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Generic Existence and Approximation of Points of Coincidence for Nonlinear Mappings

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Dedicated to the memory of Hedy Attouch.

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We study a space of nonlinear self-mappings of a complete metric space which is not necessarily bounded and show that a point of coincidence exists for a generic (typical) map.

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1. Introduction

Since the seminal result of Banach [2] the fixed point theory of nonexpansive mappings has been a rapidly growing field of research. See [3, 8, 10, 11, 12, 13, 17, 18, 19, 22, 23, 24, 27, 28] and the reference mentioned therein. A significant progress has been done, in particular, in studies of common fixed point problems, which have important applications in engineering and medical sciences [6, 7, 25, 27, 28]. The study of coincidence points of nonlinear mappings is an important topic of the fixed point theory [1, 4, 5, 14, 15, 16]. In the present paper, we study a space of nonlinear self-mappings of a complete metric space which is not necessarily bounded and show that a point of coincidence exists for a generic (typical) map. More precisely, for this space of mappings equipped with the topology of uniform convergence on bounded sets, we show the existence of a set which is a countable intersection of open everywhere dense sets and such that every its element has a point of coincidence. Such an approach, when a certain property is investigated for the whole space and not just for its single point, has already been successfully applied in many areas of Analysis and Optimization [8, 9, 20, 22, 26].

Assume that (X, ρ) is a complete metric space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

For each $x \in X$ and each set $A \subset X$ put

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

For each mapping $S : X \to X$ set $S^0(x) = x$ for each $x \in X$ and for each integer $i \ge 0$ define $S^{i+1} = S \circ S^i$.

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Assume that $S : X \to X$ and $T : X \to X$. If $x \in X$ and $S(x) = T(x)$, then the point x is called a coincidence point, while the point $y = T(x)$ is called a point of coincidence.

2. The spaces of mappings and the main result

Assume that $S : X \to X$, $S(X)$ is a closed subset of X and that the following assumption holds.

(A1) The mapping S is bounded on bounded sets and $S^{-1}(V)$ is bounded for each bounded set $V \subset X$.

Fix $\theta \in X$. Assume that for each $x \in X$ and each $\gamma \in [0,1]$, there exists a point denoted as $\gamma\theta \oplus (1-\gamma)x \in X$ such that for each $x, y \in X$ and each $\gamma \in [0,1]$,

$$\rho(\gamma\theta \oplus (1-\gamma)x, \gamma\theta \oplus (1-\gamma)y) \le (1-\gamma)\rho(x,y), \tag{1}$$

$$\rho(\gamma\theta \oplus (1-\gamma)x, x) \le \gamma\rho(x,\theta). \tag{2}$$

It should be mentioned the Hilbert ball with the hyperbolic metric [12] and the hyperbolic spaces of [21] are examples of our space X. Denote by $\mathcal{M}_S$ the set of all $T : X \to X$ such that for each $x, y \in X$,

$$\rho(T(x), T(y)) \le \rho(S(x), S(y)). \tag{3}$$

The set $\mathcal{M}_S$ is equipped with the uniformity determined by the base

$$\begin{aligned} \mathcal{U}(M,\epsilon) = \{&(T_1,T_2) \in \mathcal{M}_S \times \mathcal{M}_S : \\ &\rho(T_1(x), T_2(x)) \le \epsilon \text{ for each } x \in B(\theta, M)\}, \end{aligned} \tag{4}$$

where $M, \epsilon > 0$. Clearly, this uniform space is metrizable (by a metric d) and complete. For each $A \in \mathcal{M}_S$ and each $\gamma \in (0,1)$ define

$$A_\gamma(x) = \gamma\theta \oplus (1-\gamma)A(x),\ x \in X. \tag{5}$$

By (A1), (2) and (3), for each $A \in \mathcal{M}_S$, each $\gamma \in (0,1)$ and each $M > 0$,

