

Mosco-Convergence of Convex Sets and Unilateral Problems for Differential Operators with Lower Order Terms Having Natural Growth

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We study the stability of solutions to a class of variational inequalities posed on obstacle-type convex sets, under Mosco-convergence. Specifically, we consider problems of the form

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), & u \geq \psi \quad \text{in } \Omega, \\ \langle A(u), v - u \rangle + \int_\Omega H(x, u, Du)(v - u) \geq 0, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), & v \geq \psi \quad \text{in } \Omega. \end{cases}$$

Here, A is a Leray–Lions type operator, mapping $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, while $H(x, u, Du)$ grows like $|Du|^p$. The obstacle ψ is a function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Our main result establishes that the solutions are stable under Mosco-convergence of the constraint sets. This extends classical stability results to natural growth problems.

Keywords: Mosco-convergence, variational inequalities, obstacle-type convex sets, Leray–Lions operators, operators with natural growth terms, stability of solutions.

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1. Introduction

The main focus of the papers [16], [17] by Umberto Mosco is the study of the convergence properties of solutions to variational inequalities involving strongly monotone operators. In particular, the author investigates under which conditions on the convex sets the solutions converge, thereby introducing a notion of convergence for sequences of convex sets in reflexive Banach spaces. Since then, this condition is known as Mosco-convergence and several papers are devoted to this subject.

In a recent paper (see [7]), the stability result by Mosco has been extended to pseudomonotone operators (which, in general, are not monotone; see [6]), of differential type defined in Sobolev spaces.

On the other hand, monotone operators and convex functionals are strongly related: after the first results by Umberto Mosco on convex functionals, a paper dedicated to him (cf. [3]) studies the convergence in $W_0^{1,p}(\Omega)$ of constrained minima on convex sets C_n for integral functionals of the form

$$I(v) = \int_{\Omega} i(x, Dv) - \int_{\Omega} f(x) v(x),$$

where $i(x, \xi)$ is a convex function in ξ , assuming Mosco-convergence (see Definition 2.1 below) of the sets C_n . We point out that the proofs do not change if we consider functionals like

$$J(v) = \int_{\Omega} j(x, v, Dv) - \int_{\Omega} f(x) v(x),$$

where $j(x, s, \xi)$ is strictly convex in ξ .

Furthermore, we recall that, in general, J is convex only if $j(x, s, \xi)$ does not depend on s (see [10]) and that (roughly speaking) its Fréchet differential J' is of the form

$$\langle J'(v), w \rangle = \int_{\Omega} \partial_{\xi} j(x, u, Du) \cdot Dw + \int_{\Omega} \partial_s j(x, u, Du) w - \int_{\Omega} f w, \quad (1)$$

for all $w \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. In particular, the lower order term $\partial_s j(x, u, Du)$ has *natural growth*, meaning it has order p with respect to the gradient.

This motivates the study of variational inequalities in $W_0^{1,p}(\Omega)$, with $p \in (1, \infty)$, involving a natural growth term. This problem was studied in [4] and [5] under the assumption that the convex set was of obstacle type, specifically:

$$\mathcal{C}(\psi) = \{u \in W_0^{1,p}(\Omega) \mid u \geq \psi \text{ in } \Omega\}.$$

Building on these results, our aim is to extend the result of [7] to differential operators with a Leray–Lions type principal part and lower order terms exhibiting natural growth. More specifically, this paper examines the stability properties of solutions to

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega), & u \geq \psi \quad \text{in } \Omega, \\ \langle A(u), v - u \rangle + \int_{\Omega} a_0(x, u)(v - u) + \int_{\Omega} H(x, u, Du)(v - u) \geq 0, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega), & v \geq \psi \quad \text{in } \Omega, \end{cases} \quad (2)$$

where A is a Leray–Lions type operator and H has natural growth, under Mosco-convergence of convex sets.

We point out that problems of the type (2) are loosely related to those studied in [1], where the authors consider a quasilinear quasivariational inequality arising from a stochastic control problem with a quadratic growth Hamiltonian and obstacle.

2. Main result and outline

In this section, we present the main result of this work. Before proceeding, let us recall the definition of Mosco-convergence.

Definition 2.1. Let $\{\mathcal{C}_n\}_n$ be a sequence of closed convex subsets of a Banach space $(X, \|\cdot\|)$. We say that $\{\mathcal{C}_n\}_n$ *Mosco-converges* to a closed convex set $\mathcal{C}_0 \subseteq X$ if:

- (1) Whenever $\{n_i\}_i$ is an increasing sequence of positive integers and $v_{n_i} \in \mathcal{C}_{n_i}$ for each i , the weak convergence of $\{v_{n_i}\}_i$ to v_0 implies $v_0 \in \mathcal{C}_0$, and
- (2) for every $v_0 \in \mathcal{C}_0$, there exist $v_n \in \mathcal{C}_n$ such that $\|v_n - v_0\| \rightarrow 0$.

We will write $\mathcal{C}_n \xrightarrow{M} \mathcal{C}_0$.

In particular, we focus on a specific class of convex sets, known as obstacle-type convex sets. Given $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we define the associated obstacle-type convex set as

$$\mathcal{C}(\psi) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}.$$

We are interested in the stability properties of the solutions to the inequality

$$\begin{cases} u \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \\ \langle A(u), v - u \rangle + \int_{\Omega} a_0(x, u)(v - u) + \int_{\Omega} H(x, u, Du)(v - u) \geq 0, \\ \forall v \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \end{cases} \quad (3)$$

under Mosco-convergence of the obstacle-type convex sets.

Here, we state the assumptions on Ω , p , A , a_0 , H and ψ .

- Ω is an open, bounded subset of $\mathbb{R}^N$.
- $p \in (1, +\infty)$ and $p' = \frac{p}{p-1}$.
- The principal part A is a differential operator of second order in divergence form acting from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$:

$$A(v) = -\operatorname{div}(a(x, v, Dv)),$$

where a is a Carathéodory function with values in $\mathbb{R}^N$ satisfying the following conditions for all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$, and for almost every $x \in \Omega$:

$$a(x, s, \xi)\xi \geq \alpha|\xi|^p, \quad (4)$$

$$|a(x, s, \xi)| \leq \beta (h(x) + |s|^{p-1} + |\xi|^{p-1}), \quad (5)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0, \quad (6)$$

where α, β are positive constants and $h \in L^{p'}(\Omega)$.

- a_0 is a Carathéodory function with values in $\mathbb{R}$ such that for all $s \in \mathbb{R}$, and for almost every $x \in \Omega$:

$$|a_0(x, s)| \leq \beta [h(x) + |s|^{p-1}], \quad (7)$$

$$a_0(x, s)s \geq \alpha_0|s|^p, \quad \text{with } \alpha_0 > 0. \quad (8)$$

- $H(x, s, \xi)$ is a Carathéodory function such that for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$:

$$|H(x, s, \xi)| \leq f(x) + b(|s|)|\xi|^p, \quad (9)$$

where $f \in L^\infty(\Omega)$ is a positive function and $b(\cdot)$ is a positive, increasing, continuous function defined on $\mathbb{R}^+$.

