epi} f} (z,f(z))} \frac{\langle (x^*, t), (y - z, (\beta + r - f(z)) - r) \rangle}{\|(x - z, \beta - f(z))\|} \leq 0.$$

This implies that $(x^*, t) \in \hat{N}(\operatorname{epi} f, (z, f(z)))$ and thus (5) holds.

Let $(x^*, t) \in N(\operatorname{epi} f, (z, r))$. In consequence there exist $\epsilon_k \downarrow 0, (z_k, r_k) \xrightarrow{\operatorname{epi} f} (z, r)$ and $(x_k^*, t_k) \xrightarrow{w^*} (x^*, t)$ such that $(x_k^*, t_k) \in \hat{N}_{\epsilon_k}(\operatorname{epi} f, (z_k, r_k))$ for all k. Then for any k, one has

$$\limsup_{(x,\alpha) \xrightarrow{\operatorname{epi} f} (z_k,r_k)} \frac{\langle (x_k^*, t_k), (x - z_k, \alpha - r_k) \rangle}{\|(x - z_k, \alpha - r_k)\|} \leq \epsilon_k. \tag{7}$$

Let $(y, \beta) \xrightarrow{\operatorname{epi} f} (z_k, f(z_k))$. Then $(y, \beta + r_k - f(z_k)) \xrightarrow{\operatorname{epi} f} (z_k, r_k)$ as $f(z_k) \leq r_k$ and it follows from (7) that

$$\limsup_{(y,\beta) \xrightarrow{\operatorname{epi} f} (z_k,f(z_k))} \frac{\langle (x_k^*, t_k), (y - z, (\beta + r - f(z)) - r) \rangle}{\|(x - z_k, \beta - f(z_k))\|} \leq \epsilon_k.$$

This means that $(x_k^*, t_k) \in \hat{N}_{\epsilon_k}(\operatorname{epi} f, (z_k, f(z_k)))$. Noting that f is continuous at z, it follows that $(x^*, t) \in N(\operatorname{epi} f, (z, f(z)))$ as $z_k \to z$. Hence (6) holds. The proof is complete. □

Remark 2.6. It should be noted that (6) may not hold necessarily if dropping the continuity assumption. For example, let $\mathbb{X} := \mathbb{R}$ and $f : \mathbb{X} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1, \ x > 0, \\ x, \ x \leq 0. \end{cases}$$

Let $(z, r) := (0, 1)$. Then one can verify that f is not continuous at z and

$$N(\operatorname{epi} f, (z, f(z))) = \{(t, s) : t + s \geq 0, s \leq 0\}.$$

Further one has $(0, -1) \in N(\operatorname{epi} f, (z, r)) \backslash N(\operatorname{epi} f, (z, f(z)))$, which implies that (6) does not hold.

3. Main results

In this section, we study error bounds of a general inequality defined by a proper lower semicontinuous function on an Asplund space and we aim to give dual criteria for ensuring error bounds. As main results, it is proved that several necessary dual conditions for error bounds are provided in terms of Fréchet/Mordukhovich subdifferentials of the concerned function at points in the solution set. Further, for a convex-composite inequality defined by a composition of a convex function with a smooth mapping, such dual conditions are proved to be also sufficient for the existence of error bounds. We begin with the definition of error bounds of a general inequality.

Throughout of this section, unless stated otherwise, we always suppose that $\mathbb{X}$ is an Asplund space and $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function.

We consider the following inequality:

$$\text{Find } \ x \in \mathbb{X} \text{ such that } f(x) \leq 0. \tag{8}$$

We denote by $\mathbf{S}_f := \{x \in \mathbb{X} : f(x) \leq 0\}$ the solution set of inequality (8).

Recall that inequality (8) is said to have a *local error bound* at $\bar{x} \in \mathbf{S}_f$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$\mathbf{d}(x, \mathbf{S}_f) \leq \tau f_+(x) \ \ \forall x \in \mathbb{B}(\bar{x}, \delta), \tag{9}$$

where $f_+(x) := \max\{f(x), 0\}$. The scalar

$$\tau(f, \bar{x}) := \inf\{\tau > 0 : \exists\, \delta > 0 \text{ s.t. } \ (9) \text{ holds}\} \tag{10}$$

is called the *modulus of local error bound* of f at $\bar{x}$.

The following theorem gives a necessary condition for local error bounds of inequality (8) in terms of Mordukhovich subdifferentials and singular subdifferentials.

Theorem 3.1. *Suppose that inequality* (8) *has a local error bound at $\bar{x} \in \mathbf{S}_f$. Then there exists $\sigma \in (0, +\infty)$ such that*

$$N(\mathbf{S}_f, x) \subseteq [0, +\infty)\partial f(x) + \partial^\infty f(x) \tag{11}$$

holds for all $x \in \mathbb{B}(\bar{x}, \sigma)$ with $f(x) = 0$.