$$\begin{aligned} \rho(A(x), A_\gamma(x)) &= \rho(A(x), \gamma\theta \oplus (1-\gamma)A(x)) \\ &= \gamma\rho(\theta, A(x)) \le \gamma\rho(\theta, A(\theta)) + \gamma\rho(A(\theta), A(x)) \\ &\le \gamma\rho(\theta, A(\theta)) + \gamma\rho(S(\theta), S(x)) \to 0 \text{ as } \gamma \to 0^+ \end{aligned} \tag{6}$$

uniformly on bounded subsets of X. Together with (4) this implies that

$$A_\gamma \to A \text{ as } \gamma \to 0^+$$

in $(\mathcal{M}_s, d)$. By (1), (3) and (5), for each $A \in \mathcal{M}_S$, each $x, y \in X$ and each $\gamma \in (0,1)$,

$$\rho(A_\gamma(x), A_\gamma(y)) \le (1-\gamma)\rho(A(x), A(y)) \le (1-\gamma)\rho(S(x), S(y)). \tag{7}$$

Proposition 2.1. *Assume that $A \in \mathcal{M}_S$, $\gamma \in (0,1]$, $A_\gamma(X) \subset S(X)$ and that $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty \subset X$ satisfy for each integer $n \ge 0$,*

$$y_n = A_\gamma(x_n),\ S(x_{n+1}) = y_n. \tag{8}$$

Then there exist $y_* = \lim_{n\to\infty} y_n$ and $x_* \in X$ such that $A_\gamma(x_*) = S(x_*) = y_*$.

Proof. By (7) and (8), for each integer $n \geq 1$,

$$\rho(y_n, y_{n+1}) = \rho(A_\gamma(x_n), A_\gamma(x_{n+1})) \leq (1-\gamma)\rho(S(x_n), S(x_{n+1}))$$
$$\leq (1-\gamma)\rho(y_n, y_{n-1}). \tag{9}$$

In view of (9), for each integer $n \geq 1$ we have $\rho(y_n, y_{n+1}) \leq (1-\gamma)^{n-1}\rho(y_0, y_1)$. This implies that there exists

$$y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} S(x_{n+1}). \tag{10}$$

Since the set $S(X)$ is closed there exists $x_* \in X$ such that

$$S(x_*) = y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} S(x_{n+1}). \tag{11}$$

By (8) and (11),
$$y_* = \lim_{n\to\infty} A_\gamma(x_n). \tag{12}$$

It follows from (7) and (11) that for each integer $n \geq 0$,

$$\rho(A_\gamma(x_n), A_\gamma(x_*)) \leq (1-\gamma)\rho(S(x_n, S(x_*)) \to 0 \text{ as } n \to \infty.$$

Together with (12) this implies that $y_* = \lim_{n\to\infty} A_\gamma(x_n) = A_\gamma(x_*)$.
Proposition 2.1 is proved. □

3. The main result

In this paper we prove the following generic result.

Theorem 3.1. *Assume that $\mathcal{A}$ is a nonempty closet set in a metric space $(\mathcal{M}_S, d)$ which is also equipped with the metric d such that for each $T \in \mathcal{A}$ and each $\gamma \in (0,1)$,*

$$\rho(T(x), T(y)) \leq \rho(S(x), S(y)) \text{ for each } x, y \in X,$$

$$T(X) \subset S(X),\ T_\gamma \in \mathcal{A}.$$

Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense sets in $\mathcal{A}$ such that for each $T \in \mathcal{F}$ there exists $y_T \in X$ such that the following assertions hold.