Remark 2.2. Note that, under assumptions (4), (5), (6), A is a bounded and continuous pseudomonotone operator of Leray-Lions type.

The existence of a solution $u \in W_0^{1,p}(\Omega)$ to the variational inequalities (3) was proved in [5]. More precisely, the following theorem holds.

Theorem 2.3. *There exists a solution $u \in W_0^{1,p}(\Omega)$ to*

$$\begin{cases} u \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \\ \langle A(u), v - u \rangle + \int_{\Omega} a_0(x, u)(v - u) + \int_{\Omega} H(x, u, Du)(v - u) \geq 0, \\ \forall v \in \mathcal{C}(\psi) \cap L^\infty(\Omega). \end{cases} \quad (10)$$

We now state the main problem: let $\psi_n, \psi_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. From the argument in the previous section, we know that there exist solutions u_n and u_0 to the following variational problems:

$$\begin{cases} u_n \in \mathcal{C}(\psi_n) \cap L^\infty(\Omega), \\ \langle A(u_n), v - u_n \rangle + \int_{\Omega} a_0(x, u_n)(v - u_n) + \int_{\Omega} H(x, u_n, Du_n)(v - u_n) \geq 0, \\ \forall v \in \mathcal{C}(\psi_n) \cap L^\infty(\Omega) \end{cases} \quad (11)$$

and

$$\begin{cases} u_0 \in \mathcal{C}(\psi_0) \cap L^\infty(\Omega), \\ \langle A(u_0), v - u_0 \rangle + \int_{\Omega} a_0(x, u_0)(v - u_0) + \int_{\Omega} H(x, u_0, Du_0)(v - u_0) \geq 0, \\ \forall v \in \mathcal{C}(\psi_0) \cap L^\infty(\Omega). \end{cases} \quad (12)$$

The main result of this paper states that, if the sequence $\{\psi_n\}_n$ is such that convex sets $\mathcal{C}(\psi_n)$ Mosco-converge to $\mathcal{C}(\psi_0)$, then the solutions u_n of (11) satisfy a stability property. That is, the sequence $\{u_n\}_n$ has a subsequence that converges in $W_0^{1,p}(\Omega)$ to a solution u_0 of (12). This is formalized in the following theorem.

Theorem 2.4. *Let $\psi_n, \psi_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Consider the convex sets*

$$\mathcal{C}(\psi_n) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi_n \text{ a.e. in } \Omega\},$$

$$\mathcal{C}(\psi_0) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi_0 \text{ a.e. in } \Omega\},$$

and assume that $\{\psi_n\}_n$ is bounded in $L^\infty(\Omega)$ and that

$$\mathcal{C}(\psi_n) \xrightarrow{M} \mathcal{C}(\psi_0).$$

For every n , consider the solution u_n to (11). Then there exists a subsequence $\{u_{n_j}\}_j$ and $u_0 \in \mathcal{C}(\psi_0)$ such that $u_{n_j} \rightarrow u_0$ in $W_0^{1,p}(\Omega)$. Moreover, u_0 is a solution to (12).

Outline of the paper: in Section 3, we recall some results regarding Mosco-convergence of convex sets. In Section 4, we follow the proof from [5] to establish a quantitative estimate for u_n , the solution to (11). In Section 5, we prove Theorem 2.4.

For simplicity, in what follows, we will extract subsequences when necessary, without explicitly stating it each time.

3. Mosco-convergence

In this section we recall some basic results regarding Mosco-convergence (see [16]). We start with a simple example of the relation between the convergence of convex sets and the convergence of solutions to variational inequalities.

Let H be a Hilbert space and $\{C_n\}_n$, C_0 be closed convex subsets of H . For any $f \in H$, let $u_n(f)$ (resp $u_0(f)$) be the projection of f onto the set C_n (resp C_0), which are characterized by the inequalities:

$$u_n \in C_n : (u_n - f \mid v - u_n) \geq 0, \quad \forall v \in C_n; \quad (13)$$

$$u_0 \in C_0 : (u_0 - f \mid v - u_0) \geq 0, \quad \forall v \in C_0. \quad (14)$$

Proposition 3.1. *The following are equivalent:*

- $C_n \xrightarrow{M} C_0$,
- $\|u_n(f) - u_0(f)\| \rightarrow 0$, for all $f \in H$.

Proof. We prove that $\xrightarrow{M}$ implies $\|u_n - u_0\| \rightarrow 0$.

- Fix $w_0 \in C_0$; there exists $w_n \in C_n$ such that $\|w_n - w_0\| \rightarrow 0$; the choice $v = w_n$ in (13) yields the boundedness (in H) of the sequence $\{u_n\}_n$.
- As a consequence, there exist $u^* \in H$ and a subsequence $\{u_{n_i}\}_i$ weakly convergent to u^* ; the definition of Mosco-convergence implies that $u^* \in C_0$.
- Fix $w_0 \in C_0$; there exists $w_n \in C_n$ such that $\|w_n - w_0\| \rightarrow 0$; the choice $v = w_n$ in (13) yields

$$(u_{n_i} - f \mid w_{n_i} - u_{n_i}) \geq 0.$$

We pass to the limit (use the weak l.s.c. of the norm) and we deduce that

$$u^* \in C_0 : (u^* - f \mid w_0 - u^*) \geq 0, \quad \forall w_0 \in C_0;$$

that is u^* is a solution on C_0 : the uniqueness of the solution says that $u^* = u_0$ and thus the whole sequence weakly converges to u_0 .

- There exists $z_n \in C_n$ such that $\|z_n - u_0\| \rightarrow 0$; the choice $v = z_n$ in (13) yields

$$0 \leq (u_n - u_0 + u_0 - f \mid z_n - u_0 - u_n + u_0)$$

$$\text{and} \quad (u_n - u_0 \mid u_n - u_0) \leq (u_n - u_0 \mid z_n - u_0) + (u_0 - f \mid z_n - u_n),$$

which implies that $\|u_n - u_0\| \rightarrow 0$

Conversely, we prove that that if $\|u_n(f) - u_0(f)\| \rightarrow 0, \forall f \in H$, then $C_n \xrightarrow{M} C_0$.

- Let $\{w_n\}_n$ be a sequence that converges weakly to w_0 , with $w_n \in C_n$. Consider the problems (13) with $f = w_0$.

$$u_n \in C_n : (u_n - w_0 \mid v - u_n) \geq 0, \quad \forall v \in C_n.$$

Take $v = w_n$, then $u_n \in C_n : (u_n - w_0 \mid w_n - u_n) \geq 0$.

The assumptions imply that it is possible to pass to the limit and

$$(u_0 - w_0 \mid w_0 - u_0) \geq 0,$$

that is $w_0 = u_0$; so that $w_0 \in C_0$.

- Let $w_0 \in C_0$. Define $f = w_0$, then $u_0 = w_0$. Moreover $\{u_n\}_n$ is the required sequence. $\square$

In the specific case of obstacle-type convex sets, we can associate Mosco-convergence with the behavior of the obstacle functions. The following proposition provides sufficient conditions for the Mosco-convergence of these obstacle-type convex sets, which depend on the properties of the associated obstacles (see [3], Proposition 3).