Proof. Since inequality (8) has a local error bound at $\bar{x}$, there exist $\tau, \delta \in (0, +\infty)$ such that (9) holds. For any $(x, r) \in \mathbb{X} \times \mathbb{R}$, let $\|(x, r)\|_\tau := \frac{1}{\tau}\|x\| + |r|$. Then $(\mathbb{X} \times \mathbb{R}, \|\cdot\|_\tau)$ is a Banach space and the unit ball of the dual space of $(\mathbb{X} \times \mathbb{R}, \|\cdot\|_\tau)$ is $(\frac{1}{\tau}\mathbb{B}_{\mathbb{X}^*}) \times \mathbb{B}_{\mathbb{R}}$. Let $\delta_0 := \frac{\delta}{2}$. We claim that

$$\mathbf{d}(x, \mathbf{S}_f) \leq \tau(\mathbf{d}_{\|\cdot\|_\tau}((x, r), \operatorname{epi} f) + |r|) \ \ \forall (x, r) \in \mathbb{B}(\bar{x}, \delta_0) \times \mathbb{R}. \tag{12}$$

Indeed, suppose on the contrary that there exists $(x_0, r_0) \in \mathbb{B}(\bar{x}, \delta_0) \times \mathbb{R}$ such that

$$\mathbf{d}(x_0, \mathbf{S}_f) > \tau(\mathbf{d}_{\|\cdot\|_\tau}((x_0, r_0), \operatorname{epi} f) + |r_0|).$$

Then there exists $(u, r) \in \operatorname{epi} f$ such that

$$\mathbf{d}(x_0, \mathbf{S}_f) > \tau(\frac{1}{\tau}\|u - x_0\| + |r_0 - r| + |r_0|),$$

and thus $\mathbf{d}(x_0, \mathbf{S}_f) > \|u - x_0\| + \tau|r|$. Note that $f(u) \leq r$ and one can verify that

$$\mathbf{d}(x_0, \mathbf{S}_f) > \|u - x_0\| + \tau f_+(u). \tag{13}$$

Since $\|u - \bar{x}\| \leq \|u - x_0\| + \|x_0 - \bar{x}\| < \mathbf{d}(x_0, \mathbf{S}_f) + \|x_0 - \bar{x}\| \leq 2\|x_0 - \bar{x}\| < \delta$, it follows from (9) and (13) that

$$\mathbf{d}(x_0, \mathbf{S}_f) > \|u - x_0\| + \tau f_+(u) \geq \|u - x_0\| + \mathbf{d}(u, \mathbf{S}_f) \geq \mathbf{d}(x_0, \mathbf{S}_f),$$

and a contradiction. Hence (12) holds.

Take $\sigma := \frac{\delta_0}{2}$. Let $x \in \mathbb{B}(\bar{x}, \sigma)$ be such that $f(x) = 0$ and $x^* \in N(\mathbf{S}_f, x)$. Then there exist $x_n \xrightarrow{\mathbf{S}_f} x$ and $x_n^* \xrightarrow{w^*} x^*$ such that $x_n^* \in \hat{N}(\mathbf{S}_f, x_n)$. Applying the Banach-Steinhaus Theorem, we can assume that $\|x_n^*\| \leq M$ for some $M > 0$ and all n. Then for any n sufficiently large, by [34, Corollary 1.96], one has

$$\frac{x_n^*}{M} \in \hat{N}(\mathbf{S}_f, x_n) \cap \mathbb{B}_{\mathbb{X}^*} = \hat{\partial}\mathbf{d}(\cdot, \mathbf{S}_f)(x_n)$$

and thus for any $\varepsilon > 0$ there exists $\delta_1 \in (0, \sigma - \|x_n - x\|)$ such that

$$\left\langle \frac{x_n^*}{M}, u - x_n \right\rangle \leq \mathbf{d}(u, \mathbf{S}_f) + \tau\varepsilon\|u - x_n\| \quad \forall u \in \mathbb{B}(x_n, \delta_1). \tag{14}$$

Since $\mathbb{B}(x_n, \delta_1) \subseteq \mathbb{B}(\bar{x}, \delta_0)$, it follows from (12) and (14) that

$$\left\langle \frac{x_n^*}{M}, u - x_n \right\rangle \leq \tau(\mathbf{d}_{\|\cdot\|_\tau}((u, r), \operatorname{epi} f) + |r|) + \tau\varepsilon\|u - x_n\| \quad \forall (u, r) \in \mathbb{B}(x_n, \delta_1) \times \mathbb{R}.$$

Let $\Phi(u, r) := \mathbf{d}_{\|\cdot\|_\tau}((u, r), \operatorname{epi} f) + |r|$. Then one has

$$\left(\frac{x_n^*}{\tau M x}, 0\right) \in \hat{\partial}\Phi(x_n, 0). \tag{15}$$

By Lemma 2.1, there exist $(z_n, r_n) \in \mathbb{B}(x_n, \frac{1}{n}) \times (-\frac{1}{n}, \frac{1}{n})$ and $s_n \in (-\frac{1}{n}, \frac{1}{n})$ such that

$$(\frac{x_n^*}{\tau M}, 0) \in \hat{\partial}\mathbf{d}_{\|\cdot\|_\tau}((\cdot, \cdot), \operatorname{epi} f)(z_n, r_n) + \{0\} \times [-1, 1] + \frac{1}{n}\mathbb{B}_{\mathbb{X}^*} \times [-\frac{1}{n}, \frac{1}{n}].$$

Thus there exist $(z_n^*, -\alpha_n) \in \hat{\partial}\mathbf{d}_{\|\cdot\|_\tau}((\cdot, \cdot), \operatorname{epi} f)(z_n, r_n)$ and $\gamma_n \in [-1, 1]$ such that

$$\left(\frac{x_n^*}{\tau M}, 0\right) \in (z_n^*, -\alpha_n) + (0, \gamma_n) + \frac{1}{n}\mathbb{B}_{\mathbb{X}^*} \times [-\frac{1}{n}, \frac{1}{n}]. \tag{16}$$

