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A Note about the Linear Time-Optimal Control Problem

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In non-autonomous time-optimal control problems the question of the sufficiency of the maximum principle is considered. We also briefly discuss the question of uniqueness of optimal control. New conditions are obtained that guarantee the sufficiency of the maximum principle in terms of convex analysis, properties of matrices in the linear system, the geometry of the control set and the nonempty interior of some set-valued integrals. Examples that demonstrate the unimprovability of the obtained results are considered.

Keywords: Maximum principle, support function, set-valued integral, nonsmooth analysis.

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1. Introduction and main notations

Consider a linear control system

$$\begin{cases} x'(t) = A(t)x(t) + B(t)u(t), \quad u(t) \in \mathcal{U}(t), \quad x(t) \in \mathbb{R}^n \\ A(t), B(t) \in \mathbb{R}^{n\times n}, \quad \forall t \in [0, +\infty). \end{cases} \tag{1}$$

Here $t \geq 0$ is time, $\mathbb{R}^{n\times n}$ is the space of real $n \times n$ matrices. A permissible control in system (1) is a function $u \in L_\infty([0, T], \mathcal{U}(t))$ (i.e. $u \in L_\infty([0, T])$ and $u(t) \in \mathcal{U}(t)$ for a.a. $t \in [0, T]$) and $T > 0$ is a fixed time which depends on u. Permissible controls $u_1(t)$ and $u_2(t)$, $t \in [0, T]$, in system (1) coincide if $u_1(t) = u_2(t)$ for a.a. $t \in [0, T]$.

In this paper, we assume that the matrices $A(t)$ and $B(t)$ are continuous in t, and that the set $\mathcal{U}(t) \subset \mathbb{R}^n$ is convex and compact for each $t \geq 0$. In addition, it is assumed that $\mathcal{U}(t)$ is continuous with respect to t in the Hausdorff metric, or that $\mathcal{U}(t)$ is measurable, and that $\mathcal{U}(t)$ is uniformly bounded. Recall that a set-valued mapping $t \to \mathcal{U}(t)$ is measurable if for any open set $\mathcal{V} \subset \mathbb{R}^n$ the set $\{t \geq 0 \ : \ \mathcal{U}(t) \cap \mathcal{V} \neq \emptyset\}$ is measurable. Uniform boundedness means

$$\sup\{\|x\| \ : \ x \in \mathcal{U}(t), \ t \geq 0\} < +\infty.$$

Let $\Phi(t) \in \mathbb{R}^{n\times n}$ denote the fundamental matrix of the system $x'(t) = A(t)x(t)$, i.e. $\Phi'(t) \equiv A(t)\Phi(t)$ and $\Phi(0) = I$. Here I is the identity matrix. Note that the matrix $\left(\Phi^{-1}(t)\right)^T$ is the fundamental matrix of the system $x'(t) = -A^T(t)x(t)$ with the normalization $\left(\Phi^{-1}(0)\right)^T = I$.

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It is known (see, for example, Chapter 2, §2.2 of [11]) that for any time $t \geq 0$ and control $u : [0,t] \ni s \to u(s) \in \mathcal{U}(s)$ the value $x(t)$ has the form

$$x(t) = \Phi(t)x_0 + \Phi(t)\int\limits_0^t \Phi^{-1}(s)B(s)u(s)\,ds. \tag{2}$$

Let us introduce a set-valued integral in the sense of Aumann, i.e. let

$$\int\limits_0^t \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds := \left\{ \int\limits_0^t \Phi^{-1}(s)B(s)u(s)\,ds : u(s) \in L_\infty([0,t],\mathcal{U}(s))\ \forall s \in [0,t] \right\}.$$

The conditions of measurability and uniform boundedness for a set-valued mapping $t \to \mathcal{U}(t)$ are sufficient for the Aumann integral to exist, see [1, Ch. 8].