Proposition 3.2. (Obstacle-Type Convex Sets) *Let $\{\psi_n\}_n$ be a sequence of measurable functions such that the obstacle sets*

$$\mathcal{C}(\psi_n) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi_n \text{ a.e. in } \Omega\}$$

are nonempty. Then, $\mathcal{C}(\psi_n)$ Mosco-converges to $\mathcal{C}(\psi_0)$ if any of the following conditions hold:

1. ψ_n weakly converges in $W_0^{1,p}(\Omega)$ to ψ_0 , with $\psi_n \leq \psi_0$;
2. ψ_n strongly converges in $W_0^{1,p}(\Omega)$ to ψ_0 (see Lemma 1.7 of [16]);
3. ψ_n weakly converges in $W_0^{1,p}(\Omega)$ to ψ_0 , with $\psi_n \geq \psi_0$, and $A(\psi_n) \leq T$, where $A(v) = -\operatorname{div}(a(x, Dv))$ is a Leray-Lions operator and T belongs to the dual of $W_0^{1,p}(\Omega)$;
4. ψ_n strongly converges in $L^\infty(\Omega)$ to ψ_0 , and there exists $\Psi \in W_0^{1,p}(\Omega)$ such that $\Psi \geq \psi_n$ for all n (see [4]);
5. ψ_n weakly converges in $W_0^{1,q}(\Omega)$ to ψ_0 , with $q > p$ (see [2], [4]).

Furthermore, the paper [11] establishes some necessary and sufficient conditions for the Mosco-convergence of sequences of obstacle-type convex sets. These conditions are expressed in terms of the properties of the p -capacities of the level sets of the obstacles.

4. Variational inequalities with a natural growth term

In this section we follow the proof of Theorem 2.3 presented in [5] to gain quantitative estimates on the solutions u of

$$\begin{cases} u \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \\ \langle A(u), v - u \rangle + \int_{\Omega} a_0(x, u)(v - u) + \int_{\Omega} H(x, u, Du)(v - u) \geq 0, \\ \forall v \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \end{cases} \quad (15)$$

$$\text{where} \quad \mathcal{C} = \mathcal{C}(\psi) = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}. \quad (16)$$

First, note that the solution is not unique in general. So, when we talk about *the* solution, we mean *one* of the possible solutions. It is also important to emphasize that the existence of a solution to this problem strongly relies on the obstacle-type structure of these convex sets, and thus this assumption cannot be relaxed.

In the estimates below, for the solution u of (15) we explicitly state the dependence on ψ . The goal is to use these estimates to study the stability of the solutions to (15), where ψ is replaced by a sequence $\{\psi_n\}_n$.

The strategy to obtain estimates for the solution u of (15) relies on considering a sequence of approximating problems of (15), for which the existence of solutions u_j is known from classical theorems by Brezis (cf. [8, 14]). The next step is to derive an estimate for the sequence $\{u_j\}_j$ in both $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$. Thanks to these estimates, we can prove that the sequence $\{u_j\}_j$ converges to u , the solution to (15), which inherits the same bounds.

Consider then the following approximating problems:

$$\begin{cases} u_j \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \\ \langle A(u_j), v - u_j \rangle + \int_{\Omega} a_0(x, u_j)(v - u_j) + \int_{\Omega} H_j(x, u_j, Du_j)(v - u_j) \geq 0, \\ \forall v \in \mathcal{C}(\psi) \cap L^\infty(\Omega), \end{cases} \quad (17)$$

where
$$H_j(x, s, \xi) = \frac{H(x, s, \xi)}{1 + \frac{1}{j}|H(x, s, \xi)|}, \quad \forall j \in \mathbb{N}. \quad (18)$$

Recall that, since H_j is bounded, for every fixed $j \in \mathbb{N}$ there exists at least one solution u_j to the system (17) (cf. [13]). Furthermore, for every $j \in \mathbb{N}$ there exists a constant $\tilde{C}_j$ such that

$$\|u_j\|_{L^\infty(\Omega)} \leq \tilde{C}_j$$

(cf. [8]). The next Lemma improves this estimate, removing the dependence from j and giving a bound on the sequence $\{u_j\}_j$ in $L^\infty(\Omega)$.

Lemma 4.1. *There exists a positive constant $C_\infty = C_\infty(\|f\|_{L^\infty(\Omega)}, \alpha_0, p, \|\psi\|_{L^\infty(\Omega)})$ such that, for every $j \in \mathbb{N}$, the solutions u_j of (17) satisfy the estimate*

$$\|u_j\|_{L^\infty(\Omega)} \leq C_\infty. \quad (19)$$

More precisely, C_∞ is defined as

$$C_\infty := \max \left\{ \left(\frac{\|f\|_{L^\infty(\Omega)}}{\alpha_0} \right)^{1/(p-1)}, \|\psi\|_{L^\infty(\Omega)} \right\}. \quad (20)$$

Before proving Lemma 4.1, we need the following Remark.

Remark 4.2. Let c , d and λ be positive real numbers and φ_λ be the real function defined by

$$\varphi_\lambda(t) = te^{\lambda t^2}. \quad (21)$$

If $\lambda = c^2/4d^2$ the following inequality holds:

$$d\varphi'_\lambda(t) - c|\varphi_\lambda(t)| \geq d/2, \quad \forall t \in \mathbb{R}. \quad (22)$$

Proof of Lemma 4.1. Consider the function v_j defined by

$$v_j = u_j + \varphi_{\lambda_j}(z_j^-), \quad (23)$$

where, for any real value z , $z^- = \max\{-z; 0\}$, and

$$\lambda_j = \frac{C_j^2}{4\alpha^2}, \quad C_j = b\left(\|u_j\|_{L^\infty(\Omega)}\right), \quad z_j = u_j + \left(\frac{\|f\|_{L^\infty(\Omega)}}{\alpha_0}\right)^{1/(p-1)}.$$

As we already remarked, we have $u_j \in L^\infty(\Omega)$ for each $j \in \mathbb{N}$, whence v_j belongs to $\mathcal{C}(\psi) \cap L^\infty(\Omega)$ (recall that $\alpha, \alpha_0 > 0$) and can thus be used as a test function in (17) to get

$$\int_{\Omega} a(x, u_j, Du_j) Dz_j^- \varphi'_{\lambda_j}(z_j^-) + \int_{\Omega} a_0(x, u_j) \varphi_{\lambda_j}(z_j^-) + \int_{\Omega} H(x, u_j, Du_j) \varphi_{\lambda_j}(z_j^-) \geq 0.$$