By virtue of Lemma 2.2, there exist $(\tilde{z}_n, \tilde{r}_n) \in \operatorname{epi} f$ and $(\tilde{z}_n^*, -\tilde{\alpha}_n) \in \hat{N}(\operatorname{epi} f, (\tilde{z}_n, \tilde{r}_n))$ such that

$$\|(\tilde{z}_n, \tilde{r}_n) - (z_n, r_n)\| < \mathbf{d}_{\|\cdot\|_\tau}((z_n, r_n), \operatorname{epi} f) + \frac{1}{n}$$

and
$$\|(\tilde{z}_n^*, -\tilde{\alpha}_n) - (z_n^*, -\alpha_n)\| < \frac{1}{n}. \tag{17}$$

Then $\tilde{\alpha}_n \geq 0$ for all n. Since $\mathbb{X}$ is an Asplund space, we know that $\mathbb{B}_{\mathbb{X}^*}$ is weak* sequentially compact. Note that

$$r_n \in (-\frac{1}{n}, \frac{1}{n}),\ (z_n^*, \alpha_n) \in \hat{\partial}\mathbf{d}_{\|\cdot\|_\tau}((\cdot,\cdot), \operatorname{epi} f)(z_n, r_n) \subseteq \frac{1}{\tau}\mathbb{B}_{\mathbb{X}^*} \times [-1, 1],$$

and thus, considering subsequences if necessary, we can assume that

$$z_n^* \xrightarrow{w^*} z^* \in \frac{1}{\tau}\mathbb{B}_{\mathbb{X}^*},\ \alpha_n \to \alpha \in [0, 1], r_n \to 0 \text{ and } \gamma_n \to \gamma \in [-1, 1].$$

By (17), one has $\tilde{z}_n^* \xrightarrow{w^*} z^*$, $\tilde{r}_n \to 0$ and $\tilde{\alpha}_n \to \alpha$.

This implies that $(z^*, -\alpha) \in N(\operatorname{epi} f, (x, 0)) = N(\operatorname{epi} f, (x, f(x)))$. (18)

By taking limits with respect to the weak*-topology in (16) as $n \to \infty$, one has

$$x^* = \tau M z^*. \tag{19}$$

From (18), one has $\alpha \geq 0$. If $\alpha = 0$, then $z^* \in \partial^\infty f(x)$ and so $x \in \partial^\infty f(x)$, which implies that (11) holds. If $\alpha > 0$, then $\frac{z^*}{\alpha} \in \partial f(x)$ and

$$x^* = \tau M z^* \in [0, \tau M\alpha]\partial f(x) \subseteq [0, +\infty)\partial f(x).$$

This means that (11) holds. The proof is complete. □

In terms of Fréchet subdifferentials and singular subdifferentials, the following theorem gives a sharper necessary condition for a local error bound for nequality (8).

Theorem 3.2. *Suppose that inequality* (8) *has a local error bound at* $\bar{x} \in \mathbf{S}_f$. *Then there exist* $\tau, \delta_0 > 0$ *such that for any* $\epsilon > 0$, *the inclusion*

$$\hat{N}(\mathbf{S}_f, x) \cap \mathbb{B}_{\mathbb{X}^*} \subseteq \bigcup \left\{[0, (1+\epsilon)\tau]\hat{\partial} f(u) + \hat{\partial}^\infty f(u) : u \in \mathbb{B}(x, \epsilon)\right\} + \epsilon\mathbb{B}_{\mathbb{X}^*} \tag{20}$$

holds for all $x \in \mathbf{S}_f \cap \mathbb{B}(\bar{x}, \delta_0)$.

Proof. By (9), there exist $\tau, \delta \in (0, +\infty)$ such that (9) holds. Using the proof of Theorem 3.1, by defining $\|(x, r)\|_\tau := \frac{1}{\tau}\|x\| + |r|$ for any $(x, r) \in \mathbb{X} \times \mathbb{R}$, one has that (12) holds with $\tau > 0$ and $\delta_0 = \frac{\delta}{2} > 0$. Let $x \in \mathbb{B}(\bar{x}, \delta_0) \cap \mathbf{S}_f$ and $x^* \in \hat{N}(\mathbf{S}_f, x) \cap \mathbb{B}_{\mathbb{X}^*}$.

Then $x^* \in \hat{\partial}\mathbf{d}(\cdot, \mathbf{S}_f)(x)$ by [34, Corollary 1.96]. Let $\epsilon > 0$. Take $\epsilon_1 > 0$ such that

$$\max\{2\tau\epsilon_1, 2\epsilon_1, \frac{1}{\tau}\epsilon_1 + \epsilon_1\} < \epsilon.$$

Let $\Phi(u,r) := \mathbf{d}_{\|\cdot\|_\tau}((u,r), \mathrm{epi}\, f) + |r|$. Similar to the proof of Theorem 3.1, one can verify that

$$\left(\frac{x^*}{\tau}, 0\right) \in \hat{\partial}\Phi(x,0). \tag{21}$$

By Lemma 2.3, there exist $(z_1, r_1) \in \mathbb{B}(x, \epsilon_1) \times (-\epsilon_1, \epsilon_1)$ and $s_1 \in (-\epsilon_1, \epsilon_1)$ such that

$$\left(\frac{x^*}{\tau}, 0\right) \in \hat{\partial}\mathbf{d}_{\|\cdot\|_\tau}((\cdot,\cdot), \mathrm{epi}\, f)(z_1, r_1) + \{0\} \times [-1,1] + \epsilon_1 \mathbb{B}_{\mathbb{X}^*} \times [-\epsilon_1, \epsilon_1].$$