(1) *For each pair of sequences $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty \subset X$ satisfying for each integer $n \geq 0$, the conditions $y_n = T(x_n)$, $S(x_{n+1}) = y_n$, the relation $\lim_{n\to\infty} y_n = y_T$ holds.*

(2) *If $x \in X$ satisfies $T(x) = S(x)$, then $T(x) = y_T$.*

(3) *There exist $x_T \in X$ such that $T(x_T) = S(x_T) = y_T$.*

(4) *For each $\epsilon \in (0,1)$ and each $M > 1$ there exist a neighborhood $\mathcal{U}$ of T in $\mathcal{M}_S$, $\delta \in (0, \epsilon)$ and a natural number n_0 such that for each $A \in \mathcal{U}$ and each pair of sequences $\{x_n\}_{n=0}^\infty, \{y_n\}_{i=0}^\infty \subset X$ which satisfy $\rho(x_0, \theta) \leq M$ and for each integer $n \geq 0$,*

$$\rho(y_n, A(x_n)) \leq \delta,\ \rho(S(x_{n+1}, y_n)) \leq \delta$$

the inequality $\rho(y_n, y_T) \leq \epsilon$ holds for each integer $n \geq n_0$.

4. Auxiliary results

Assume that $\gamma \in (0,1)$ and $T \in \mathcal{A}$, $x_*, y_* \in X$ and that (see Proposition 2.1)

$$T_\gamma(x_*) = S(x_*) = y_*. \tag{13}$$

Lemma 4.1. *Assume that $\delta \in (0,1]$ and that $\{x_n\}_{n=0}^\infty, \{y_n\}_{i=0}^\infty \subset X$ satisfy each integer $n \geq 0$,*

$$\rho(y_n, T_\gamma(x_n)) \leq \delta,\ \rho(S(x_{n+1}), y_n)) \leq \delta. \tag{14}$$

Then for each integer $n \geq 0$,

$$\rho(y_*, y_{n+1}) \leq (1-\gamma)\rho(y_*, y_n) + \delta(2-\gamma)$$

and

$$\rho(y_*, y_n) \leq (1-\gamma)^n \rho(y_*, y_0) + \delta(2-\gamma)\sum_{i=1}^{n-1}(1-\gamma)^i.$$

Proof. Let $n \geq 0$ be an integer. By (7), (13) and (14),

$$\begin{aligned}\rho(y_*, y_{n+1}) &\leq \rho(T_\gamma(x_*), T_\gamma(x_{n+1})) + \rho(T_\gamma(x_{n+1}), y_{n+1}) \leq \rho(T_\gamma(x_*), T_\gamma(x_{n+1})) + \delta \\ &\leq \delta + (1-\gamma)\rho(S(x_*), S(x_{n+1})) \leq \delta + (1-\gamma)(\rho(y_*, y_n) + \delta) \\ &\leq (1-\gamma)\rho(y_*, y_n) + \delta(2-\gamma). \end{aligned} \tag{15}$$

In view of (15), we have $\rho(y_*, y_1) \leq (1-\gamma)\rho(y_*, y_0) + \delta(2-\gamma)$ and

$$\begin{aligned}\rho(y_*, y_2) &\leq (1-\gamma)\rho(y_*, y_1) + \delta(2-\gamma) \\ &\leq (1-\gamma)^2\rho(y_*, y_0) + \delta(2-\gamma)(1 + (1-\gamma)).\end{aligned}$$

We show by induction that for each integer $n \geq 1$,

$$\rho(y_*, y_n) \leq (1-\gamma)^n \rho(y_0, y_*) + \delta(2-\gamma)\sum_{i=0}^{n-1}(1-\gamma)^i. \tag{16}$$

By the relations above (16) holds for $n = 1, 2$. Assume that $k \geq 2$ is an integer and (16) holds for $n = k$. It follows from (15) and (16) (with $n = k$) that

$$\rho(y_*, y_{k+1}) \leq (1-\gamma)\rho(y_*, y_k) + \delta(2-\gamma)$$

$$\leq (1-\gamma)^{k+1}\rho(y_*, y_0) + \delta(2-\gamma)\sum_{i=0}^{k-1}(1-\gamma)^{i+1} + \delta(2-\gamma)$$

$$= (1-\gamma)^{k+1}\rho(y_*, y_0) + \delta(2-\gamma)\sum_{i=0}^{k}(1-\gamma)^i.$$

Thus (16) holds for $n = k+1$ too. Therefore we showed by induction that (16) holds for each integer $n \geq 1$. Lemma 4.1 is proved. □