Using (9), we get

$$\begin{aligned} \int_{\Omega} a(x, u_j, Du_j) Dz_j^- \varphi'_{\lambda_j}(z_j^-) + \int_{\Omega} a_0(x, u_j) \varphi_{\lambda_j}(z_j^-) \\ + \int_{\Omega} \left(\|f\|_{L^\infty(\Omega)} + C_j |Du_j|^p \right) \varphi_{\lambda_j}(z_j^-) \geq 0, \end{aligned}$$

which, since $Dz_j = Du_j$ and $\varphi_{\lambda_j}(z_j^-) = 0$ a.e. where $z_j \geq 0$,

$$\begin{aligned} \int_{\Omega} a(x, u_j, Dz_j) Dz_j^- \varphi'_{\lambda_j}(z_j^-) + \int_{\Omega} a_0(x, u_j) \varphi_{\lambda_j}(z_j^-) \\ + \int_{\Omega} \left(\|f\|_{L^\infty(\Omega)} + C_j |Dz_j^-|^p \right) \varphi_{\lambda_j}(z_j^-) \geq 0. \end{aligned}$$

This, using (4) and rearranging terms, leads to

$$\int_{\Omega} \left\{ \alpha \varphi'_{\lambda_j}(z_j^-) - C_j \varphi_{\lambda_j}(z_j^-) \right\} |Dz_j^-|^p \leq \int_{\Omega} \left[a_0(x, u_j) + \|f\|_{L^\infty(\Omega)} \right] \varphi_{\lambda_j}(z_j^-). \quad (24)$$

Note that the previous integrals are non-zero only in the set

$$\left\{ x \in \Omega : u_j \leq -\left(\frac{\|f\|_{L^\infty(\Omega)}}{\alpha_0} \right)^{1/(p-1)} \right\},$$

where we have, by (8), $a_0(x, u_j) \leq \alpha_0 |u_j|^{p-2} u_j \leq -\|f\|_{L^\infty(\Omega)}$.

In particular, we have

$$\int_{\Omega} \left\{ \alpha \varphi'_{\lambda_j}(z_j^-) - C_j \varphi_{\lambda_j}(z_j^-) \right\} |Dz_j^-|^p \leq 0. \quad (25)$$

It follows (using Remark 4.2 with $d = \alpha$ and $c = C_j$) that we have

$$\alpha \int_{\Omega} |Dz_j^-|^p \leq 0,$$

which implies $z_j^- = 0$.

In conclusion, we have
$$u_j \geq - \left(\frac{\|f\|_{L^\infty(\Omega)}}{\alpha_0} \right)^{1/(p-1)}. \quad (26)$$

We now consider the test function v_j defined by

$$v_j = u_j - \delta_j \varphi_{\lambda_j} (z_j^+), \quad (27)$$

where
$$z_j = u_j - C_\infty, \quad C_\infty = \max \left\{ \left(\frac{\|f\|_{L^\infty(\Omega)}}{\alpha_0} \right)^{1/(p-1)}; \|\psi\|_{L^\infty(\Omega)} \right\},$$

$$\delta_j = e^{\{-\lambda_j \|u_j\|_{L^\infty(\Omega)}^2\}} \quad \text{and} \quad z_j^+ = \max\{z; 0\}.$$

Note that $0 \leq \delta_j \varphi_{\lambda_j} (z_j^+) \leq z_j^+$ and $v_j \in \mathcal{C}(\psi)$. It follows that, substituting into the inequality (17) and using the bound for the function H (9), we have

$$\begin{aligned} \delta_j \int_{\Omega} a(x, u_j, Du_j) D z_j^+ \varphi'_{\lambda_j} (z_j^+) + \delta_j \int_{\Omega} a_0(x, u_j) \varphi_{\lambda_j} (z_j^+) \\ \leq \delta_j \int_{\Omega} \left(\|f\|_{L^\infty(\Omega)} + C_j |Du_j|^p \right) \varphi_{\lambda_j} (z_j^+). \end{aligned}$$

Using (4) (recall that $\varphi'_\lambda \geq 0$), this implies

$$\int_{\Omega} \left\{ \alpha \varphi'_{\lambda_j} (z_j^+) - C_j \varphi_{\lambda_j} (z_j^+) \right\} |D z_j^+|^p + \int_{\Omega} \left[a_0(x, u_j) - \|f\|_{L^\infty(\Omega)} \right] \varphi_{\lambda_j} (z_j^+) \leq 0.$$

Recall that the previous integrals are non-zero only in $\{x \in \Omega : u_j \geq C_\infty\}$. In this set, by (8), we have

$$a_0(x, u_j) \geq \alpha_0 |u_j|^{p-2} u_j \geq \alpha_0 (C_\infty)^{p-1} \geq \|f\|_{L^\infty(\Omega)}.$$

In particular, we have

$$\int_{\Omega} \left\{ \alpha \varphi'_{\lambda_j} (z_j^+) - C_j \varphi_{\lambda_j} (z_j^+) \right\} |D z_j^+|^p \leq 0. \quad (28)$$

Once again, we apply Remark 4.2 to conclude that $z_j^+ = 0$, which implies

$$u_j \leq C_\infty. \quad (29)$$

The inequalities (26) and (29) give the desired estimate (19). $\square$

The next lemma gives a uniform estimate on the norms of $\{u_j\}_j$ in $W_0^{1,p}(\Omega)$.

Lemma 4.3. *There exists a positive constant $M = M(\alpha, p, |\Omega|, \beta)$ (depending only on the data in the assumptions) such that, for every $\Psi \in \mathcal{C}(\psi) \cap L^\infty(\Omega)$,*

$$\begin{aligned} \int_{\Omega} |Du_j|^p \leq M(p, \alpha, |\Omega|, \beta) \left[(1 + \lambda \tilde{C}^2) e^{4\lambda \tilde{C}^2} \right]^p \left\{ \|f\|_{L^\infty(\Omega)} + \|h\|_{L^{p'}(\Omega)} \right. \\ \left. + \|D\Psi\|_{L^p(\Omega)}^p + \|D\Psi\|_{L^p(\Omega)} \left(\|h\|_{L^{p'}(\Omega)} + C_\infty^{p-1} \right) + C_\infty^{p-1} \right\}, \end{aligned} \quad (30)$$

with C_∞ given by (20) and

$$B_\infty = b(C_\infty), \quad \tilde{C} = C_\infty + \|\Psi\|_{L^\infty(\Omega)}, \quad \text{and} \quad \lambda = \frac{B_\infty^2}{4\alpha^2}. \quad (31)$$

Proof. Let C_∞ be as in (20). Given Ψ in $\mathcal{C}(\psi) \cap L^\infty(\Omega)$, we define v_j as

$$v_j = \left[1 - \delta e^{\{\lambda|u_j - \Psi|^2\}} \right] u_j + \delta e^{\{\lambda|u_j - \Psi|^2\}} \Psi, \quad (32)$$

where
$$\lambda = \frac{b(C_\infty)^2}{4\alpha^2}, \quad \delta = e^{-\{\lambda(C_\infty + \|\Psi\|_{L^\infty(\Omega)})^2\}}.$$

Note that, by Lemma 4.1 v_j belongs to $\mathcal{C}(\psi) \cap L^\infty(\Omega)$. We can thus use it as a test function in (17). Since $v_j - u_j = -\delta\varphi_\lambda(u_j - \Psi)$, we obtain

$$\begin{aligned} & -\delta \int_{\Omega} a(x, u_j, Du_j) D(u_j - \Psi) \varphi'_\lambda(u_j - \Psi) - \delta \int_{\Omega} a_0(x, u_j) \varphi_\lambda(u_j - \Psi) \\ & - \delta \int_{\Omega} H_j(x, u_j, Du_j) \varphi_\lambda(u_j - \Psi) \geq 0. \end{aligned}$$

