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On the Weak* Convergence of Gâteaux Derivatives of Upper Envelopes

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Dedicated to the memory of Hedy Attouch.

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For convex functions on Banach space endowed with a uniformly Gâteaux differentiable norm two observations are presented: first, the Moreau envelope of a proper lower semicontinuous convex function is Gâteaux differentiable; second, if the Moreau envelopes of a sequence of lower semicontinuous convex functions $\{f_n\}_{n=1}^{\infty}$ create a pointwise convergent sequence, then Gâteaux derivatives of the envelopes (computed at a given point) are weakly* convergent.

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1. Introduction

To investigate properties of convex functions we need appropriate tools. An important one of them is the subdifferential with its subgradients, see for example [2]. However, even though that definitions of subdifferential and subgradients are natural and easy to understand, and several properties of these notions were revealed, see [2], it is rather hard to evaluate precisely subgradients except in commonly known cases, for example, whenever the convex function is well recognized like the distance from a convex set. In general, we know in advance that subdifferential of a proper lower semicontinuous convex function on a Banach space is cyclically maximal monotone operator, see [9] or [12, Chapter 3] and likely do not know more about it, whenever we want to apply subgradients to investigate open problems. A remedy to overcome this obstacle is to approximate a given subgradient by subgradients with known virtues. The best idea is to find a "good" subgradient close enough, for example, the distance between a subgradient and the "good" one should be small, that is close concerning the strong topology. Results of this type were first observed by H. Attouch in the reflexive Banach space setting, see for example [1]. Of course, the reflexive Banach space setting is enough to deal with a wide important class of problems which are well elaborated in this case, see Attouch's monograph [1]. Unfortunately, this does not work beyond this type of space except specific cases, rather not applicable. A way out is to change the demand on the topology to get a "good" approximant in a neighborhood. A natural candidate for this, since subgradients are in the dual space, is the weak* topology. The first step in this direction

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was done in [13]. In the case the space is weakly compactly generated such a result was obtained under additional restriction that the function under consideration is continuous at some point, namely, it has been shown that for each subgradient x^* of f at x, the Mosco convergence of $\{f_n\}_{n=1}^{\infty}$ to f with the additional assumption that the sequence of functions is uniformly bounded from above on some open set, ensures the existence of a sequence of subgradients $x_n^* \in \partial f_n(x_n)$, which is weakly* convergent to x^*, $\lim_{n\to\infty} |f_n(x_n) - f(x)| = 0$ and $\lim_{n\to\infty} \|x_n - x\| = 0$, see [13, 14, 15]. Herein it is shown that the subgradients convergence, mentioned above, can be also preserved for convex functions on Banach space, whenever f is Gîteaux differentiable at x, see Theorem 3.1 below. A consequence of this result, whenever the Banach space has additionally uniformly Gâteaux differentiable norm, is that if a sequence of Moreau envelopes of lower semicontinuous convex functions $\{f_n\}_{n=1}^{\infty}$ is a pointwise convergent sequence, then Gâteaux derivatives of the envelopes (computed at a given point) are weakly* convergent, see Theorem 4.2 below.

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2. Preliminaries

Throughout, unless otherwise stated, $\mathbb{R}$ is the real line, E is a *real Banach space* with its norm $\|\cdot\|$, see [6] for information on Banach spaces, E^* its topological dual, and $B_E(x, \delta)$ (resp. $B_E[x, \delta]$) the open (resp. closed) ball of centre x and radius δ. The topological closure of subset $D \subset E$ with respect to the norm of E is denoted by $\mathrm{cl}\, D$ and $\mathrm{cl}^{w^*} A$ stands for the closure of $A \subset E^*$ with respect to the weak* topology, denoted by $w(E^*, E)$, see [11] for more on weak and weak* topologies.

For an extended real-valued function $f : U \to \mathbb{R} \cup \{+\infty\}$ defined on a nonempty subset U of E, its *effective domain* is defined by $\mathrm{dom}\, f := \{x \in U \mid f(x) < +\infty\}$. The directional derivative of f at $x \in \mathrm{dom}\, f$ in the direction of d is defined by

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$

when this limit exists, see [2, Section 4] for some information on the directional derivative. If there is $x^* \in E^*$ such that

$$\langle x^*, d \rangle = f'(x; d) \tag{1}$$

for all $d \in E$, then f is said to be Gâteaux differentiable, where $\langle \cdot, \cdot \rangle$ stands for the duality pairing, namely $\langle z^*, z \rangle$ denotes the value of the continuous linear functional $z^* \in E^*$ at $z \in E$.