We will define the inner product of vectors $p, x \in \mathbb{R}^n$ by $\langle p, x\rangle$, the norm of the vector $x \in \mathbb{R}^n$ by $\|x\| = \sqrt{\langle x, x\rangle}$, and the support function of a compact set $\mathcal{Q} \subset \mathbb{R}^n$ by $c(p, \mathcal{Q}) = \max_{x\in\mathcal{Q}}\langle p, x\rangle$, where $p \in \mathbb{R}^n$. By $\operatorname{bd}\mathcal{Q}$, $\operatorname{int}\mathcal{Q}$, $\operatorname{cl}\mathcal{Q}$ and $\operatorname{ri}\mathcal{Q}$ we denote the boundary, interior, closure and relative interior of the set $\mathcal{Q} \subset \mathbb{R}^n$, respectively. For a convex compact set $\mathcal{Q} \subset \mathbb{R}^n$ and a vector $p \in \mathbb{R}^n$ we put $\mathcal{Q}(p) = \{x \in \mathcal{Q} : \langle p, x\rangle = c(p, \mathcal{Q})\}$. Note that for $p = 0$ the equality $\mathcal{Q}(0) = \mathcal{Q}$ holds.

Let's put $\|\mathcal{Q}\| = \max\limits_{x\in\mathcal{Q}} \|x\|$ and $\operatorname{co}\mathcal{Q}$ as the convex hull of the set $\mathcal{Q}$.

A closed Euclidean ball of radius $r > 0$ centered at $a \in \mathbb{R}^n$ is denoted by $\mathcal{B}(a, r)$. The distance in the Hausdorff metric between compact sets $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^n$ is denoted by

$$h(\mathcal{P}, \mathcal{Q}) = \max\left\{ \max_{x\in\mathcal{P}}\min_{y\in\mathcal{Q}} \|x - y\|,\ \max_{x\in\mathcal{Q}}\min_{y\in\mathcal{P}} \|x - y\| \right\}.$$

The sum of sets $\mathcal{P}$ and $\mathcal{Q}$ according to Minkowski is the set of sums

$$\mathcal{P} + \mathcal{Q} = \{p + q\ :\ p \in \mathcal{P},\ q \in \mathcal{Q}\}.$$

For a convex closed set $\mathcal{Q} \subset \mathbb{R}^n$ we define the normal cone at a point $x_0 \in \mathcal{Q}$ by

$$\mathcal{N}(\mathcal{Q}, x_0) = \{p \in \mathbb{R}^n\ :\ \langle p, x - x_0\rangle \leq 0\}.$$

A closed subset $\mathcal{U} \subset \mathbb{R}^n$ is strictly convex if it is convex and its boundary $\operatorname{bd}\mathcal{U}$ does not contain segments of nonzero length. If the set $\mathcal{U}$ is strictly convex but is not a point, then it has nonempty interior. If a compact set $\mathcal{U}$ is strictly convex then for any nonzero vector $p \in \mathbb{R}^n$ the value $\max\limits_{u\in\mathcal{U}}\langle p, u\rangle$ is achieved at a single point $u(p) \in \mathcal{U}$.

The above mentioned definitions can be found in standard monographs (see, for example, [1, 9]).

Due to the linearity of the Aumann integral, for any time-independent constant matrix C of suitable size, we have for any $t \geq 0$ the equality

$$C\int\limits_0^t \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds = \int\limits_0^t C\Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds.$$

Moreover, for any vector $p \in \mathbb{R}^n$ the identity holds:

$$c\left(p, \int_0^t \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds\right) = \int_0^t c\left(p, \Phi^{-1}(s)B(s)\mathcal{U}(s)\right)\,ds. \tag{3}$$

Formula (3) is a well known property of integrals of set-valued mappings, see for example Proposition 8.6.2, item 4), [1].