Thus there exist $(z_1^*, -\alpha_1) \in \hat{\partial}\mathbf{d}_{\|\cdot\|_\tau}((\cdot,\cdot), \mathrm{epi}\, f)(z_1, r_1)$ and $\gamma_1 \in [-1,1]$ such that

$$\left(\frac{x^*}{\tau}, 0\right) \in (z_1^*, -\alpha_1) + (0, \gamma_1) + \epsilon_1 \mathbb{B}_{\mathbb{X}^*} \times [-\epsilon_1, \epsilon_1]. \tag{22}$$

By virtue of Lemma 2.2, there are $(\tilde{z}_1, \tilde{r}_1) \in \mathrm{epi}\, f$ and $(\tilde{z}_1^*, -\tilde{\alpha}_1) \in \hat{N}(\mathrm{epi}\, f, (\tilde{z}_1, \tilde{r}_1))$ such that $\|(\tilde{z}_1, \tilde{r}_1) - (z_1, r_1)\| < \mathbf{d}_{\|\cdot\|_\tau}((z_1, r_1), \mathrm{epi}\, f) + \epsilon_1$ and

$$\|(\tilde{z}_1^*, -\tilde{\alpha}_1) - (z_1^*, -\alpha_1)\| < \epsilon_1. \tag{23}$$

Then $(\tilde{z}_1^*, -\tilde{\alpha}_1) \in \hat{N}(\mathrm{epi}\, f, (\tilde{z}_1, f(\tilde{z}_1)))$ by Lemma 2.5 and

$$\|(\tilde{z}_1, \tilde{r}_1) - (x, 0))\| \leq \|(\tilde{z}_1, \tilde{r}_1) - (z_1, r_1)\| + \|(z_1, r_1) - (x, 0))\| < \frac{1}{\tau}\epsilon_1 + \epsilon_1.$$

From $(\tilde{z}_1^*, -\tilde{\alpha}_1) \in \hat{N}(\mathrm{epi}\, f, (\tilde{z}_1, f(\tilde{z}_1)))$, one has $\tilde{\alpha}_1 \geq 0$. If $\tilde{\alpha}_1 = 0$, then $\tilde{z}_1^* \in \hat{\partial}^\infty f(\tilde{z}_1)$. By virtue of (22) and (23), one has

$$\|\frac{x^*}{\tau} - \tilde{z}_1^*\| \leq \|\frac{x^*}{\tau} - z_1^*\| + \|z_1^* - \tilde{z}_1^*\| < 2\epsilon_1$$

and consequently $x^* \in \hat{\partial}^\infty f(\tilde{z}_1) + 2\tau\epsilon_1 \mathbb{B}_{\mathbb{X}^*} \subseteq \hat{\partial}^\infty f(\tilde{z}_1) + \epsilon \mathbb{B}_{\mathbb{X}^*}$ (thanks to the choice of ϵ_1). This implies that (20) holds.

If $\tilde{\alpha}_1 > 0$, then $\frac{\tilde{z}_1^*}{\tilde{\alpha}_1} \in \hat{\partial} f(\tilde{z}_1)$ and $\tilde{z}_1^* \in [0, 1+2\epsilon_1]\hat{\partial} f(\tilde{z}_1)$ by (23). By virtue of (22) and (23), one has

$$x^* \in [0, (1+2\epsilon_1)\tau]\hat{\partial} f(\tilde{z}_1) + 2\tau\epsilon_1 \mathbb{B}_{\mathbb{X}^*} \subseteq [0, (1+\epsilon)\tau]\hat{\partial} f(\tilde{z}_1) + \epsilon \mathbb{B}_{\mathbb{X}^*}.$$

Hence (20) holds. The proof is complete. □

In Theorem 3.2, a necessary condition for the etrror bound of inequality (8) was established in terms of Fréchet subdifferentials of the given function and normal cones at those points inside the solution set in (20). This gives rise to a natural question whether ϵ in (20) can be taken as 0. We have no answer to this question. However, we are now in a position to consider the circumstance when $\epsilon \downarrow 0$. The following theorem shows that this can be done in finite-dimensional spaces.

Theorem 3.3. *Suppose that $\mathbb{X}$ is finite-dimensional, inequality (8) has a local error bound at $\bar{x} \in \mathbf{S}_f$. Then there exist $\tau, \sigma \in (0, +\infty)$ such that*

$$N(\mathbf{S}_f, x) \cap \mathbb{B}_{\mathbb{X}^*} \subseteq [0, \tau]\partial f(x) + \partial^\infty f(x) \tag{24}$$

holds for all $x \in \mathbb{B}(\bar{x}, \sigma)$ with $f(x) = 0$.