For any convex function $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ finite at x and $\varepsilon \geq 0$, by $\partial_\varepsilon f(x)$ we denote its ε-subdifferential at x i.e.

$$\partial_\varepsilon f(x) := \{x^* \in E^* \mid < x^*, h > \leq \varepsilon + f(x + h) - f(x) \text{ for every } h \in E\}.$$

When $\varepsilon = 0$ then ε-subdifferential is called subdifferential and it is denoted by $\partial f(x)$, see [1, 2, 8, 12] for properties of subdifferentials.

In the lemma below it is stated that under a mild set of assumptions a sequence of convex functions is uniformly bounded on some open set.

Lemma 2.1. *Let E be a Banach space and $U \subset E$ be an open nonempty subset. Assume that functions $f, f_1, f_2, \dots : U \longrightarrow \mathbb{R}$ are continuous and convex on U, $\bar{x} \in U$ is fixed and*

(a) $\lim_{n\longrightarrow\infty} f_n(\bar{x}) = f(\bar{x})$;

(b) $\limsup_{n\longrightarrow\infty} f_n(u) \leq f(u)$ *for all $u \in U$.*

Then there is an open neighbourhood of $\bar{x}$, say $U' \subset U$ such that

$$f_n(u) \leq c \tag{2}$$

for some $c \in \mathbb{R}$ and all $u \in U'$, $n \in \mathbb{N}$. Moreover we have: if $\{u_n\}_{n=1}^{\infty}$ is a sequence converging to $\bar{x}$, then $\lim_{n\longrightarrow\infty} f_n(u_n) = f(\bar{x})$ and any sequence $\{(v_n, v_n^*)\}_{n\in\mathbb{N}}$, such that $\lim_{n\longrightarrow\infty} v_n = \bar{x}$ and $v_n^* \in \partial f_n(v_n)$ for all $n \in \mathbb{N}$, is bounded with respect to the strong topology in $E \times E^*$, see for example [6, Definitions 1.28 and 1.33] for more on the strong topologies in the space and its dual.

Proof. First, let us show that there is an open neighbourhood of $\bar{x}$, say $U' \subset U$ such that

$$f_n(u) \leq c \tag{3}$$

for some $c \in \mathbb{R}$ and all $u \in U'$, $n \in \mathbb{N}$. Fix $r > 0$ such that $B_E[\bar{x}, r] \subset U$. For each $k \in \mathbb{N}$ put

$$C_k := \{x \in B_E[\bar{x}, r] \mid f_k(x) \leq f(x) + 1 \text{ and } f_k(2\bar{x} - x) \leq f(2\bar{x} - x) + 1\}.$$

Sets C_k are closed in $B_E[\bar{x}, r]$, hence are closed in E as well as in U, since the involved functions are continuous. Let us fix $x \in B_E[\bar{x}, r]$ and observe that by assumption (b) we infer the existence of some $k(x) \in \mathbb{N}$ such that

$$f_k(x) \leq f(x) + \frac{1}{2} \quad \text{and} \quad f_k(2\bar{x} - x) \leq f(2\bar{x} - x) + \frac{1}{2}$$

for all $k \geq k(x)$. Thus

$$B_E[\bar{x}, r] = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} C_k.$$

By Baire's theorem, see for example [11], there are $n_0 \in \mathbb{N}$, $x_0 \in E$ and $\delta_0 > 0$ such that

$$B_E(x_0, \delta_0) \subset \bigcap_{k\geq n_0} C_k.$$

We may without loss of generality assume that both function $f(\cdot)$ and $f(2\bar{x} - \cdot)$ are bounded on the ball $B_E(x_0, \delta_0)$ by a some constant $m \in \mathbb{R}$, otherwise we take a smaller δ_0 to preserve this property. Now by the definition of sets C_k and convexity of functions f_k for all $u \in B_E(\bar{x}, \delta_0)$ and $k \geq n_0$ we get

$$f_k(u) \leq m + 1,$$

which implies (3).