The convexity of the compact set $\mathcal{U}(s)$ implies the convexity of the reachability set $\mathcal{R}(t, x_0)$ for any $t \geq 0$ and $x_0 \in \mathbb{R}^n$, here

$$\mathcal{R}(t, x_0) = \Phi(t)x_0 + \Phi(t)\int_0^t \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds.$$

Note also that by Lyapunov's theorem on vector measures (see, for example, [9, Theorem 2, Ch. 8, §8.2]), the set $\mathcal{R}(t, x_0)$ is a convex compact set for an arbitrary almost everywhere continuous and uniformly bounded family of sets $\mathcal{U}(s)$, $s \geq 0$. Notice also that even without the assumption of convexity of the compact set $\mathcal{U}(s)$, by the Lyapunov theorem on vector measures, the set $\mathcal{R}(t, x_0)$ is a convex compact set. In this case, the reachability sets of the system (1) for control sets $\mathcal{U}(t)$ and $\operatorname{co}\mathcal{U}(t)$ coincide, see (for a fixed set $\mathcal{U}$) [11, Theorem 4, Chapter 2, §2.2].

The linear time-optimal control problem consists in the fastest possible transfer of the controlled system (1) from the initial point x_0 to the origin, that is,

$$T \to \min_{T \geq 0}, \quad x(T) = 0, \tag{4}$$

and the equation (1) with the initial condition $x(0) = x_0$ holds. The solution to the problem (4) will be denoted by $T(x_0)$. Naturally, $T(x_0)$ is a function of the initial condition x_0. It is usually called the optimal time function or the Bellman function.

Let us define for $\psi, x \in \mathbb{R}^n$, $u \in \mathcal{U}(t)$ and $t \geq 0$ the Hamiltonian

$$H = H(t, \psi, x, u) = \langle \psi, A(t)x + B(t)u \rangle$$

and the maximum function $M(t, \psi, x) = \max_{u \in \mathcal{U}(t)} H(t, \psi, x, u)$. Let us write the Hamiltonian system for $i = 1, \dots, n$ for some admissible control $u(\cdot)$, trajectory $x(\cdot)$ and function $\psi(\cdot)$

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial \psi_i}, \tag{5}$$

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x_i}. \tag{6}$$

Note that system (5) coincides with system from (1). System (6) has the form

$$\psi'(t) = -A^T(t)\psi(t). \tag{7}$$

If $\psi(s) \neq 0$ for some $s \in [0, T]$, then $\psi(t) \neq 0$ for any $t \in [0, T]$.

Definition 1.1. Let $u(t)$ be a permissible control in (1), $t \in [0, T]$, that takes a point from position x_0 to point $x(T) = 0$, and $x(t)$ is the trajectory corresponding to this control. Let $\psi(0) \neq 0$, where $\psi(t)$ is some solution of (7).

We will say that $u(t)$ satisfies Pontryagin's maximum principle, i.e. $u(t)$ is an extremal control, if the next condition

$$H(t, \psi(t), x(t), u(t)) = M(t, \psi(t), u(t)) \qquad \text{for a.a. } t \in [0, T] \tag{8}$$

is satisfied. □

Note that under the assumption of continuity of the set-valued mapping $t \to \mathcal{U}(t)$ one can prove the inequality

$$M(T, \psi(T), x(T)) = M(T, \psi(T), 0) \geq 0. \tag{9}$$

A permissible control $u(t)$, $t \in [0, T(x_0)]$, that provides a minimum in problem (4) is called an optimal control. As is known, the corresponding optimal control exists, see for example [6, Theorem 1].

The maximum principle in optimal control theory arose in the works of Pontryagin and his colleagues [14]. The linear time-optimal control problem was considered by Gamkrelidze [14, Chapter 3]. Later, linear time-optimal control problem were considered in a large number of works, see for example [3, 7, 8, 11] and [4, 5, 12, 13, 15], where the uniform continuity of the Bellman function was mainly studied.

In this paper, we consider the question when condition (8) is sufficient for the control $u(\cdot)$ in the problem (4) to be optimal? We also briefly discuss the question of uniqueness of optimal control.

2. Criteria of optimality

By $\mathcal{X}(t)$ we denote the set of initial conditions $x(0) = x_0 \in \mathbb{R}^n$ of system (1), from which it is possible to get to the origin in time $t \geq 0$. We call this set the *controllability set* of system (1) at time t.