Dividing by δ both sides and rearranging terms, we get

$$\begin{aligned} & \int_{\Omega} a(x, u_j, Du_j) Du_j \varphi'_\lambda(u_j - \Psi) \leq \int_{\Omega} a(x, u_j, Du_j) D\Psi \varphi'_\lambda(u_j - \Psi) \\ & - \int_{\Omega} a_0(x, u_j) \varphi_\lambda(u_j - \Psi) - \int_{\Omega} H_j(x, u_j, Du_j) \varphi_\lambda(u_j - \Psi). \end{aligned}$$

Using (9) and (4), we get

$$\begin{aligned} & \alpha \int_{\Omega} |Du_j|^p \varphi'_\lambda(u_j - \Psi) \leq \int_{\Omega} |a_0(x, u_j)| |\varphi_\lambda(u_j - \Psi)| \\ & + \int_{\Omega} a(x, u_j, Du_j) D\Psi \varphi'_\lambda(u_j - \Psi) + \int_{\Omega} \left(\|f\|_{L^\infty(\Omega)} + b(C_\infty) |Du_j|^p \right) |\varphi_\lambda(u_j - \Psi)|. \end{aligned}$$

That is,

$$\begin{aligned} & \int_{\Omega} |Du_j|^p \left[\alpha \varphi'_\lambda(u_j - \Psi) - b(C_\infty) |\varphi_\lambda(u_j - \Psi)| \right] \\ & \leq \int_{\Omega} \left[|a_0(x, u_j)| + \|f\|_{L^\infty(\Omega)} \right] |\varphi_\lambda(u_j - \Psi)| + \int_{\Omega} |a(x, u_j, Du_j)| |D\Psi| |\varphi'_\lambda(u_j - \Psi)|. \end{aligned}$$

By Remark 4.2 (with $d = \alpha$ and $c = b(C_\infty)$) and the choice of λ , we have

$$\begin{aligned} & \frac{\alpha}{2} \int_{\Omega} |Du_j|^p \leq \int_{\Omega} \left[|a_0(x, u_j)| + \|f\|_{L^\infty(\Omega)} \right] |\varphi_\lambda(u_j - \Psi)| \\ & + \int_{\Omega} |a(x, u_j, Du_j)| |D\Psi| |\varphi'_\lambda(u_j - \Psi)|, \end{aligned}$$

which implies, by (5), (8) and using the fact that φ'_λ is increasing,

$$\begin{aligned} & \frac{\alpha}{2} \int_{\Omega} |Du_j|^p \leq \left(\beta \int_{\Omega} (h(x) + |u_j|^{p-1}) + |\Omega| \|f\|_{L^\infty(\Omega)} \right) \varphi_\lambda(C_\infty + \|\Psi\|_{L^\infty(\Omega)}) \\ & + \varphi'_\lambda(C_\infty + \|\Psi\|_{L^\infty(\Omega)}) \beta \int_{\Omega} |D\Psi| (h(x) + |u_j|^{p-1} + |Du_j|^{p-1}). \end{aligned}$$

Thus, by Hölder's inequality and Lemma 4.1

$$\begin{aligned} \frac{\alpha}{2} \int_{\Omega} |Du_j|^p &\leq \left[\int_{\Omega} \beta (h(x) + C_{\infty}^{p-1}) + |\Omega| \|f\|_{L^{\infty}(\Omega)} \right] \varphi_{\lambda}(C_{\infty} + \|\Psi\|_{L^{\infty}(\Omega)}) \\ &\quad + \varphi'_{\lambda}(C_{\infty} + \|\Psi\|_{L^{\infty}(\Omega)}) \beta \|D\Psi\|_{L^p(\Omega)} \left(\|h\|_{L^{p'}(\Omega)} + |\Omega|^{\frac{1}{p'}} C_{\infty}^{p-1} \right) \\ &\quad + \varphi'_{\lambda}(C_{\infty} + \|\Psi\|_{L^{\infty}(\Omega)}) \beta \int_{\Omega} |D\Psi| |Du_j|^{p-1}. \end{aligned}$$

Let $\tilde{C} = C_{\infty} + \|\Psi\|_{L^{\infty}(\Omega)}$. Note that an application of Young's inequality leads to

$$\varphi'_{\lambda}(\tilde{C}) \beta |D\Psi| |Du_j|^{p-1} \leq \frac{\alpha}{4} |Du_j|^p + \tilde{c}(\alpha, p) \left[\varphi'_{\lambda}(\tilde{C}) \beta |D\Psi| \right]^p,$$

which implies (applying Young's inequality to the other terms) that there exists a positive constant $M = M(\alpha, p, |\Omega|, \beta)$ such that

$$\begin{aligned} \frac{\alpha}{4} \int_{\Omega} |Du_j|^p &\leq M(p, \alpha, |\Omega|, \beta) \left(C_{\infty}^{p-1} + \|f\|_{L^{\infty}(\Omega)} + \|h\|_{L^{p'}(\Omega)} \right) \varphi_{\lambda}(\tilde{C}) \\ &\quad + \beta \varphi'_{\lambda}(\tilde{C}) \|D\Psi\|_p \left(\|h\|_{L^{p'}(\Omega)} + |\Omega|^{\frac{1}{p'}} C_{\infty}^{p-1} \right) \\ &\quad + [\beta \varphi'_{\lambda}(\tilde{C})]^p \tilde{c}(p, \alpha) \|D\Psi\|_{L^p(\Omega)}^p. \end{aligned}$$

Now, since $\varphi_{\lambda}(t) \leq \varphi'_{\lambda}(t) = (1 + 2\lambda t^2)e^{\lambda t^2}$ and $\varphi'_{\lambda}(t) \geq 1$, we obtain (possibly changing the constant M),

$$\begin{aligned} \int_{\Omega} |Du_j|^p &\leq M(p, \alpha, |\Omega|, \beta) \left[(1 + \lambda \tilde{C}^2) e^{4\lambda \tilde{C}^2} \right]^p \left\{ \|f\|_{L^{\infty}(\Omega)} + \|h\|_{L^{p'}(\Omega)} \right. \\ &\quad \left. + \|D\Psi\|_{L^p(\Omega)}^p + \|D\Psi\|_{L^p(\Omega)} \left(\|h\|_{L^{p'}(\Omega)} + C_{\infty}^{p-1} \right) + C_{\infty}^{p-1} \right\}, \end{aligned}$$

which concludes the proof. $\square$

From Lemma 4.1 and Lemma 4.3, it follows that there exists $u \in \mathcal{C}(\psi) \cap L^{\infty}(\Omega)$ and a subsequence (still denoted by u_j) such that:

$$\begin{aligned} u_j &\overset{*}{\rightharpoonup} u \quad \text{in } L^{\infty}(\Omega), \\ u_j &\rightarrow u \quad \text{a.e. in } \Omega, \\ u_j &\rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega). \end{aligned} \tag{33}$$

Here, $\rightharpoonup$ denotes the weak convergence and $\overset{*}{\rightharpoonup}$ denotes the weak* convergence in the corresponding spaces.