A simple consequence of assumption (a) and (3), and the reasoning in the proof of [2, Lemma 4.1.3] is that there are a constant $L \geq 0$ and an open ball of centre $\bar{x}$ and radius $\delta > 0$ such that all functions f_n are Lipschitz continuous with the constant L on $B_E(\bar{x}, \delta)$, that is, for all $x, y \in B_E(\bar{x}, \delta)$ and all $n \in\in N$ we have

$$f_n(y) - f_n(x) \leq L\|y - x\|,$$

which implies that: if $\{u_n\}_{n=1}^{\infty}$ is a sequence convergent to $\bar{x}$, then

$$\lim_{n \longrightarrow \infty} f_n(u_n) = f(\bar{x})$$

and any sequence $\{(v_n, v_n^*)\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \longrightarrow \infty} v_n = \bar{x} \text{ and } v_n^* \in \partial f_n(v_n)$$

for all $n \in \mathbb{N}$, must be bounded with respect to the strong topology in $E \times E^*$. $\square$

It is known (and easily seen) that for all x in a Banach space E

$$\partial(\frac{1}{2}\| \cdot \|^2)(x) = \{x^* \in E^* \mid \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\}, \tag{4}$$

and the graph of this set-valued mapping is $\| \cdot \| \times w(E^*, E)$ closed in $E \times E^*$. Further, the set $\partial(\frac{1}{2}\| \cdot \|^2)(x)$ is a singleton for all $x \in E$ if and only if the norm $\| \cdot \|$ is Gâteaux differentiable outside zero, and in such a case writing $J(x) := \partial(\frac{1}{2}\| \cdot \|^2)(x)$ the mapping

$$J : E \to E^* \text{ is norm to weak}^* \text{ continuous}, \tag{5}$$

that is, the mapping J (usually called the duality (single valued) mapping) is continuous with respect to the norm topology on E and the weak* topology on E^*. Obviously, one has

$$J(tx) = tJ(x) \quad \text{for all } t \in \mathbb{R},\ x \in E. \tag{6}$$

We recall that a norm $\| \cdot \|$ is a uniformly Gâteaux differentiable norm (off zero) in a direction $h \in E$ if for any $\varepsilon > 0$, there exists a real $\delta(h, \varepsilon) > 0$ such that for every $u \in X$ with $\|u\| = 1$, there is a continuous linear functional $x_u^* \in E^*$ for which

$$|\frac{\|u + th\| - \|u\|}{t} - \langle x_u^*, h\rangle| < \varepsilon$$

for all $t \in]-\delta(h, \varepsilon), \delta(h, \varepsilon)[\setminus\{0\}$. The norm is uniformly Gâteaux differentiable when it is uniformly Gâteaux differentiable in each direction $h \in E$ or equivalently (see [5, Definition 1.1, Definition 6.5 and Lemma 6.6])

$$\lim_{t \downarrow 0} \sup_{u \in X, \|u\|=1} t^{-1}(\|u - th\| + \|u + th\| - 2) = 0 \text{ for all } h \in E,$$

see also [6, page 335 and Lemma 14.17]. Of course, in such a case $x_u^*(h) = \langle J(u), h\rangle$. The following observation was done in [7, Remark 3.1].

Remark 2.2. Usually when we want to apply the above equality we are not in the unit sphere. Moreover it is often more convenient to use the norm squared instead of the norm itself. Under the assumption that the norm $\| \cdot \|$ on the Banach space E is uniformly Gâteaux differentiable (off zero) in a direction $h \in E$, for all reals $0 < \rho < r$ we have

$$\lim_{t \downarrow 0} \sup_{y \in X,\, \rho \leq \|y\| \leq r} t^{-1}\big(\|y + th\|^2 - \|y\|^2 - 2\langle J(y), th\rangle\big) = 0.$$

Below, the Clarke subdifferential (called also the Clarke generalized gradient) is defined for locally Lipschitz continuous functions via the generalized directional derivative. Recall that, for a locally Lipschitz continuous function $f : U \to \mathbb{R}$ on an open set U of a normed vector space E, its *Clarke generalized directional derivative* (see [3]) is defined for $x \in U$ by

$$f^o(x; h) := \limsup_{u \to x; t \downarrow 0} t^{-1}[f(u + th) - f(u)]$$

and then its *Clarke subdifferential* at x can be defined similarly to the convex setting by

$$\partial_C f(x) := \{x^* \in E^* \mid \langle x^*, h\rangle \le f^o(x; h), \forall h \in E\}.$$

It is not difficult to see that for a locally Lipschitz continuous function f we have on one hand

$$f^o(x; -h) = (-f)^o(x; h), \tag{7}$$

and on the other hand

$$f^o(x; h) = \limsup_{(u,w) \to (x,h); t \downarrow 0} t^{-1}[f(u + tw) - f(u)],$$

and the latter equality entails that the function $f^o(\cdot; \cdot)$ is upper semicontinuous on $U \times E$, we refer to [3] for details.