Lemma 4.2. *Let $M > 0$,*

$$M_1 > M + 2\gamma^{-2}, \tag{17}$$

$$S(B(x_*, M)) \subset B(y_*, M_1 - 1), \tag{18}$$

$$M_2 > M_1, \quad S^{-1}(B(y_*, M_1 + 1)) \subset B(x_*, M_2), \tag{19}$$

a natural number n_0 satisfy

$$(1-\gamma)^{n_0} M_1 \le \gamma, \tag{20}$$

$$A \in \mathcal{M}_S, \quad \rho(A(y), T_\gamma(y)) \le 2^{-1}, \ y \in B(x_*, M_2), \tag{21}$$

$$\{x_n\}_{n=0}^\infty, \{y_n\}_{i=0}^\infty \subset X, \quad \rho(x_0, x_*) \le M, \tag{22}$$

and for each integer $n \ge 0$,

$$\rho(y_n, A(x_n)) \le 2^{-1}, \ \rho(S(x_{n+1}), y_n) \le 2^{-1}. \tag{23}$$

Then

$$\rho(x_n, x_*) \le M_2, \ \rho(y_*, y_n) \le M_1 + 1, \ n = 0, 1, \ldots$$

for each integer $n \ge 0$,

$$\rho(y_*, y_n) \le (1-\gamma)^n \rho(y_*, y_0) + 2\gamma^{-1} \le (1-\gamma)^n M_1 + 2\gamma^{-1}$$

and for each integer $n \ge n_0$,

$$\rho(y_*, y_n) \le (1-\gamma)^{n_0} M_1 + 2\gamma^{-1} \le 3\gamma^{-1}.$$

Proof. By (7), (13), (18) and (21)–(23),

$$\begin{aligned}
\rho(y_*, y_0) &\le \rho(y_*, T_\gamma(x_0)) + \rho(T_\gamma(x_0), A(x_0)) + \rho(A(x_0), y_0) \\
&\le \rho(T_\gamma(x_*), T_\gamma(x_0)) + 1 \\
&\le (1-\gamma)\rho(S(x_*), S(x_0)) + 1 \le M_1.
\end{aligned}$$

In view of (22) and (23),

$$\rho(S(x_1), y_*) \le \rho(S(x_1), y_0) + \rho(y_0, y_1) \le M_1 + 2^{-1}. \tag{24}$$

It follows from (19) and (24) that

$$\rho(S(x_1), x_*) \le M_2.$$

Assume that $n \ge 0$ is an integer and that

$$\rho(y_*, y_n) \le M_1. \tag{25}$$

By (19), (23) and (25),

$$\rho(S(x_{n+1}), y_*) \le \rho(S(x_{n+1}), y_n) + \rho(y_n, y_*) \le M_1 + 2^{-1}. \tag{26}$$

Equations (25) and (26) imply that

$$\rho(x_{n+1}, x_*) \le M_2. \tag{27}$$

It follows from (7), (13), (17), (23), (25)–(27) that

$$\begin{aligned}\rho(y_*, y_{n+1}) &\leq \rho(y_*, T_\gamma(x_{n+1})) + \rho(T_\gamma(x_{n+1}), A(x_{n+1})) + \rho(A(x_{n+1}), y_{n+1}) \\ &\leq \rho(T_\gamma(x_*), T_\gamma(x_{n+1})) + 2^{-1} + 2^{-1} \\ &\leq (1-\gamma)\rho(S(x_*), S(x_{n+1})) + 1 \\ &\leq (1-\gamma)\rho(y_*, y_n) + (1-\gamma)\rho(y_n, S(x_{n+1})) + 1 \\ &\leq (1-\gamma)\rho(y_*, y_n) + 2 \leq (1-\gamma)M_1 + 2 \leq M_1.\end{aligned}$$