At this point, the authors of [5] show that it is possible to recover strong convergence in $W_0^{1,p}(\Omega)$, up to subsequences, of $\{u_j\}_j$ towards u . By passing to the limit in the approximating problems (17), it then becomes straightforward to conclude that u is a solution to (15). For the details of the proof, we refer to the aforementioned article, specifically Lemma 4, Lemma 5, and the proof of Theorem 1.

For the convenience of the reader, we state here the final result with the precise estimates obtained in Lemma 4.1 and Lemma 4.3.

Lemma 4.4. *The sequence $\{u_j\}_j$ of solutions to (17) converges, up to subsequences, in $W_0^{1,p}(\Omega)$. Moreover, if u is a limit point of $\{u_j\}_j$, then it is a solution to (15), and the following estimate holds:*

$$\|u\|_{L^\infty(\Omega)} \leq C_\infty, \quad (34)$$

with C_∞ depending continuously (see (20)) on $\|f\|_{L^\infty(\Omega)}$, α_0 , p and $\|\psi\|_{L^\infty(\Omega)}$.

Moreover, there exists a positive constant $M = M(\alpha, p, |\Omega|, \beta)$ (depending only on the data in the assumptions) such that, given any $\Psi \in \mathcal{C}(\psi) \cap L^\infty(\Omega)$,

$$\begin{aligned} \int_{\Omega} |Du_j|^p &\leq M(p, \alpha, |\Omega|, \beta) \left[(1 + \lambda \tilde{C}^2) e^{4\lambda \tilde{C}^2} \right]^p \left\{ \|f\|_{L^\infty(\Omega)} + \|h\|_{L^{p'}(\Omega)} \right. \\ &\quad \left. + \|D\Psi\|_{L^p(\Omega)}^p + \|D\Psi\|_{L^p(\Omega)} \left(\|h\|_{L^{p'}(\Omega)} + C_\infty^{p-1} \right) + C_\infty^{p-1} \right\}, \end{aligned} \quad (35)$$

where $\lambda = \frac{b(C_\infty)^2}{4\alpha^2}$ and $\tilde{C} = C_\infty + \|\Psi\|_{L^\infty(\Omega)}$.

5. Stability under Mosco-convergence of obstacles

The aim of this section is to prove Theorem 2.4. From now on, we consider the sequence $\{\psi_n\}_n$ and the function ψ_0 satisfying the assumptions of Theorem 2.4.

Remark 5.1. Observe that in Theorem 2.4, we require the sequence $\{\psi_n\}_n$ to be bounded in L^∞ . In view of point 3 of Proposition 3.2, this can be seen as a natural requirement. However, this condition will be needed to apply the techniques from [5], since the boundedness results from the previous section, Lemma 4.1 and Lemma 4.3, rely on the L^∞ norm of the obstacle ψ .

From the results obtained in the previous section and summarized in Lemma 4.4, we derive the following conclusions.

Lemma 5.2. *The sequence $\{u_n\}_n$ satisfies the estimate*

$$\|u_n\|_{L^\infty(\Omega)} \leq \max \left\{ \left(\frac{\|f\|_{L^\infty(\Omega)}}{\alpha_0} \right)^{1/(p-1)} ; \|\psi_n\|_{L^\infty(\Omega)} \right\}. \quad (36)$$

Therefore, there exists a constant C_∞ depending on $\|f\|_{L^\infty(\Omega)}$, $\sup_n \|\psi_n\|_{L^\infty(\Omega)}$, α_0 and p such that

$$\|u_n\|_{L^\infty(\Omega)} \leq C_\infty.$$

In particular, $\{u_n\}_n$ is bounded in $L^\infty(\Omega)$.

Lemma 5.3. *The sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$.*

The proof of Lemma 5.3 is an immediate consequence of Lemma 5.2 and Lemma 4.4, in view of the following proposition.

Proposition 5.4. *Let $w_0 \in \mathcal{C}(\psi_0) \cap L^\infty(\Omega)$ and let $w_n \in \mathcal{C}(\psi_n)$ be such that $w_n \rightarrow w_0$ in $W_0^{1,p}(\Omega)$. Then there exists a sequence $\{\tilde{w}_n\}_n$, with $\tilde{w}_n \in \mathcal{C}(\psi_0) \cap L^\infty(\Omega)$ such that $\tilde{w}_n \rightarrow w_0$ in $W_0^{1,p}(\Omega)$. In particular, if $\{\psi_n\}_n$ is bounded in $L^\infty(\Omega)$, then the sequence $\{\tilde{w}_n\}_n$ is also bounded in $L^\infty(\Omega)$.*

Proof. It suffices to define the truncated functions $\tilde{w}_n = T_{\lambda_n}(w_n)$ with

$$\lambda_n = \max\{\|\psi_n\|_{L^\infty(\Omega)}; \|w_0\|_{L^\infty(\Omega)}\}.$$

With this choice, the function $\tilde{w}_n$ still belongs to $\mathcal{C}(\psi_n)$. Moreover, we have

$$\|Dw_n - D\tilde{w}_n\|_{L^p(\Omega)}^p = \int_{|w_n| \geq \lambda_n} |Dw_n|^p \rightarrow 0. \quad \square$$

Proof of Lemma 5.3. It suffices to apply Lemma 4.4 for each n , using the uniform bound of Lemma 5.2 and choosing an appropriate function Ψ_n in each convex set in the following way. By Mosco-convergence and Proposition 5.4, there exists a sequence $\Psi_n \in \mathcal{C}(\psi_n) \cap L^\infty(\Omega)$ such that $\Psi_n \rightarrow \psi_0$ in $W_0^{1,p}(\Omega)$ and $\{\Psi_n\}_n$ is bounded in $L^\infty(\Omega)$. Hence, the sequence $\{\Psi_n\}_n$ is also bounded in $W_0^{1,p}(\Omega)$. By Lemma 4.4, this gives a bound on the $W_0^{1,p}$ norm of the sequence $\{u_n\}_n$. $\square$

From Lemma 5.2 and Lemma 5.3 there exists $u^* \in W_0^{1,\infty}(\Omega) \cap L^\infty(\Omega)$ and a subsequence (not relabeled) such that

$$\begin{aligned} u_n &\xrightarrow{*} u^* && \text{in } L^\infty(\Omega), \\ u_n &\rightarrow u^* && \text{a.e. in } \Omega, \\ u_n &\rightharpoonup u^* && \text{in } W_0^{1,p}(\Omega). \end{aligned} \quad (37)$$

Moreover, by Mosco-convergence of the convex sets $\mathcal{C}(\psi_n)$, we know that $u^* \in \mathcal{C}(\psi_0)$.

We now provide the proof of the main result of this work, stated in Theorem 2.4. We will show that the sequence $\{u_n\}_n$, which satisfies (37), converges, up to subsequences, in $W_0^{1,p}(\Omega)$ to a solution u^* to problem (12). Before doing this, it is useful to recall the following simple lemma.