It is easy to see that the sets $\mathcal{X}(t)$ and $\mathcal{R}(t, x_0)$ are related by the equality

$$\mathcal{X}(t) = \{x_0 \in \mathbb{R}^n \ : \ 0 \in \mathcal{R}(t, x_0) \text{ for system (1)}\}.$$

If $x_0 \in \mathcal{X}(t)$, then from Formula (2) due to the inclusion $0 \in \mathcal{R}(t, x_0)$ and the non-degeneracy of the fundamental matrix $\Phi(t)$ we obtain that

$$-x_0 \in \int_0^t \Phi^{-1}(s)B(s)\mathcal{U}(s)\, ds.$$

Thus,

$$\mathcal{X}(t) = -\int_0^t \Phi^{-1}(s)B(s)\mathcal{U}(s)\, ds. \tag{10}$$

From formulas (2) and (10) the next relationship between the sets $\mathcal{R}(t, x_0)$ and $\mathcal{X}(t)$ follows:

$$\mathcal{R}(t, x_0) = \Phi(t)(x_0 + (-\mathcal{X}(t))). \tag{11}$$

Notice also that if $0 \in \mathcal{U}(t)$ for a.a. t then the sets $\{\mathcal{X}(t)\}_{t\geq 0}$ are are ordered by inclusion: for $0 \leq t_1 \leq t_2$ we have $\mathcal{X}(t_1) \subset \mathcal{X}(t_2)$. The sets $\{\mathcal{R}(t,x)\}_{t\geq 0}$ are not ordered by inclusion.

Lemma 2.1. *Consider a system* (1) *in which the set-valued mapping $t \to \mathcal{U}(t)$ has convex compact values, is uniformly bounded and measurable. Then the control $u : [0,T] \to \mathcal{U}(t)$ is extremal in the problem* (4) *if and only if*

$$x = \int_0^T \Phi^{-1}(t)B(t)u(t)\,dt \in \mathrm{bd} \int_0^T \Phi^{-1}(t)B(t)\mathcal{U}(t)\,dt = \mathrm{bd}\,(-\mathcal{X}(T)). \tag{12}$$

Moreover, for any extremal control $u : [0,T] \to \mathcal{U}(t)$ from formula (12) *we have $0 \in \mathrm{bd}\,\mathcal{R}(T,x)$.*

Proof. Lemma 2.1 is simply rephrasing of formula (3) and equality (11). □

Lemma 2.2. *Let us take an arbitrary x_0 for which in the time-optimal control problem with a linear control system* (1) *the time $T := T(x_0)$ is finite. Let the mapping $t \to \mathcal{U}(t)$ be measurable, uniformly bounded, and have convex compact values, $0 \in \mathcal{U}(t)$ for a.a. $t \geq T$.*
Then, if for any $\Delta t > 0$ the inclusion holds

$$0 \in \mathrm{int} \int_T^{T+\Delta t} \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds, \tag{13}$$

then Pontryagin's maximum principle is a sufficient condition for optimality in a problem with the initial condition $x(0) = x_0$.

Proof. For $\Delta t > 0$ we put

$$\mathcal{I}(\Delta t) = \int_T^{T+\Delta t} \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds. \tag{14}$$

Assume that for any $\Delta t > 0$ the condition (13) holds. According to formula (10), due to the additivity of the integral over the integration limits, we obtain

$$\mathcal{X}(T+\Delta t) = \mathcal{X}(T) + \left(- \int_T^{T+\Delta t} \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds \right) = \mathcal{X}(T) + (-\mathcal{I}(\Delta t)).$$

Since $\Delta t > 0$, then $0 \in \mathrm{int}\,\mathcal{I}(\Delta t)$, and therefore $0 \in \mathrm{int}\,(-\mathcal{I}(\Delta t))$. Hence there exists $\delta > 0$ which depends on Δt such that $\mathcal{B}(0,\delta) \subset (-\mathcal{I}(\Delta t))$. Moreover, it is obvious that $x_0 \in \mathcal{X}(T)$, since by condition $T = T(x_0)$. Therefore

$$x_0 + \mathcal{B}(0,\delta) \subset \mathcal{X}(T) + \mathcal{B}(0,\delta) \subset \mathcal{X}(T) + (-\mathcal{I}(\Delta t)) = \mathcal{X}(T+\Delta t).$$