Proof. According to the local error bound property at $\bar{x}$, there exist $\tau, \delta \in (0, +\infty)$ such that (9) holds. Take $\sigma := \frac{\delta}{2}$. Let $x \in \mathbb{B}(\bar{x}, \sigma)$ be such that $f(x) = 0$ and $x^* \in N(\mathbf{S}_f, x) \cap \mathbb{B}_{\mathbb{X}^*}$ with $\|x^*\| > 0$. Since $\mathbb{X}$ is finite-dimensional, there exist $x_n \xrightarrow{\mathbf{S}_f} x$ and $x_n^* \xrightarrow{\|\cdot\|} x^*$ such that $x_n^* \in \hat{N}(\mathbf{S}_f, x_n)$. Then for any n sufficiently large, one has $\frac{x_n^*}{\|x_n^*\|} \in \hat{\partial}\mathbf{d}(\cdot, \mathbf{S}_f)(x_n)$ by [34, Corollary 1.96] and thus for any $\epsilon > 0$ there is $\delta_n \in (0, \sigma - \|x - x_n\|)$ such that

$$\left\langle \frac{x_n^*}{\|x_n^*\|}, u - x_n \right\rangle \leq \mathbf{d}(u, \mathbf{S}_f) + \tau\epsilon\|u - x_n\|, \quad \forall u \in \mathbb{B}(x_n, \delta_n).$$

This and (9) imply that for any $u \in \mathbb{B}(x_n, \delta_n)$, one has

$$\left\langle \frac{x_n^*}{\|x_n^*\|}, u - x_n \right\rangle \leq \tau f_+(u) + \tau\epsilon\|u - x_n\|$$

and thus

$$\frac{x_n^*}{\tau\|x_n^*\|} \in \hat{\partial} f_+(x_n).$$

(thanks to $f_+(x_n) = 0$). Since $\mathbb{X}$ is finite-dimensional, it follows from Lemma 2.4 that there exist $\lambda_n \in [0, 1 + \frac{1}{n}]$, $u_n \in \mathbb{B}(x_n, \frac{1}{n})$ with $|f(u_n) - f(x_n)| < \frac{1}{n}$ and $u_n^* \in \hat{\partial} f(u_n)$ such that

$$\left\| \frac{x_n^*}{\tau\|x_n^*\|} - \lambda_n u_n^* \right\| < \frac{1}{n}. \tag{25}$$

Without loss of generality, considering a subsequence if necessary, we can assume that $\lambda_n \to \lambda \in [0, 1]$. We divide λ into two cases:

(a) $\lambda = 0$. Noting that $f(x) \leq \liminf\limits_{n\to\infty} f(u_n)$ by the lower semicontinuity and

$$\limsup_{n\to\infty} f(u_n) \leq \limsup_{n\to\infty} (f(x_n) + \frac{1}{n}) \leq 0,$$

it follows that $f(u_n) \to f(x)$ as $f(x) = 0$. By letting $n \to \infty$ in (25), one has

$$\frac{x^*}{\tau\|x^*\|} \in \partial^\infty f(x)$$

and thus (24) holds.

(b) $\lambda > 0$. By letting $n \to \infty$ in (25), one has

$$u_n^* \to \frac{x^*}{\lambda\tau\|x^*\|} \in \partial f(x)$$

(as $f(u_n) \to f(x)$). This means that

$$x^* \in \lambda\tau\|x^*\|\partial f(x) \subseteq [0, \tau]\partial f(x)$$

(thanks to $\lambda \leq 1$ and $\|x^*\| \leq 1$). Hence (24) holds.

The proof is complete. □

Remark 3.4. It is known from Theorem 3.3 that the validity of (24) is necessary for the local error bound. However, this condition may not be sufficient for the local error bound even in the finite-dimensional space.

For example, let $\mathbb{X} := \mathbb{R}^2$,

$$A_1 := (\mathbb{R}_- \times \mathbb{R}_+) \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 0, x_1 \geq 0\}$$

and $$A_2 := \left\{(x_1, x_2) \in \mathbb{R}^2_+ : (x_1 - 1)^2 + x_2^2 \leq 1 \ \text{ and } \ x_1^2 + (x_2 - 1)^2 \leq 1\right\}.$$

Define $f(x) := \max\{\mathbf{d}(x, A_1), \mathbf{d}(x, A_2)\}$ for all $x \in X$. Take $\bar{x} = (0, 0)$, $e = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Then $\mathbf{S}_f = A_1 \cap A_2 = \{\bar{x}\}$ and by computation, one has

$$\partial \mathbf{d}(\cdot, A_1)(\bar{x}) \supseteq ([0, 1] \times \{0\}) \cup (\{0\} \times [-1, 0]) \cup \{e\}$$

and $\partial \mathbf{d}(\cdot, A_2)(\bar{x}) = N(A_2, \bar{x}) \cap \mathbb{B}_{\mathbb{X}^*} = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_1 \leq 0, x_2 \leq 0\}$.

Then one can verify that there exists $\varepsilon > 0$ such that

$$\partial f(\bar{x}) \supseteq \varepsilon \mathbb{B}_{\mathbb{X}^*} = N(\mathbf{S}_f, \bar{x}) \cap \varepsilon \mathbb{B}_{\mathbb{X}^*},$$

which implies that (24) holds for $x = \bar{x}$.