Thus if $n \geq 0$ is an integer and $\rho(y_*, y_n) \leq M_1$, then

$$\rho(y_*, y_{n+1}) \leq M_1, \quad \rho(x_*, x_{n+1}) \leq M_2.$$

Thus by induction we showed that for each integer $n \geq 0$,

$$\rho(x_*, x_n) \leq M_2, \quad \rho(y_*, y_n) \leq M_1. \tag{28}$$

Let $n \geq 0$ be an integer. By (21), (23) and (28),

$$\rho(y_n, T_\gamma(x_n)) \leq \rho(y_n, A(x_n)) + \rho(A(x_n), T_\gamma(x_n)) \leq 1,$$
$$\rho(S(x_{n+1}), y_n) \leq 2^{-1}.$$

By the relation above and Lemma 4.1 with $\delta = 1$, for each integer $n \geq 0$,

$$\rho(y_*, y_{n+1}) \leq (1-\gamma)\rho(y_*, y_n) + 2 - \gamma,$$
$$\rho(y_*, y_n) \leq (1-\gamma)^n \rho(y_*, y_0) + (2-\gamma)\gamma^{-1}$$

and in view of (20), (28), for each integer $n \geq n_0$,

$$\rho(y_*, y_n) \leq (1-\gamma)^{n_0} M_1 + 2\gamma^{-1}.$$ □

Lemma 4.3. *Let $\epsilon > 0$, $M > 1$,*

$$M_1 > M + 2\gamma^{-2}, \tag{29}$$

$$S(B(x_*, M)) \subset B(y_*, M_1 - 1), \tag{30}$$

$$M_2 > M_1, \quad S^{-1}(B(y_*, M_1 + 1)) \subset B(x_*, M_2) \tag{31}$$

and let a natural number n_0 satisfy

$$(1-\gamma)^{n_0} M_1 \leq \gamma\epsilon/4, \ \delta \in (0, 8^{-1}\epsilon\gamma). \tag{32}$$

Assume that $A \in \mathcal{M}_S$, $\{x_n\}_{n=0}^\infty, \{y_n\}_{i=0}^\infty \subset X$,

$$\rho(A(y), T_\gamma(y)) \leq \delta, \ y \in B(x_*, M_2), \tag{33}$$

$$\rho(x_0, x_*) \leq M \tag{34}$$

and that for each integer $n \geq 0$,

$$\rho(y_n, A(x_n)) \leq \delta, \tag{35}$$

$$\rho(S(x_{n+1}, y_n) \leq \delta. \tag{36}$$

Then for each integer $n \geq n_0$, we have $\rho(y_n, y_*) \leq \epsilon$.

Proof. By our assumptions and Lemma 4.2, for each integer $n \geq 0$,

$$\rho(x_*, x_n) \leq M_2, \ \rho(y_*, y_n) \leq M_1 \tag{37}$$

and for each integer $n \geq n_0$, $\quad \rho(y_*, y_n) \leq 3\gamma^{-1}$. (38)

It follows from (33), (36) and (37) that for each integer $n \geq 0$,

$$\rho(y_n, T_\gamma(x_n)) \leq \rho(y_n, A(x_n)) + \rho(A(x_n), T_\gamma(x_n)) \leq 2\delta. \tag{39}$$

Lemma 4.1 and (39) imply that for each integer $n \geq 0$,

$$\rho(y_*, y_{n+1}) \leq (1-\gamma)\rho(y_*, y_n) + 2\delta(2-\gamma),$$
$$\rho(y_*, y_n) \leq (1-\gamma)^n \rho(y_*, y_0) + 2\delta(2-\gamma)\gamma^{-1}.$$