The lower Dini directional derivative of f at $u \in \operatorname{dom} f$ is given by

$$d^- f(u; h) := \liminf_{w \to h; t \downarrow 0} t^{-1}[f(u + tw) - f(u)]$$

and when f is Lipschitz continuous near u we obviously have for all $h \in E$

$$d^- f(u; h) = \liminf_{t \downarrow 0} t^{-1}[f(u + th) - f(u)].$$

The Dini subdifferential of f at x is the set

$$\partial^- f(x) = \{x^* \in E^* \mid \langle x^*, h\rangle \le d^- f(x; h), \forall h \in E\}$$

for $x \in \operatorname{dom} f$ and $\partial^- f(x) = \emptyset$ if $x \notin \operatorname{dom} f$.

Let $\lambda > 0$ be given and $\|\cdot\|$ be a norm. Following Rockafellar and Wets (see [10]) we will call the infimal-convolution of f and $\frac{1}{2\lambda}\|\cdot\|^2$ *Moreau envelope of f of index* λ and we will denote it as $e_\lambda f$, that is,

$$e_\lambda f(x) := \inf_{z \in E}\left(f(z) + \frac{1}{2\lambda}\|x - z\|^2\right),$$

see [7, 12] for several information on Moreau envelope. Let us recall that the Gâteaux differentiability of Moreau envelope is known property, whenever E is a reflexive Banach space such that its norm is both strictly convex and Gâteaux differentiable off zero, see for example [12, Theorem 3.267]. Below this property is obtained in the case of any Banach space with its norm uniformly Gâteaux differentiable.

It is also known that convex lower semicontinuous functions are weak lower semicontinuous, see for example [2, Corollary 4.1.14], as well as they are bounded from below by affine continuous functions, see [2, Theorem 4.1.21] for example. Thus, as a simple consequence of [7, Theorem 3.2] we get the following corollary.

Corollary 2.3. *Let E be a Banach space with its norm uniformly Gâteaux differentiable and let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then $e_\lambda f$ is locally Lipschitz continuous for each $\lambda > 0$ and for all $x, h \in E$ the following equalities hold true*

$$d^-(-e_\lambda f)(x; -h) = \lim_{t \uparrow 0} \frac{e_\lambda f(x+th) - e_\lambda f(x)}{t} = (e_\lambda f)^o(x; h) \tag{8}$$

and
$$\partial_C(e_\lambda f)(x) = -\partial^-(-e_\lambda f)(x). \tag{9}$$

Hence
$$\partial(e_\lambda f)(x) = \partial_C(e_\lambda f)(x) = -\partial^-(-e_\lambda f)(x) \tag{10}$$

and $e_\lambda f$ is Gâteaux differentiable at x.

Proof. Equalities in (8) and (9) were obtained in [7, Theorem 3.2], whereas the first equality in (10) is a simple consequence of (9) and [3, Propostion 2.2.7], see also [4, Lemma 3.2]. Let us observe that for any $x^* \in \partial e_\lambda f(x)$ and all $h \in E$ using (8) and (10) we get

$$\begin{aligned}\langle x^*, h\rangle \le (e_\lambda f)^o(x; h) = \lim_{t \uparrow 0} \frac{e_\lambda f(x+th) - e_\lambda f(x)}{t} &= d^-(-e_\lambda f)(x; -h) \\ = \liminf_{t \downarrow 0} \frac{-e_\lambda f(x-th) + e_\lambda f(x)}{t} &= \lim_{t \downarrow 0} \frac{-e_\lambda f(x-th) + e_\lambda f(x)}{t}.\end{aligned}$$

Since $x^* \in \partial(e_\lambda f)(x)$, so we get

$$\langle x^*, h\rangle \le \lim_{t \downarrow 0} \frac{e_\lambda f(x+th) - e_\lambda f(x)}{t} = (e_\lambda f)'(x; h)$$

and
$$\langle x^*, -h\rangle \ge \lim_{t \downarrow 0} \frac{e_\lambda f(x+t(-h)) - e_\lambda f(x)}{t} = (e_\lambda f)'(x; -h)$$

for all $h \in E$, which yields the equality

$$\langle x^*, h\rangle = (e_\lambda f)'(x; h)$$

for all $h \in E$. The proof is completed, see (1). □

3. Weak* convergence of subgradients

Let us assume that a continuous convex function f is the pointwise limit of a sequence of convex functions. Below, a sufficient condition for the weak* convergence of subgradients of these functions to the Gâteaux derivative of f at a given point $\bar{x}$ is provided.