Thus, the inclusion $x_0 \in \mathrm{int}\,\mathcal{X}(T+\Delta t)$ holds. By Lemma 2.1 the control leading to the point $x_0 \in \mathrm{int}\,\mathcal{X}(T+\Delta t)$ is not extremal for any $\Delta t > 0$. Therefore, Pontryagin's maximum principle is a sufficient condition for optimality. □

Lemma 2.3. *Let a set-valued mapping $t \to \mathcal{U}(t)$ be measurable, uniformly bounded, and have convex compact values, $0 \in \mathcal{U}(t)$ for a.a. $t \geq 0$. The Pontryagin maximum principle is a sufficient optimality condition in the time-optimal control problem with a linear control system* (1) *for any initial condition $x(0) = x_0 \in \mathbb{R}^n$, for which $T(x_0) < \infty$, if and only if for all $t \geq 0$ and $\Delta t > 0$ the inclusion*

$$0 \in \operatorname{int} \int_t^{t+\Delta t} \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds \tag{15}$$

is fulfilled.

Proof. For $t \geq 0$ and $\Delta t > 0$ we set

$$\mathcal{I}(t,\Delta t) = \int_t^{t+\Delta t} \Phi^{-1}(s)B(s)\mathcal{U}(s)\,ds.$$

Sufficiency follows from Lemma 2.2. Let us prove necessity.

Suppose that there exist numbers $t \geq 0$ and $\Delta t > 0$ such that $0 \in \operatorname{bd}\mathcal{I}(t,\Delta t)$. So $0 \in \operatorname{bd}(-\mathcal{I}(t,\Delta t))$. The set $(-\mathcal{I}(t,\Delta t))$ is a convex compact set and contains zero on its boundary. By virtue of the finite-dimensional theorem on the separation of a boundary point of a convex compact set from the same set, there exists a unit vector $p \in \mathbb{R}^n$ such that $\langle p, 0\rangle = c(p,(-\mathcal{I}(t,\Delta t)))$.

Let us fix a point $x_0 \in \operatorname{bd}\mathcal{X}(t)$ for which $\langle p, x_0\rangle = c(p,\mathcal{X}(t))$.

From formula (10) it follows the equality $\mathcal{X}(t+\Delta t) = \mathcal{X}(t) + (-\mathcal{I}(t,\Delta t))$. Hence, since $0 \in (-\mathcal{I}(t,\Delta t))$ and $x_0 \in \mathcal{X}(t)$, we obtain that the point $x_0 = x_0 + 0$ belongs to the set $\mathcal{X}(t+\Delta t)$. From the equality of support functions at point p

$$c\big(p,\mathcal{X}(t+\Delta t)\big) = c\big(p,\mathcal{X}(t)\big) + c\big(p,(-\mathcal{I}(t,\Delta t))\big) = \langle p, x_0\rangle + \langle p, 0\rangle = \langle p, x_0\rangle$$

it follows that $\langle p, x_0\rangle = c\big(p,\mathcal{X}(t+\Delta t)\big)$. Therefore, the point x_0 is a boundary point of the set $\mathcal{X}(t+\Delta t)$.

By Lemma 2.1, there exists an extremal control u defined on the interval $[0, t+\Delta t]$ such that the solution (1) with this control takes the point x_0 to the origin in time $t+\Delta t$. But this extremal control u is not optimal, since $x_0 \in \mathcal{X}(t)$ and $\Delta t > 0$. □

Lemma 2.3 is a mathematical folklore. It can be found (sometimes partially and in different notations) in some classic monographs, e.g. [3, Ch. 2, §5, i.23].