By the relation above, (32) and (37), for each integer $n \geq n_0$,

$$\rho(y_*, y_n) \leq (1-\gamma)^{n_0} M_1 + 4\delta\gamma^{-1} < \epsilon.$$ □

5. Proof of Theorem 3.1

In view of (7), $\{T_\gamma : T \in \mathcal{A}, \ \gamma \in (0,1)\}$ is an everywhere dense set in $\mathcal{A}$. Let $T \in \mathcal{A}$, $\gamma \in (0,1)$ and m be a natural number. By Proposition 2.1, there exists $x_{T,\gamma}, y_{T,\gamma} \in X$ such that

$$S(x_{T,\gamma}) = y_{T,\gamma} = T(x_{T,\gamma}). \tag{40}$$

Lemmas 4.2 and 4.3 imply that there exist

$$M_1(T,\gamma,m) > m + 2\gamma^{-2}, \ M_2(T,\gamma,m) > M_1(T,\gamma,m), \ \delta(T,\gamma,m) \in (0, 4^{-m-1})$$

and natural numbers $n_1(T,\gamma,m)$, $n_0(T,\gamma,m)$ such that the following properties hold:

(i) for each $A \in \mathcal{M}_S$ satisfying

$$\rho(A(y), T_\gamma(y)) \leq 2^{-1}, \ y \in B(x_{T,\gamma}, M_2(T,\gamma,m))$$

and for each $\{x_n\}_{n=0}^\infty, \{y_n\}_{i=0}^\infty \subset X$ satisfying $\rho(x_0, x_{T,\gamma}) \leq M$, and for each $n \geq 0$,

$$\rho(y_n, A(x_n)) \leq 2^{-1}, \ \rho(S(x_{n+1}), y_n) \leq 2^{-1}$$

we have $\quad \rho(x_n, x_{T,\gamma}) \leq M_2, \ \rho(y_{T,\gamma}, y_n) \leq M_1 + 1, \ n = 0, 1, \ldots.$

and for each integer $n \geq n_1(T,\gamma,m)$, $\ \rho(y_{T,\gamma}, y_n) \leq 3\gamma^{-1}$.

(ii) for each $A \in \mathcal{M}_S$ satisfying

$$\rho(A(y), T_\gamma(y)) \leq \delta(T,\gamma,m), \ y \in B(\theta, M_2(T,\gamma,m)),$$

and for each $\{x_n\}_{n=0}^\infty, \{y_n\}_{i=0}^\infty \subset X$ satisfying $\rho(x_0, \theta) \leq m$, and for each $n \geq 0$,

$$\rho(y_n, A(x_n)) \leq \delta(T,\gamma,m), \ \rho(S(x_{n+1}), y_n) \leq \delta(T,\gamma,m)$$

we have for each integer $n \geq n_0(T,\gamma,m)$, $\ \rho(y_{T,\gamma}, y_n) \leq 2^{-m-4}$.

There exists an open neighborhood $\mathcal{U}(T,\gamma,m)$ of T_γ in $\mathcal{M}_S$ such that for each $y \in B(\theta.M_2(T,\gamma,m))$,

$$\rho(A(y), T_\gamma(y)) \leq \delta(T,\gamma,m)/2. \tag{41}$$

Define

$$\mathcal{F} = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,m) \cap \mathcal{A} :\ T \in \mathcal{M}_S,\ \gamma \in (0,1),\ m \geq p(s) \text{ is an integer}\}. \quad (42)$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\mathcal{A}$.
Assume that $T \in \mathcal{F},\ \epsilon \in (0,1),\ M > 1$.

Choose an integer
$$p > \epsilon^{-1} + M + 1. \quad (44)$$

In view of (42) and (43), there exist $T_p \in \mathcal{A}$, $\gamma_p \in (0,1)$ and an integer $m \geq p$ such that
$$T \in \mathcal{U}(T_p, \gamma_p, m). \quad (45)$$

Assume that
$$A \in \mathcal{U}(T_p, \gamma_p, m), \quad (46)$$

$$\{x_n\}_{n=0}^{\infty}, \{y_n\}_{i=0}^{\infty} \subset X \ \text{ satisfy } \ \rho(x_0, \theta) \leq m, \quad (47)$$

and that for each integer $n \geq 0$,

$$\rho(y_n, A(x_n)) \leq \delta(T_p, \gamma_p, m),\ \rho(S(x_{n+1}), y_n) \leq \delta(T_p, \gamma_p, m). \quad (48)$$

