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$\mathcal{F}$-Convexity of Real Functions

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Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called *$\mathcal{F}$-convex* if for any points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *$\mathcal{F}$-convex* if its epigraph is $\mathcal{F}$-convex. In this paper we investigate various $\mathcal{F}$-convexities for real functions.

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1. Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if its epigraph is convex. In order to obtain a generalization via $\mathcal{F}$-convexity, it is natural to ask the epigraph to be $\mathcal{F}$-convex. The investigation of $\mathcal{F}$-convex functions has been suggested by Constantin Niculescu [5].

We denote by $g(f)$ and $e(f)$ the *graph* and the *epigraph* of f, where

$$e(f) = \{(x, t) | x \in \mathbb{R}, f(x) \le t\}.$$

The first investigation of a kind of $\mathcal{F}$-convexity was done in [1], where $\mathcal{F}$ is the set of all (non-degenerate) rectangles. This $\mathcal{F}$-convexity is called *rectangular convexity*. This concept was generalized in [4], by taking $\mathcal{F}$ to be the family of all rectangular quadruples, which are vertex sets of rectangles. This is the *rq-convexity*.

But what does it mean for a set $M \subset \mathbb{R}^d$ to be $\mathcal{F}$-convex, once a collection $\mathcal{F}$ of sets in $\mathbb{R}^d$ is singled out? It means that, for any pair $x, y \in M$, there exists $F \in \mathcal{F}$ with $x, y \in F$ and $F \subset M$.

Accordingly, $f : \mathbb{R} \to \mathbb{R}$ is called $\mathcal{F}$-convex, if $e(f) \subset \mathbb{R}^2$ is $\mathcal{F}$-convex. In particular, it is *rectangularly convex* if its epigraph is rectangularly convex, and *rq-convex* if its epigraph is rq-convex.

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Every rectangularly convex function is, of course, convex, while rq-convex functions may be very different from being convex, as we shall see.

Several further $\mathcal{F}$-convexities have been investigated during the last few years. So, for example, $\mathcal{F}$ can be taken to be the family of all right triangles – this leads to the *right convexity* [7]. Or, $\mathcal{F}$ can be the family of all half-circles (*$\mathcal{C}$-convexity*) [2], or that of all unions of the two smaller sides of an isosceles right triangle (*Γ-convexity*) [3].

For $x, y \in \mathbb{R}^2$, xy denotes the line-segment from x to y, C_{xy} the circle of diameter xy and $B(x, r)$ the closed ball of centre x and radius $r \in \mathbb{R}$.

For $a, b, c, d \in \mathbb{R}^2$, put $abc = \mathrm{conv}\{a, b, c\}$ and $abcd = \mathrm{conv}\{a, b, c, d\}$.

2. Rectangular and rq-convexity

The following result characterizes rectangularly convex functions inside the family of all convex ones.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and put*

$$\alpha = \lim_{x\to-\infty} f'(x) \qquad \beta = \lim_{x\to+\infty} f'(x),$$

where x takes only values where f' exists. Then, f is rectangularly convex, if and only if $\alpha = 0$ or $\beta = 0$ or $\alpha\beta \geq -1$.

Proof. We use the characterization of unbounded rectangularly convex sets given in [1]. By Theorem 2 in [1], $e(f)$ is rectangularly convex if and only if its recession cone is an angle measuring at least $\pi/2$. This happens precisely when $\alpha = 0$ or $\beta = 0$ or $\alpha\beta \geq -1$. $\square$

Thus, the linear functions, the exponential function and the absolute value function are all rectangularly convex. On the other hand, familiar convex functions, like $f(x) = x^2$, are not rectangularly-convex.

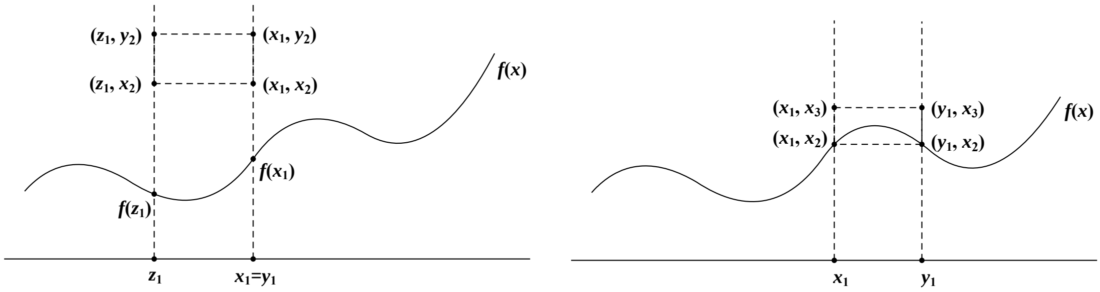
The family of all rq-convex functions includes, of course, that of all rectangularly convex ones. Moreover, it contains a large variety of other functions, as the next result uncovers.