On the other hand, if $x^{(k)} = \left(\frac{1}{k}, (\frac{2}{k} - \frac{1}{k^2})^{\frac{1}{2}}\right)$ for each $k \in \mathbb{N}$, one has

$$\mathbf{d}(x^{(k)}, \mathbf{S}_f) = \left(\frac{2}{k}\right)^{\frac{1}{2}}, \mathbf{d}(x^{(k)}, A_1) = \frac{1}{k} \ \text{ and } \ \mathbf{d}(x^{(k)}, A_2) = 0,$$

and thus $$\frac{\mathbf{d}(x^{(k)}, \mathbf{S}_f)}{f(x^{(k)})} = (2k)^{\frac{1}{2}} \to \infty.$$

This means that inequality $f(x) \leq 0$ does not have a local error bound at $\bar{x}$. □

Next, we consider error bounds for a convex-composite inequality defined by a convex-composite function (i.e. the composition of a convex function with a continuously differentiable mapping), and we use Fréchet subdifferentials and normal cones to prove dual characterizations of error bounds.

Throughout the remainder of this section, we suppose that $\mathbb{Y}$ is an Asplund space, $\psi : \mathbb{X} \to \mathbb{Y}$ is continuously differentiable and $g : \mathbb{Y} \to \mathbb{R} \cup \{+\infty\}$ is a proper semicontinuous convex function on $\mathbb{Y}$.

Let $f := g \circ \psi$. We consider the following convex-composite inequality:

$$(g \circ \psi)(x) \leq 0. \tag{26}$$

We denote by $\mathcal{S}$ the solution set of (26), and denote by

$$\mathbf{S}_g := \{y \in \mathbb{Y} : g(y) \leq 0\} \tag{27}$$

the solution set of the convex inequality $g(y) \leq 0$.

Now, we are in a position to present dual characterizations for error bounds of the convex-composite inequality (26) when ψ is smooth.

Theorem 3.5. *Let $f := g \circ \psi$ be such that $\mathrm{bd}(\mathbf{S}_g) \subseteq g^{-1}(0)$ and $\bar{x} \in \mathcal{S}$ such that $\nabla\psi(\bar{x})$ is surjective. Then convex-composite inequality (26) has a local error bound at $\bar{x}$ if and only if there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\hat{N}(\mathcal{S}, x) \cap \mathbb{B}_{\mathbb{X}^*} \subseteq [0, \tau]\hat{\partial} f(x) \tag{28}$$

holds for all $x \in \mathcal{S} \cap \mathbb{B}(\bar{x}, \delta)$.

Proof. Since $\mathrm{bd}(\mathbf{S}_g) \subseteq g^{-1}(0)$, then it follows that

$$\mathrm{bd}(\mathcal{S}) \subseteq \psi^{-1}(\mathrm{bd}(\mathbf{S}_g)) \subseteq \psi^{-1}(g^{-1}(0)) = f^{-1}(0). \tag{29}$$

Note that $\mathcal{S} = \psi^{-1}(\mathbf{S}_g)$ and by applying [47, Lemma 4.2], there are $\kappa, L > 0$ and $r_0 > 0$ such that

$$\hat{N}(\mathcal{S}, x) \cap \kappa\mathbb{B}_{\mathbb{X}^*} \subseteq \nabla\psi(x)^*(N(\mathbf{S}_g, \psi(x)) \cap \mathbb{B}_{\mathbb{Y}^*}) \subseteq \hat{N}(\mathcal{S}, x) \cap L\mathbb{B}_{\mathbb{X}^*} \tag{30}$$

holds for all $x \in \mathbb{B}(\bar{x}, r_0) \cap \mathcal{S}$.

The necessity part. By the local error bound, there exist $\tau > 0$ and $\delta \in (0, r_0)$ such that (9) holds. Let $x \in \mathcal{S} \cap \mathbb{B}(\bar{x}, \delta)$ and $x^* \in \hat{N}(\mathcal{S}, x) \cap \mathbb{B}_{\mathbb{X}^*}$. Then $x^* \in \hat{\partial}\mathbf{d}(\cdot, \mathcal{S})(x)$ by [34, Corollary 1.96] and thus for any $\epsilon > 0$, there is $\delta_1 \in (0, \delta - \|x - \bar{x}\|)$ such that

$$\langle x^*, u - x\rangle \leq \mathbf{d}(u, \mathcal{S}) + \tau\epsilon\|u - x\|, \quad \forall u \in \mathbb{B}(x, \delta_1).$$

By using (9), one has

$$\langle x^*, u - x\rangle \leq \tau f_+(u) + \tau\epsilon\|u - x\|, \quad \forall u \in \mathbb{B}(x, \delta_1).$$

This implies that $\frac{x^*}{\tau} \in \hat{\partial} f_+(x)$ and it follows from [34, Theorem 3.36 and Theorem 3.46] that

$$\frac{x^*}{\tau} \in [0, 1]\partial f(x),$$

By virtue of [47, Lemma 4.1], one has $\partial f(x) = \hat{\partial} f(x)$ and thus

$$x^* \in [0, \tau]\hat{\partial} f(x),$$

which implies that (28) holds.