Property (ii) and (41), (46)–(48) imply that for each integer $n \geq n_0(T_p, \gamma_p, m)$,

$$\rho(y_m, y_{T_p,\gamma_p}) \leq 2^{-m-4}. \quad (49)$$

This implies that the following property holds:

(iii) For each $A \in \mathcal{U}(T_p, \gamma_p, m)$ and each $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{i=0}^{\infty} \subset X$ satisfying (47) and (48) for each integer $n \geq 0$, we have for each integer $n \geq n_0(T_p, \gamma_p, m)$ the inequality $\rho(y_n, y_{T_p,\gamma_p}) \leq \epsilon/4$.

Assume that $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{i=0}^{\infty} \subset X$ satisfy $\rho(x_0, \theta) \leq M$ and that for each integer $n \geq 0$, $y_n = A(x_n)$ and $S(x_{n+1}) = y_n$.

Property (iii) implies that for each integer $n \geq n_0(T_p, \gamma_p, m)$,

$$\rho(y_n, y_{T_p,\gamma_p}) \leq \epsilon/4. \quad (50)$$

In view of (50), since ϵ is any element of $(0,1)$, we conclude that $\{y_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists $\lim_{i\to\infty} y_i$ in (X,ρ). By (50),

$$\rho(\lim_{i\to\infty} y_i, y_{T_p,\gamma_p}) \leq \epsilon/4. \quad (51)$$

Assume that $\{\tilde{x}_n\}_{n=0}^{\infty}, \{\tilde{y}_n\}_{i=0}^{\infty} \subset X$ satisfy $\rho(\tilde{x}_0, \theta) \leq M$ and that for each integer $n \geq 0$ we have $\tilde{y}_n = T(\tilde{x}_n)$ and $S(\tilde{x}_{n+1}) = \tilde{y}_n$.

Clearly, there exists $\lim_{i\to\infty} \tilde{y}_i$ in (X,ρ) and $\rho(\lim_{i\to\infty} \tilde{y}_i, y_{T_p,\gamma_p}) \leq \epsilon/4$.

Together with (51) this implies that $\rho(\lim_{i\to\infty} \tilde{y}_i, \lim_{i\to\infty} y_i) \leq 2^{-1}\epsilon$.

Since ϵ is any element of $(0,1)$ and M is any element of $(1,\infty)$ we conclude that there exists $y_T \in X$ such that for each pair of sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{i=0}^{\infty} \subset X$ satisfying for each integer $n \geq 0$,

$$y_n = T(x_n), \quad S(x_{n+1}) = y_n \quad (52)$$

we have
$$\lim_{i\to\infty} y_i = y_T. \quad (53)$$

In view of (50),
$$\rho(y_T, y_{T_p,\gamma_p}) \le \epsilon/4. \tag{54}$$

This implies that if $x \in X$ and $T(x) = S(x)$, then $T(x) = y_T$. Since $g(X)$ is closed there exists $x_T \in X$ such that
$$S(x_T) = y_T. \tag{55}$$

Then for each for each pair of sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{i=0}^{\infty} \subset X$ satisfying (52) for each integer $i \ge 0$, it follows from (53) that for each integer $n \ge 0$,

$$\rho(T(x_T), T(x_n)) \le \rho(S(x_T), y_n) \le \rho(y_T, y_n) \to 0 \text{ as } n \to \infty,$$

$$T(x_T) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} y_n = y_T.$$

Assume that $A \in \mathcal{U}(T_p, \gamma_p, m)$, $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{i=0}^{\infty} \subset X$, $\rho(x_0, \theta) \le M$ and that for each integer $n \ge 0$ (48) holds. Then (see (49)) for each integer $n \ge n_0(T_p, \gamma_p, m)$,

$$\rho(y_m, y_{T_p,\gamma_p}) \le \epsilon/4$$

and together with (54) this implies that for each integer $n \ge n_0(T_p, \gamma_p, m)$,

$$\rho(y_n, y_T) \le \epsilon/2.$$

Theorem 3.1 is proved.

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