Theorem 2.2. *Any bounded function $f : \mathbb{R} \to \mathbb{R}$ which doesn't realize its infimum, i.e.* $\inf f(\mathbb{R}) \notin f(\mathbb{R})$*, is rq-convex.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in e(f)$. Thus, $x_2 \geq f(x_1)$, $y_2 \geq f(y_1)$.

Case 1. $x_1 = y_1$.
We suppose w.l.o.g. $x_2 < y_2$. Since $f(x_1) > \inf f(\mathbb{R})$, we find $z_1 \neq x_1$ such that $f(z_1) < f(x_1) \leq x_2$, where $(z_1, x_2) \in e(f)$, and $(z_1, y_2) \in e(f)$. Moreover, (x_1, x_2), (x_1, y_2), $(z_1, y_2), (z_1, x_2)$ are vertices of a rectangle, as shown in Figure 1(a).

Case 2. $x_2 = y_2$.
For any $x_3 > x_2$, clearly, $(x_1, x_3), (y_1, x_3) \in e(f)$ and (x_1, x_2), (y_1, x_2), (y_1, x_3), (x_1, x_3) are vertices of a rectangle, as shown in Figure 1(b).



(a) (b)

Figure 1: $x_1 = y_1$ or $x_2 = y_2$.

Case 3. Neither $x_1 = y_1$, nor $x_2 = y_2$.

Assume w.l.o.g that $x_1 < y_1 < 1$ and $x_2 > y_2$. For every $n \in \mathbb{N}$, consider the points (n, u_n), $(n+y_1-x_1, v_n)$ such that (x_1, x_2), (y_1, y_2), $(n+y_1-x_1, v_n)$, (n, u_n) be vertices of a rectangle, see Figure 2.

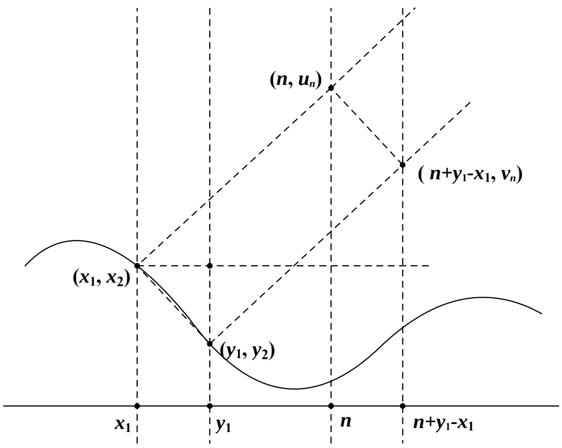


Figure 2: Neither $x_1 = y_1$, nor $x_2 = y_2$.

Suppose that $e(f)$ is not rq-convex. Then, for every $n \in \mathbb{N}$, $(n + y_1 - x_1, v_n) \notin e(f)$ or $(n, u_n) \notin e(f)$. This yields $f(n + y_1 - x_1) > v_n$ or $f(n) > u_n$. Since

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n = \infty,$$

we have
$$\lim_{n\to\infty} f(n + y_1 - x_1) = \infty \quad \text{or} \quad \lim_{n\to\infty} f(n) = \infty,$$

contradicting the boundedness of f. Hence, $e(f)$ is rq-convex. □

To illustrate Theorem 2.2, take arctan and the Gaussian $f(x) = e^{-x^2}$.

The condition in Theorem 2.2 about the infimum of f can be dropped, but only by replacing it with some other conditions.

Theorem 2.3. *Suppose the bounded function $f : \mathbb{R} \to \mathbb{R}$ has an absolute minimum at x. If f is differentiable in a neighborhood of x and f'' exists and vanishes at x, then f is rq-convex.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in e(f)$.

Case 1. $x_1 = y_1 = x$ and $x_2 = f(x) < y_2$.

Let $V \subset \mathbb{R}^2$ be a disc around $(x, f(x))$ not containing (x, y_2). Choose an interval $W \subset \mathbb{R}$ with midpoint x, such that $(x^*, f(x^*)) \in V$ for any $x^* \in W$. Since $f''(x) = 0$, the curvature of $g(f)$ vanishes at x. This implies the existence of $x' \in W$ such that $(x', f(x')) \in (V \setminus B((x, y_2), \|y_2 - x_2\|)$. Then, for some $y' > f(x')$, $\angle(x, f(x))(x', y')(x, y_2) = \pi/2$.