The sufficiency part. Let $\epsilon > 0$ be such that $\epsilon < \kappa$. Since ψ is continuously differentiable at $\bar{x}$, there exists $r_1 > 0$ such that $r_1 < \min\{r_0, \delta\}$ and

$$\|\psi(x) - \psi(u) - \nabla\psi(u)(x - u)\| < \epsilon\|x - u\|, \ \forall x, u \in \mathbb{B}(\bar{x}, r_1). \tag{31}$$

Take $\delta_1 := \frac{r_1}{2}$ and let $x \in \mathbb{B}(\bar{x}, \delta_1)\backslash\mathcal{S}$. Then $\mathbf{d}(x, \mathcal{S}) \leq \|x - \bar{x}\| < \delta_1$. Choose $\beta \in (0, 1)$ such that $\beta > \max\{\frac{\mathbf{d}(x,\mathcal{S})}{\delta_1}, \frac{\epsilon}{\kappa}\}$. By virtue of Lemma 2.3, there exist $a \in \mathrm{bd}(\mathcal{S})$ and $a^* \in \hat{N}(\mathcal{S}, a)$ with $\|a^*\| = 1$ such that

$$\beta\|x - a\| \leq \min\{\mathbf{d}(x, \mathcal{S}), \langle a^*, x - a\rangle\}. \tag{32}$$

Then
$$\|a - \bar{x}\| \leq \|a - x\| + \|x - \bar{x}\| < \frac{\mathbf{d}(x, \mathcal{S})}{\beta} + \delta_1 < 2\delta_1 = r_1$$

and it follows from (28) and [47, Lemma 4.1] that there exist $\lambda \in [0, \tau], y^* \in \partial g(\psi(a))$ such that

$$a^* = \nabla\psi(a)^*(\lambda y^*).$$

By (30), there is $y_1^* \in N(\mathbf{S}_g, \psi(a)) \cap \mathbb{B}_{\mathbb{Y}^*}$ such that

$$\kappa a^* = \nabla\psi(a)^*(y_1^*).$$

Since $\nabla\psi(a)^*$ is one-to-one, then one has $\lambda y^* = \frac{1}{\kappa} y_1^*$ and it follows from (31) that

$$\begin{aligned}
\langle a^*, x-a\rangle &= \lambda\langle \nabla\psi(a)^*(y^*), x-a\rangle = \langle \lambda y^*, \nabla\psi(a)(x-a)\rangle \\
&= \langle \lambda y^*, \psi(x)-\psi(a)\rangle + \langle \lambda y^*, \nabla\psi(a)(x-a)+\psi(a)-\psi(x)\rangle \\
&\le \lambda\big(g(\psi(x)) - g(\psi(a))\big) + \frac{1}{\kappa}\|\psi(x)-\psi(a)-\nabla\psi(a)(x-a))\| \\
&\le \tau(f(x)-f(a)) + \frac{\epsilon}{\kappa}\|x-a\| \\
&= \tau f_+(x) + \frac{\epsilon}{\kappa}\|x-a\|
\end{aligned}$$

(thanks to $f(a)=0$ by (29)). Then (32) gives that

$$(\beta - \tfrac{\epsilon}{\kappa})\|x-a\| \le \tau f_+(x).$$

By taking limits as $\beta \uparrow 1$, one gets $\mathbf{d}(x,\mathcal{S}) \le \dfrac{\kappa\tau}{\kappa-\epsilon} f_+(x)$.

This means that the convex-composite inequality system (26) has a local error bound with constant $\frac{\kappa\tau}{\kappa-\epsilon} > 0$. The proof is complete. □

The following corollary is a consequence result of Theorem 3.5. It gives a precise estimate of the local error bound modulus.

Corollary 3.6. *Let $f := g\circ\psi$ be such that $\mathrm{bd}(\mathbf{S}_g) \subseteq g^{-1}(0)$ and $\bar{x}\in\mathcal{S}$ such that $\nabla\psi(\bar{x})$ is surjective. Denote*

$$\tau^*(f,\bar{x}) := \{\tau > 0 : \exists\, \delta > 0 \textit{ s.t. } (28) \textit{ holds for all } x\in\mathcal{S}\cap\mathbb{B}(\bar{x},\delta)\}. \tag{33}$$

Then
$$\tau(f,\bar{x}) = \tau^*(f,\bar{x}). \tag{34}$$

Proof. Consider the case that $\tau(f,\bar{x}) < +\infty$. Let $\tau\in(0,\tau(f,\bar{x}))$. Then there exists $\delta>0$ such that (9) holds. Using the proof of the necessity part in Theorem 3.5, one can verify that

$$\tau^*(f,\bar{x}) \le \tau$$

and by letting $\tau\uparrow\tau(f,\bar{x})$, one has $\tau^*(f,\bar{x}) \le \tau(f,\bar{x})$.

On the other hand, let $\tau\in(0,\tau^*(f,\bar{x}))$. Then for any $\epsilon>0$ sufficiently small, by the proof of the sufficiency part in Theorem 3.5, one has

$$\tau(f,\bar{x}) \le \frac{\kappa\tau}{\kappa-\epsilon} \to \tau \ (\text{as } \epsilon\to 0^+)$$

and thus $\tau(f,\bar{x})\le\tau$. By letting $\tau\uparrow\tau^*(f,\bar{x})$, one has

$$\tau(f,\bar{x}) \le \tau^*(f,\bar{x}).$$

Hence (34) holds. If $\tau(f,\bar{x}) = +\infty$, then we claim that $\tau^*(f,\bar{x}) = +\infty$ (otherwise, one can verify that $\tau(f,\bar{x}) < +\infty$ by the proof of the sufficiency part in Theorem 3.5, a contradiction). The proof is complete. □

Theorem 3.7. *Let $f := g\circ\psi$ be such that $\mathrm{bd}(\mathbf{S}_g)\subseteq g^{-1}(0)$ and $\bar{x}\in\mathcal{S}$ such that $\nabla\psi(\bar{x})$ is surjective. Then convex-composite inequality (26) has a local error bound at $\bar{x}$ if and only if the convex inequality $g(y)\le 0$ has a local error bound at $\psi(\bar{x})$.*