Lemma 5.5. *Let $\{y_j\}_j$ be a sequence of measurable functions in Ω such that*

$$y_j \geq 0, \quad y_j \rightarrow y \quad \text{a.e. in } \Omega, \quad y \in L^1(\Omega), \quad \int_{\Omega} y_j \rightarrow \int_{\Omega} y.$$

Then

$$y_j \rightarrow y \quad \text{in } L^1(\Omega).$$

Proof. We can write: $\int_{\Omega} |y_j - y| = 2 \int_{0 \leq y_j \leq y} (y - y_j) + \int_{\Omega} (y_j - y).$

Observing that $(y - y_j) \chi_{\{0 \leq y_j \leq y\}} \leq y \in L^1(\Omega)$, by the Lebesgue Dominated Convergence Theorem, we conclude:

$$y_j \rightarrow y \quad \text{in } L^1(\Omega). \quad \square$$

We can now prove Theorem 2.4.

Proof of Theorem 2.4. From Proposition 5.4, we know that there exists $w_n \in \mathcal{C}(\psi_n) \cap L^\infty(\Omega)$ such that $w_n \rightarrow u^*$ in $W_0^{1,p}(\Omega)$ and $\|w_n\|_{L^\infty(\Omega)} \leq \hat{C}$.

Step 1: We want to prove that

$$\int_{\Omega} [a(x, u_n, Du_n) - a(x, u_n, Dw_n)] D(u_n - w_n) \rightarrow 0. \quad (38)$$

Let us define a sequence $\{v_n\}_n$ as

$$v_n = \left[1 - \delta_n e^{\{\lambda|u_n - w_n|^2\}}\right] u_n + \delta_n e^{\{\lambda|u_n - w_n|^2\}} w_n \in \mathcal{C}(\psi_n) \cap L^\infty(\Omega), \quad (39)$$

where

$$B_\infty = b(\sup_n \|u_n\|_{L^\infty(\Omega)}), \quad \lambda = \frac{B_\infty^2}{4\alpha^2}, \quad \delta_n = e^{-\{\lambda(\|u_n\|_{L^\infty} + \|w_n\|_{L^\infty(\Omega)})^2\}}.$$

Using v_n as a test function in (11), and noting that $v_n - u_n = -\delta_n \varphi_\lambda(u_n - w_n)$, we apply (9) to obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, Du_n) D(u_n - w_n) \varphi'_\lambda(u_n - w_n) + \int_{\Omega} a_0(x, u_n) \varphi_\lambda(u_n - w_n) \\ & \leq \int_{\Omega} (\|f\|_{L^\infty(\Omega)} + B_\infty |Du_n|^p) |\varphi_\lambda(u_n - w_n)|, \end{aligned} \quad (40)$$

which, subtracting $\int_{\Omega} a(x, u_n, Dw_n) D(u_n - w_n) \varphi'_\lambda(u_n - w_n)$

from both sides and rearranging terms, leads to

$$\begin{aligned} & \int_{\Omega} [a(x, u_n, Du_n) - a(x, u_n, Dw_n)] D(u_n - w_n) \varphi'_\lambda(u_n - w_n) \\ & \leq - \int_{\Omega} a_0(x, u_n) \varphi_\lambda(u_n - w_n) + \int_{\Omega} \|f\|_{L^\infty(\Omega)} |\varphi_\lambda(u_n - w_n)| \\ & \quad + \int_{\Omega} B_\infty |Du_n|^p |\varphi_\lambda(u_n - w_n)| - \int_{\Omega} a(x, u_n, Dw_n) D(u_n - w_n) \varphi'_\lambda(u_n - w_n). \end{aligned} \quad (41)$$

Applying property (4) to the term

$$\int_{\Omega} B_\infty |Du_n|^p |\varphi_\lambda(u_n - w_n)|$$

and taking the absolute value (which is then brought inside the integral) in the first term of the right-hand side, we get

$$\begin{aligned} & \int_{\Omega} \left[a(x, u_n, Du_n) - a(x, u_n, Dw_n) \right] D(u_n - w_n) \varphi'_\lambda(u_n - w_n) \\ & \leq - \int_{\Omega} a(x, u_n, Dw_n) D(u_n - w_n) \varphi'_\lambda(u_n - w_n) + \\ & \quad + \int_{\Omega} [|a_0(x, u_n)| + \|f\|_{L^\infty(\Omega)}] |\varphi_\lambda(u_n - w_n)| + \\ & \quad + \frac{B_\infty}{\alpha} \int_{\Omega} a(x, u_n, Du_n) Du_n |\varphi_\lambda(u_n - w_n)|. \end{aligned}$$

Now we subtract

$$\int_{\Omega} \left[a(x, u_n, Du_n) - a(x, u_n, Dw_n) \right] D(u_n - w_n) \frac{B_\infty}{\alpha} |\varphi_\lambda(u_n - w_n)|,$$

from both sides, which leads to

$$\begin{aligned}
& \int_{\Omega} [a(x, u_n, Du_n) - a(x, u_n, Dw_n)] D(u_n - w_n) \\
& \quad [\varphi'_\lambda(u_n - w_n) - \frac{B_\infty}{\alpha} |\varphi_\lambda(u_n - w_n)|] \\
& \leq - \int_{\Omega} a(x, u_n, Dw_n) D(u_n - w_n) \varphi'_\lambda(u_n - w_n) \\
& \quad + \frac{B_\infty}{\alpha} \int_{\Omega} a(x, u_n, Du_n) Dw_n |\varphi_\lambda(u_n - w_n)| \\
& \quad + \frac{B_\infty}{\alpha} \int_{\Omega} a(x, u_n, Dw_n) D(u_n - w_n) |\varphi_\lambda(u_n - w_n)| \\
& \quad + \int_{\Omega} [|a_0(x, u_n)| + \|f\|_{L^\infty(\Omega)}] |\varphi_\lambda(u_n - w_n)|.
\end{aligned}$$

Because of the choice of $\lambda = B_\infty^2 / (4\alpha^2)$, we can apply Remark 4.2 to infer that the left-hand side is greater than

$$\frac{1}{2} \int_{\Omega} [a(x, u_n, Du_n) - a(x, u_n, Dw_n)] D(u_n - w_n).$$

Moreover, using Lebesgue's Dominated Convergence Theorem, we see that the right-hand side converges to zero. Hence

$$\int_{\Omega} [a(x, u_n, Du_n) - a(x, u_n, Dw_n)] D(u_n - w_n) \rightarrow 0. \quad (42)$$

Step 2: We want to show that $\{Du_n\}_n$ converges a.e. to Du^* in Ω .