Corollary 2.4. *It follows from Lemma* 2.3 *that if Pontryagin's maximum principle is a sufficient condition for optimality in the linear time-optimal control problem* (4) *for any initial condition $x(0) = x_0 \in \mathbb{R}^n$ with $T(x_0) < \infty$, then $0 \in \operatorname{int}\mathcal{X}(t)$ for any $t > 0$.*

Consider some criteria of optimality in terms of convex analysis.

Theorem 2.5. *Let for the initial condition $x(0) = x_0$ in the time-optimal control problem with a linear control system* (1) *the time $T := T(x_0)$ be finite. Suppose that the mapping $t \to \mathcal{U}(t)$ is measurable, uniformly bounded, and has convex compact values, $0 \in \mathcal{U}(t)$ for a.a. $t \geq T$.*

Then Pontryagin's maximum principle is a sufficient condition for an extremum in the time-optimal control problem with the initial condition x_0 if and only if for any $\Delta t > 0$ the equality holds

$$\mathcal{N}(\mathcal{X}(T), x_0) \cap \mathcal{N}((-\mathcal{I}(\Delta t)), 0) = \{0\}, \tag{16}$$

where $\mathcal{I}(\Delta t)$ is given by formula (14).

Proof. Sufficiency. Let for any $\Delta t > 0$ the equality (13) be satisfied. If we have $0 \in \operatorname{int} \mathcal{I}(\Delta t)$ for all $\Delta t > 0$, then as follows from Lemma 2.2 Pontryagin's maximum principle is a sufficient condition.

Therefore, suppose that there exists a number $\sigma > 0$ such that for all $\Delta t \in (0, \sigma)$ the inclusion $0 \in \operatorname{bd}(-\mathcal{I}(\Delta t))$ holds.

From condition (13) it follows that for any number $\Delta t \in (0, \sigma)$ and unit vector $p \in \mathcal{N}(\mathcal{X}(T), x_0)$ there exists a point $x_{p,\Delta t} \in (-\mathcal{I}(\Delta t))$ such that $\langle p, x_{p,\Delta t}\rangle > 0$. We define the sets

$$\mathcal{M}(\Delta t) = \{0\} \cup \bigcup_{\|p\|=1,\ p \in \mathcal{N}(\mathcal{X}(T), x_0)} \{x_{p,\Delta t}\} \subset (-\mathcal{I}(\Delta t))$$

and

$$\mathcal{Z}(\Delta t) = \operatorname{co}(\mathcal{X}(T) + \mathcal{M}(\Delta t)).$$

Notice that

$$\mathcal{Z}(\Delta t) \subset \mathcal{X}(T) + (-\mathcal{I}(\Delta t)) = \mathcal{X}(T + \Delta t).$$

If $x_0 \in \operatorname{int} \mathcal{Z}(\Delta t)$ for all $\Delta t \in (0, \sigma)$, then from the inclusion $\mathcal{Z}(\Delta t) \subset \mathcal{X}(T + \Delta t)$ it follows that $x_0 \in \operatorname{int} \mathcal{X}(T+\Delta t)$. By lemma 2.2 it follows that Pontryagin's maximum principle is a sufficient condition in the time-optimal control problem with the initial condition $x(0) = x_0$.

Let us assume that there exists a number $\Delta t \in (0, \sigma)$ such that $x_0 \in \operatorname{bd} \mathcal{Z}(\Delta t)$ or, what is the same due to the convexity of the set $\mathcal{Z}(\Delta t)$, $x_0 \in \operatorname{bd} \operatorname{cl} \mathcal{Z}(\Delta t)$. By the finite-dimensional theorem on the separation of a boundary point of a convex compact set from this set, there exists a unit vector $q \in \mathbb{R}^n$ such that

$$\langle q, x_0\rangle = c(q, \operatorname{cl} \mathcal{Z}(\Delta t)). \tag{17}$$

Since $x_0 \in \mathcal{X}(T) \subset \operatorname{cl} \mathcal{Z}(\Delta t)$, then, using (17), we have

$$c(q, \operatorname{cl} \mathcal{Z}(\Delta t)) \geq c(q, \mathcal{X}(T)) \geq \langle q, x_0\rangle = c(q, \operatorname{cl} \mathcal{Z}(\Delta t)).$$

Thus, in the previous estimate, equalities hold everywhere. Also note that we have $q \in \mathcal{N}(\mathcal{X}(T), x_0)$.