Proof. Since $\nabla\psi(\bar{x})$ is surjective, it follows from [34, Theorem 1.57] that there exist $\kappa, \delta_0 > 0$ such that

$$\mathbf{d}(x, \psi^{-1}(y)) \leq \kappa\|y - \psi(x)\| \quad \forall (x, y) \in \mathbb{B}(\bar{x}, \delta_0) \times \mathbb{B}(\psi(\bar{x}), \delta_0). \tag{35}$$

Note that $\mathcal{S} = \psi^{-1}(\mathbf{S}_g)$ and by applying [47, Lemma 4.2], there are $\kappa, L > 0$ and $r_0 \in (0, \delta_0)$ such that (30) holds for all $x \in \mathbb{B}(\bar{x}, r_0) \cap \mathcal{S}$.

The necessity part. By Theorem 3.5, there exist $\tau, \delta \in (0, +\infty)$ such that (28) holds. We next show that there exist $\tau_1, \delta_1 > 0$ such that

$$N(\mathbf{S}_g, y) \cap \mathbb{B}_{\mathbb{Y}^*} \subseteq [0, \tau_1]\partial g(y), \quad \forall y \in \mathbf{S}_g \cap \mathbb{B}(\bar{y}, \delta_1). \tag{36}$$

Granting this, it follows from [50, Theorem 2.2] that the convex inequality $g(y) \leq 0$ has a local error bound at $\psi(\bar{x})$.

Take $\delta_1 \in (0, r_1)$ such that $\kappa\delta_1 < \delta$. Let $y \in \mathbf{S}_g \cap \mathbb{B}(\psi(\bar{x}), \delta_1)$ and $y^* \in N(\mathbf{S}_g, y) \cap \mathbb{B}_{\mathbb{Y}^*}$. Then by (35), one has

$$\mathbf{d}(\bar{x}, \psi^{-1}(y)) \leq \kappa\|\psi(\bar{x}) - y\| < \kappa\delta_1$$

and so there is $x \in \mathbb{B}(\bar{x}, \kappa\delta_1) \subseteq \mathbb{B}(\bar{x}, \delta)$ such that $y = \psi(x)$. By virtue of (30), one has

$$\nabla\psi(x)^*(y^*) \in \nabla\psi(x)^*(N(\mathbf{S}_g, \psi(x)) \cap \mathbb{B}_{\mathbb{Y}^*}) \subseteq \hat{N}(\mathcal{S}, x) \cap L\mathbb{B}_{\mathbb{X}^*}$$

and thus (28) implies that

$$\begin{aligned} \nabla\psi(x)^*\left(\frac{y^*}{L}\right) &\in \hat{N}(\mathcal{S}, x) \cap \mathbb{B}_{\mathbb{X}^*} \subseteq [0, \tau]\hat{\partial} f(x) \\ &= [0, \tau]\nabla\psi(x)^*(\partial g(\psi(x))) = \nabla\psi(x)^*\big([0, \tau]\partial g(\psi(x))\big). \end{aligned}$$

Since $\nabla\psi(x)^*$ is one-to-one, it follows that

$$y^* \in [0, L\tau]\partial g(\psi(x)).$$

This means that (36) holds with $\tau_1 := L\tau$ and $\delta_1 > 0$.

The sufficiency part. Denote $\bar{y} := \psi(\bar{x})$. Then by virtue of [50, Theorem 2.2], there exist $\tau_1, \delta_1 > 0$ such that (36) holds. Based on Theorem 3.5, it suffices to show that there exist $\tau, \delta > 0$ such that (28) holds.

Take $\delta > 0$ such that $\psi(\mathbb{B}(\bar{x}, \delta)) \subseteq \mathbb{B}(\psi(\bar{x}), \delta_1)$. Let $x \in \mathcal{S} \cap \mathbb{B}(\bar{x}, \delta)$ and $x^* \in \hat{N}(\mathcal{S}, x) \cap \mathbb{B}_{\mathbb{X}^*}$. By virtue of (30), there is $y^* \in N(\mathbf{S}_g, \psi(x)) \cap \mathbb{B}_{\mathbb{Y}^*}$ such that

$$\kappa x^* = \nabla\psi(x)^*(y^*). \tag{37}$$

By virtue of (36) and [47, Lemma 4.1], one has

$$\nabla\psi(x)^*(y^*) \in \nabla\psi(x)^*\big([0, \tau_1]\partial g(\psi(x))\big) = [0, \tau_1]\nabla\psi(x)^*\big(\partial g(\psi(x))\big) = [0, \tau_1]\hat{\partial} f(x).$$

This and (37) imply that $x^* \in [0, \frac{\tau_1}{\kappa}]\hat{\partial} f(x)$ and consequently (28) holds with $\tau := \frac{\tau_1}{\kappa}$ and $\delta > 0$. The proof is complete. □

4. Conclusions

This paper deals with error bounds of a general inequality defined by a proper lower semicontinuous function on an Asplund space. It is described by the subdifferentials of the given function at points inside the solution set. Although in absence of convexity assumption the dual characterization result on error bounds for the general inequality may not hold, several necessary dual conditions for error bounds may still be obtained in terms of Fréchet / Mordukhovich subdifferentials. Further, such conditions are also proved to be sufficient for error bounds of a convex-composite inequality. Our work is an extension of dual characterizations of error bounds via subdifferentials to the possible non-convex case.

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