Here we follow [13]: let L_n be defined by

$$L_n = [a(x, u_n, Du_n) - a(x, u_n, Dw_n)] D(u_n - w_n).$$

Then, by (6), L_n is a positive function, and (42) implies

$$L_n \rightarrow 0 \quad \text{in } L^1(\Omega) \text{ strongly.}$$

Extracting a subsequence (still denoted by $\{u_n\}_n$ and $\{w_n\}_n$ respectively), we have

$$\begin{aligned}
u_n &\rightarrow u^* && \text{a.e. in } \Omega, \\
w_n &\rightarrow u^* && \text{a.e. in } \Omega, \\
Dw_n &\rightarrow Du^* && \text{a.e. in } \Omega, \\
L_n &\rightarrow 0 && \text{a.e. in } \Omega.
\end{aligned}$$

Then there exists a subset Z of Ω , of zero measure, such that, for $x \in \Omega \setminus Z$,

$$|u^*(x)| < \infty, \quad |Du^*(x)| < \infty, \quad |h(x)| < \infty,$$

and

$$u_n(x) \rightarrow u^*(x), \quad w_n(x) \rightarrow u^*(x), \quad Dw_n(x) \rightarrow Du^*(x), \quad L_n(x) \rightarrow 0.$$

Now we fix $x \in \Omega \setminus Z$ and define

$$\xi_n = Du_n(x), \quad \zeta_n = Dw_n(x),$$

so that $L_n(x) = [a(x, u_n(x), \xi_n) - a(x, w_n(x), \zeta_n)] (\xi_n - \zeta_n)$.

By assumption (4), we have, after dropping a positive term on the right-hand side

$$L_n(x) \geq \alpha |\xi_n|^p - a(x, u_n(x), \xi_n) \zeta_n - a(x, w_n(x), \zeta_n) \xi_n.$$

Since $\{w_n(x)\}_n$ and $\{\zeta_n\}_n$ are convergent sequences, there exist two positive constants (depending on x) C_1 and C_2 such that

$$|w_n(x)| \leq C_1(x), \quad |\zeta_n| \leq C_2(x),$$

Therefore, using assumption (5) and the fact that the sequence $\{a(x, w_n(x), \zeta_n)\}$ is bounded, we obtain

$$L_n(x) \geq \alpha |\xi_n|^p - c(x) [1 + |\xi_n|^{p-1} + |\xi_n|],$$

where $c(x)$ is a constant which depends on x , but does not depend on n . Since $L_n(x) \rightarrow 0$, it follows that the sequence $\{\xi_n\}_n$ is bounded. Let ξ^* be a cluster point of ξ_n . We have $|\xi^*| < \infty$ and

$$[a(x, u^*(x), \xi^*) - a(x, u^*(x), Du^*(x))] (\xi^* - Du^*(x)) = 0,$$

because of the continuity of a with respect to s and ξ . By (6), this implies that $\xi^* = Du^*(x)$. It follows that the cluster point is unique (that is, the limit does not depend on the chosen subsequence) and thus

$$Du_n(x) \rightarrow Du^*(x), \quad \forall x \in \Omega \setminus Z,$$

i.e.

$$Du_n \rightarrow Du^*, \quad \text{a.e. in } \Omega.$$

Step 3: We want to prove that $\{Du_n\}_n$ converges to Du^* strongly in $(L^p(\Omega))^N$.

Recall that

$$a(x, u_n, Du_n) Du_n \geq 0, \tag{43}$$

$$a(x, u_n, Du_n) Du_n \rightarrow a(x, u^*, Du^*) Du^* \quad \text{a.e. in } \Omega. \tag{44}$$

Moreover, since the sequence $\{a(x, u_n, Du_n)\}_n$ is bounded in $(L^{p'}(\Omega))^N$ and converges almost everywhere to $a(x, u^*, Du^*)$, we have that

$$a(x, u_n, Du_n) \rightarrow a(x, u^*, Du^*) \quad \text{weakly in } (L^{p'}(\Omega))^N \text{ and a.e. in } \Omega.$$

This property, together with assumptions (37) and (42), leads us to conclude that

$$\int_{\Omega} a(x, u_n, Du_n) Du_n \rightarrow \int_{\Omega} a(x, u^*, Du^*) Du^*. \tag{45}$$

Indeed, by Vitali's Theorem we have

$$\int_{\Omega} a(x, u_n, Dw_n) Dw_n \rightarrow \int_{\Omega} a(x, u^*, Du^*) Du^*,$$

$$\begin{aligned} \int_{\Omega} a(x, u_n, Du_n) Dw_n &\rightarrow \int_{\Omega} a(x, u^*, Du^*) Du^*, \\ \int_{\Omega} a(x, u_n, Dw_n) Du &\rightarrow \int_{\Omega} a(x, u^*, Du^*) Du^*. \end{aligned}$$

Now, by (43), (44), (45) we can take

$$y_n = a(x, u_n, Du_n) Du_n, \quad y = a(x, u^*, Du^*) Du^*,$$

in Lemma 5.5 and conclude that

$$a(x, u_n, Du_n) Du_n \rightarrow a(x, u, Du^*) Du^* \quad \text{in } L^1(\Omega), \quad (46)$$

Now from assumption (4) we deduce, using Vitali's Theorem, that

$$Du_n \rightarrow Du^* \quad \text{in } (L^p(\Omega))^N \text{ strong.} \quad (47)$$

This concludes the proof of the strong convergence of u_n to u^* in $W_0^{1,p}(\Omega)$.

Step 4: It remains to show that u^* is a solution to (12).

Let $v \in \mathcal{C}(\psi_0) \cap L^\infty(\Omega)$ be fixed as a test function in (12). By Proposition 5.4 (recall that the sequence of obstacles $\{\psi_n\}_n$ is bounded in $L^\infty(\Omega)$), there exists a sequence $\{v_n\}_n$, with $v_n \in \mathcal{C}(\psi_n) \cap L^\infty(\Omega)$, such that $v_n \rightarrow v$ in $W_0^{1,p}(\Omega)$ and weakly* in $L^\infty(\Omega)$. Choosing v_n as a test function in (11), we obtain

$$\begin{aligned} \int_{\Omega} a(x, u_n, Du_n) D(v_n - u_n) + \int_{\Omega} a_0(x, u_n)(v_n - u_n) \\ + \int_{\Omega} H(x, u_n, Du_n)(v_n - u_n) \geq 0. \end{aligned} \quad (48)$$

$$\text{Recall that, by (37),} \quad u_n \rightarrow u^* \quad \text{weakly* in } L^\infty(\Omega). \quad (49)$$

Moreover, using Vitali's Theorem and the inequality (9), we deduce that

$$H(x, u_n, Du_n) \rightarrow H(x, u^*, Du^*) \quad \text{in } L^1(\Omega) \quad (50)$$

In addition, by (5) and (7), we obtain

$$\begin{aligned} a(x, u_n, Du_n) &\rightarrow a(x, u^*, Du^*) \quad \text{in } (L^{p'}(\Omega))^N, \\ a_0(x, u_n) &\rightarrow a_0(x, u^*) \quad \text{in } L^{p'}(\Omega). \end{aligned} \quad (51)$$

We can thus pass to the limit in (48), obtaining

$$\int_{\Omega} a(x, u^*, Du^*) D(v - u^*) + \int_{\Omega} a_0(x, u^*)(v - u^*) + \int_{\Omega} H(x, u^*, Du^*)(v - u^*) \geq 0.$$

By the arbitrariness of v , we conclude that u^* is a solution to (12). This completes the proof of Theorem 2.4. $\square$

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