Moreover, $2x - x' \in W$ and $f(2x - x') < u$, where $(2x - x', u)$ is the fourth vertex of the rectangle with vertices at $(x, f(x))$, (x', y') and (x, y_2).

All other cases can be treated as in Theorem 2.2. □

An example of a function which is rq-convex by Theorem 2.3 is

$$f(x) = \begin{cases} x^4 & \text{for } |x| \leq 1 \\ 1 & \text{otherwise} \end{cases}.$$

Theorem 2.4. *If the bounded function $f : \mathbb{R} \to \mathbb{R}$ realizes its absolute minimum at more than one point, then f is rq-convex.*

Proof. Let (x_1, x_2), $(y_1, y_2) \in e(f)$.
Case 1. $x_1 = y_1$ and $x_2 = f(x_1) < y_2$.

If $f(x_1) > \inf f(\mathbb{R})$, we follow the proof in Case 1 of Theorem 2.2.

If $f(x_1) = \inf f(\mathbb{R})$, then there exists a point $(z_1, f(x_1)) \in g(f)$ with $z_1 \neq x_1$, by the hypothesis. Then, $(x_1, f(x_1))$, $(z_1, f(x_1))$, (z_1, y_2), (x_1, y_2) are vertices of a rectangle.

The other cases are as in Theorem 2.2. □

Theorem 2.5. *Any bounded monotone function $f : \mathbb{R} \to \mathbb{R}$ is rq-convex.*

Proof. Assume f is non-decreasing, i.e. $x < y$ implies $f(x) \leq f(y)$. Let again $(x_1, x_2), (y_1, y_2) \in e(f)$.

Case 1. $x_1 = y_1$.
Suppose w.l.o.g. $x_2 < y_2$. Choose arbitrarily $z_1 < x_1$. Since f is non-decreasing, we get $x_2 = f(x_1) \geq f(z_1)$, whence $(z_1, x_2), (z_1, y_2) \in e(f)$. Moreover, (x_1, x_2) (x_1, y_2) (z_1, y_2) (z_1, x_2) is a rectangle.

Two more cases can appear, they (and their treatment) being identical with those in the previous proof. □

We remark that, for strictly monotone f, Theorem 2.5 follows from Theorem 2.2.

3. Other $\mathcal{F}$-convex functions

Let us consider the family of Γ-convex functions.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If $0 \leq f' \leq 1$ everywhere or $f' \geq 1$ everywhere, then f is Γ-convex.*

Proof. Clearly, f is non-decreasing. Let $x, y \in g(f)$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $x_1 < y_1$.

First, assume $0 \leq f' \leq 1$. We have $y_2 - x_2 \leq y_1 - x_1$. Let $L(\alpha)$ be the line of positive slope through x, which makes an angle α with the horizontal coordinate axis. Let $y(\alpha)$ be the orthogonal projection of y onto $L(\alpha)$, see Figure 3.

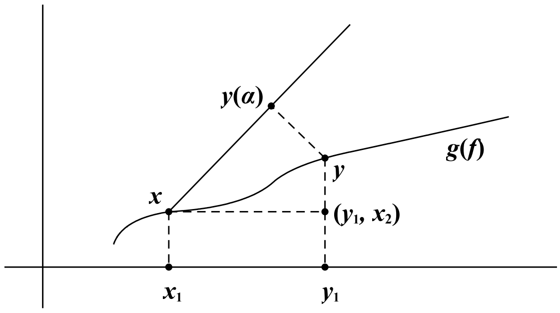


Figure 3: $0 \leq f' \leq 1$.

Clearly, $\|y - y(\alpha)\|$ is not larger than the distance from (y_1, x_2) to $L(\alpha)$, whence

$$\|y - y(\tfrac{\pi}{4})\| \leq \|x - y(\tfrac{\pi}{4})\|.$$

But
$$\|y - y(\tfrac{\pi}{2})\| \geq \|x - y(\tfrac{\pi}{2})\|.$$

Thus, for some $\alpha \in [\pi/4, \pi/2]$,

$$\|y - y(\alpha)\| = \|x - y(\alpha)\|.$$

Moreover, $yy(\alpha) \cup y(\alpha)x \subset e(f)$. This proves the Γ-property at x, y. For all other locations of x, y, the proof follows, or is straightforward.