Theorem 3.1. *Let E be a Banach space and $U \subset E$ be an open nonempty subset. Assume that functions $f, f_1, f_2, \ldots : U \longrightarrow \mathbb{R}$ are continuous and convex on U, $\bar{x} \in U$ is fixed and*

(i) $\lim_{n \longrightarrow \infty} f_n(\bar{x}) = f(\bar{x})$;

(ii) $\limsup_{n \longrightarrow \infty} f_n(u) \le f(u)$ *for all $u \in U$.*

If f is Gâteaux differentiable at $\bar{x}$, that is, there is $x^* \in E$ such that

$$\langle x^*, d\rangle = f'(\bar{x}; d)$$

for all $d \in E$, then, for any sequence $\{(v_n, v_n^*)\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\longrightarrow\infty} v_n = \bar{x} \quad \textit{and} \quad v_n^* \in \partial f_n(v_n)$$

for all $n \in \mathbb{N}$, we have, for all $h \in E$,

$$\lim_{n\longrightarrow\infty} \langle x^* - v_n^*, h\rangle = 0. \tag{11}$$

Proof. Let us fix any sequence $\{(v_n, v_n^*)\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\longrightarrow\infty} v_n = \bar{x} \quad \text{and} \quad v_n^* \in \partial f_n(v_n)$$

for all $n \in \mathbb{N}$. It follows from Lemma 2.1 that

$$\lim_{n\longrightarrow\infty} f_n(u_n) = f(\bar{x}) \quad \text{and} \quad \{(v_n, v_n^*)\}_{n\in\mathbb{N}}$$

is bounded with respect to the strong topology in $E\times E^*$, so $\{v_1^*, v_2^*, \ldots\}$ is contained in some ball in E^*, that is this set is relatively weak* compact, see for example [11].

Suppose that there are $\bar{\varepsilon} > 0$, $\bar{h} \in E$ and integers $1 \leq n_1 < n_2 < n_3 \ldots$ such that for all $i \in \mathbb{N}$ we have

$$\langle x^* - v_{n_i}^*, \bar{h}\rangle < -\bar{\varepsilon}. \tag{12}$$

By the Banach-Alaoglu theorem, see for example [11], we have

$$\bigcap_{i\in\mathbb{N}} \mathrm{cl}^{w^*} \{v_{n_i}^*, v_{n_{i+1}}^*, \ldots\} \neq \emptyset.$$

Let us fix any $y^* \in \bigcap_{i\in\mathbb{N}} \mathrm{cl}^{w^*} \{v_{n_i}^*, v_{n_{i+1}}^*, \ldots\}$. By what precedes we have

$$\lim_{i\longrightarrow\infty} f_{n_i}(v_{n_i}) = f(\bar{x}),$$

which combined with (i) and (ii) gives

$$0 \leq \liminf_{i\longrightarrow\infty} \left(f_{n_i}(u) - f_{n_i}(v_{n_i}) - \langle v_{n_i}^*, u - v_{n_i}\rangle\right) \leq f(u) - f(\bar{x}) - \langle y^*, u - \bar{x}\rangle$$

for all $u \in U$, thus $y^* \in \partial f(\bar{x})$, so $y^* = x^*$, since the convex function f is Gâteaux differentiable at $\bar{x}$ and continuous at this point. Hence

$$\bigcap_{i\in\mathbb{N}} \mathrm{cl}^{w^*} \{v_{n_i}^*, v_{n_{i+1}}^*, \ldots\} = \{x^*\},$$

which contradicts (12), so (11) is valid. □

4. On weak* convergence of derivatives of Moreau envelopes

In this section it is observed that the sequence of Gâteaux derivatives of a sequence of convex functions is weakly* convergent under no restrictive assumptions. First we observe that Moreau envelope may be also defined for functions which are not defined on the whole space.