Now suppose $f' \geq 1$. Let C be the half-circle of C_{xy} from x to y passing through $z = (x_1, y_2)$. For $u \in C$ close to x, $\|u - x\| < \|u - y\|$. But $\|z - x\| \geq \|z - y\|$. Hence, for some point $v \in C$, $\|v - x\| = \|v - y\|$. Moreover, $vz \cup vy \subset e(f)$. $\square$

Regarding the $\mathcal{C}$-convex functions, we establish that monotonicity is a simple sufficient condition. But it can be mentioned that it is far from being necessary!

Theorem 3.2. *Every monotone function is $\mathcal{C}$-convex.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be non-decreasing. Take arbitrarily $x, y \in g(f)$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $x_1 < y_1$. Of course, $xz \cup zy \subset e(f)$, where $z = (x_1, y_2)$, as shown in Figure 4. Hence, the whole half-circle of C_{xy} from x to y passing through z lies in $e(f)$. $\square$

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If $f'(x)f'(y) \neq 1$ for all pairs $x, y \in \mathbb{R}$, then f is $\mathcal{C}$-convex.*

Proof. Suppose, for some $x, y \in \mathbb{R}$, the half-circle C^+ of C_{XY} above $\overline{XY}$, where $X = (x, f(x))$ and $Y = (y, f(y))$, is not included in $e(f)$. Then, there exists a point $Z \in C^+$ below $g(f)$.

Case 1. $f(x) < f(y)$.

The vertical line through X intersects C_{XY} a second time in W, say. Since W is above $g(f)$ and Z below it, there exists a point $V \in g(f) \cap C_{XY}$, of coordinates $(v, f(v))$. Clearly, $\overline{VX} \perp \overline{VY}$, i.e.

$$\frac{f(v)-f(x)}{v-x} \cdot \frac{f(v)-f(y)}{v-y} = -1.$$

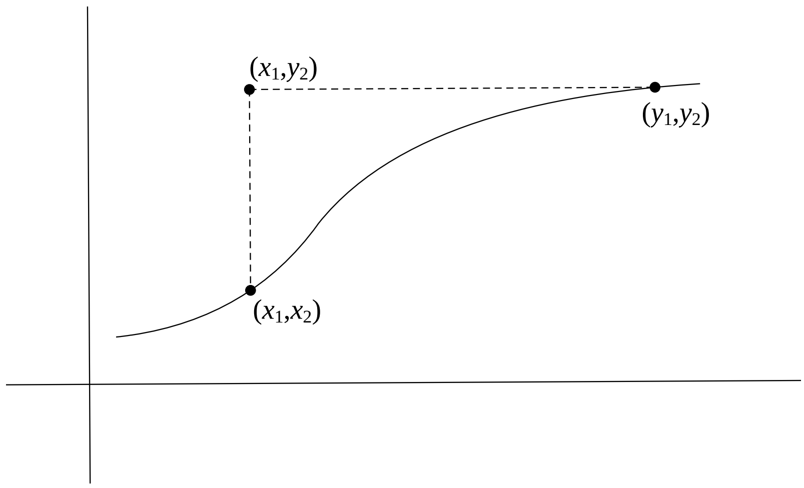


Figure 4: f is a non-decreasing function.

Now, f being differentiable, there exists a point $x' \in]x, v[$ such that

$$f'(x') = \frac{f(v)-f(x)}{v-x}$$

and another point $y' \in]v, y[$ such that

$$f'(y') = \frac{f(v)-f(y)}{v-y},$$

so that $f'(x') \cdot f'(y') = -1$, which is impossible.

Case 2. $f(x) = f(y)$.

Since $f'(x) \neq \infty$, for any point $x^* > x$ close to x, C_{XY} is above $(x^*, f(x^*))$. This yields the existence of a point $V \in g(f) \cap C_{XY}$ as in Case 1.

Case 3. $f(x) > f(y)$.

This is analogous to Case 1. □

Every function $f : \mathbb{R} \to \mathbb{R}$ is poidge-convex. For the definition of a poidge-convex set, see [6].

Many convex sets are not rightly convex. Interestingly, however, every convex function is rightly convex!

Theorem 3.4. *Every convex function is rightly convex.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be convex. Choose arbitrarily $x, y \in g(f)$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $x_1 < y_1$. Either the slope $(y_2 - x_2)/(y_1 - x_1)$ is non-positive, or it is non-negative, say it is non-negative. Then, we easily find a point z above $\overline{xy}$, such that xyz be a right triangle (with $\angle xyz = \pi/2$). $\square$

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