Remark 4.1. Let E be a Banach space and $U \subset E$ be an open nonempty convex subset, and $g : U \longrightarrow \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous proper and convex function on U. Putting ∞ outside U we extend g on the whole space E, that is $p(u) := g(u)$ for $u \in U$ and $p(u) := \infty$, whenever $u \notin U$. Following [12, see p. 288] by $\operatorname{cl} p$ it is denoted the pointwise supremum of all lower semicontinuous functions from E into $\mathbb{R} \cup \{\infty\}$ majorized by p. For the proper lower semicontinuous and convex function $\operatorname{cl} p : E \longrightarrow \mathbb{R} \cup \{\infty\}$ we may apply Corollary 2.3. Thus, its Moreau envelope $e_\lambda \operatorname{cl} p$ is locally Lipschitz continuous and convex function for all $\lambda > 0$. Let us put

$$e_\lambda g := e_\lambda \operatorname{cl} p$$

for all $\lambda > 0$. Thus we can define in a natural way Moreau envelopes, for any convex lower semicontinuous functions on some open set, preserving several useful properties. For more information on Moreau envelope of functions (not necessarily lower semicontionous) we refer to [12, Chapter 3.21].

A direct consequence of Theorem 3.1 is the following result characterizing sequential properties of derivatives Moreau envelopes on Banach spaces with norms uniformly Gâteaux differentiable. For example, if a sequence of Moreau envelopes of convex lower semicontinuous functions is pointwise converging, then the sequence of its Gateuax derivatives, calculated at a given point, is weak* convergent.

Theorem 4.2. *Let E be a Banach space with its norm uniformly Gâteaux differentiable and $U \subset E$ be an open nonempty convex subset. Assume that the functions $f, f_1, f_2, \ldots : U \longrightarrow \mathbb{R} \cup \{\infty\}$ are lower semicontinuous proper and convex on U, $\bar{x} \in U$ and $\lambda > 0$ are fixed, and*

(i) $\lim_{n\longrightarrow\infty} e_\lambda f_n(\bar{x}) = e_\lambda f(\bar{x})$;

(ii) $\limsup_{n\longrightarrow\infty} e_\lambda f_n(u) \le e_\lambda f(u)$ *for all* $u \in U$;

(iii) $e_\lambda f$ *has its Gâteaux derivative at* $\bar{x}$ *equal to* x^*, *see Corollary* 2.3.

Then, for any sequence $\{(v_n, v_n^*)\}_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\longrightarrow\infty} v_n = \bar{x} \quad \textit{and} \quad v_n^* \in \partial(e_\lambda f_n)(v_n)$$

for all $n \in N$, we have

$$\lim_{n\longrightarrow\infty} \langle x^* - v_n^*, h\rangle = 0 \tag{13}$$

for all $h \in E$.

It is natural to ask questions when assumptions of Theorem 4.2 are satisfied. In the lemma below it is pointed out that, under natural assumption on the sequence of functions, the inequality in (ii) holds true. It seems that in a general case the most difficult task is to verify condition (i).

Lemma 4.3. *Let E be a Banach space. Assume that the functions*

$$f, f_1, f_2, \ldots : U \longrightarrow \mathbb{R} \cup \{\infty\}$$

are lower semicontinuous proper on an open convex subset U, $\bar{x} \in U$ and $\lambda > 0$ are fixed, and

(a) $\lim_{n \longrightarrow \infty} e_\lambda f_n(\bar{x}) = e_\lambda f(\bar{x})$;

(b) $\limsup_{n \longrightarrow \infty} f_n(u) \leq f(u)$ *for all* $u \in U$.

Then

(i) *if $\{u_n\}_{n=1}^{\infty}$ is a sequence converging to $\bar{x}$, then* $\lim_{n \longrightarrow \infty} e_\lambda f_n(u_n) = e_\lambda f(\bar{x})$;

(ii) $\limsup_{n \longrightarrow \infty} e_\lambda f_n(u) \leq e_\lambda f(u)$ *for all* $u \in U$;

(iii) *any sequence $\{(v_n, v_n^*)\}_{n \in \mathbb{N}}$ such that* $\lim_{n \longrightarrow \infty} v_n = \bar{x}$ *and* $v_n^* \in \partial(e_\lambda f_n)(v_n)$ *for all $n \in N$ must be bounded with respect to the strong topology in the product $E \times E^*$.*

Proof. Since the implication (b) $\Longrightarrow$ (ii) is easy to verify, it follows from Lemma 2.1 that the statement in (i) holds true. The statement in (iii) can be obtained from Lemma 2.1 by applying it to $e_\lambda f_n$ and $e_\lambda f$ instead of f_n and f. □

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