On the other hand we have

$$c(q, \mathcal{X}(T)) = c(q, \operatorname{cl} \mathcal{Z}(\Delta t)) \geq \langle q, x_0 + x_{q,\Delta t}\rangle > \langle q, x_0\rangle = c(q, \mathcal{X}(T)).$$

The obtained contradiction shows that the inclusion $x_0 \in \operatorname{bd} \mathcal{Z}(\Delta t)$ is impossible for any $\Delta t \in (0, \sigma)$. Sufficiency is proved.

Necessity. Let the unit vector $p \in \mathbb{R}^n$ be contained in the intersection of normal cones from (13) for some $\Delta t > 0$. Then $\mathcal{X}(T + \Delta t) = \mathcal{X}(T) + (-\mathcal{I}(\Delta t))$, and also

$x_0 \in \operatorname{bd} \mathcal{X}(T)$, $0 \in \operatorname{bd}(-\mathcal{I}(\Delta t))$ and $\langle p, x_0 \rangle = c(p, \mathcal{X}(T))$, $\langle p, 0 \rangle = c(p, (-\mathcal{I}(\Delta t))$. Hence $x_0 \in \mathcal{X}(T + \Delta t)$ and $\langle p, x_0 \rangle = c(p, \mathcal{X}(T)) + c(p, (-\mathcal{I}(\Delta t)) = c(p, \mathcal{X}(T + \Delta t))$, i.e. $x_0 \in \operatorname{bd} \mathcal{X}(T + \Delta t)$.

By Lemma 2.1 there exists an extremal control $u : [0, T + \Delta t] \to \mathcal{U}(t)$ with the property $0 \in \operatorname{bd} \mathcal{R}(T + \Delta t, x_0)$. Thus, Pontryagin's maximum principle is not a sufficient condition in the considered linear time-optimal control problem with the initial condition $x(0) = x_0$. □

It easily follows from Lemma 2.1 that strict convexity of any reachable set $\mathcal{R}(t, x_0)$ for all $t > 0$, $x_0 \in \mathbb{R}^n$, and inclusion $0 \in \operatorname{ri} \mathcal{U}(s)$ for a.a. $s \in [0, t]$ gives sufficiency of the maximum principle and uniqueness of the optimal control. This fact is very well known for the case when $\mathcal{U}(t) \equiv \mathcal{U}$ under the so-called generality condition [14, Ch. 3, §23, (89)] and for normal linear control systems with fixed $\mathcal{U}$ of special form, see [7, 8]. Obviously, by Lemma 2.1, the reachable set for the system (1) with $B(t) = I$ is strictly convex if and only if for any unit vector $p \in \mathbb{R}^n$ the set $(\Phi^{-1}(s)\mathcal{U}(s))(p)$ is a singleton for a.a. $s \in [0, t]$. The last property is equivalent to the fact that for any non-trivial solution $\psi(t)$ of the equation (7) the extremal control $u_0(t)$, defined by the formula

$$\max_{u \in \mathcal{U}(s)} \langle \psi(s), u \rangle = \langle \psi(s), u_0(s) \rangle, \tag{18}$$

is unique on the segment $[0, t]$. Unfortunately, usually the uniqueness of u_0 in (18) is not easy to check, just like condition (13).

In the next section we will specify some sufficient conditions of optimality for the maximum principle in terms of the controllability set $\mathcal{X}(t)$. In subsection 3.2, we also prove the non-emptiness of the interior for some set-valued integrals, which is of independent interest.

3. Simple control and analytic matrices

3.1. The case of a linear autonomous control system with simple control

We will call a set-valued mapping $t \to \mathcal{U}(t)$, $t \geq 0$, with convex compact values *simple,* if

$$\mathcal{U}(t) = \sum_{k=1}^{m} \lambda_k(t) \mathcal{U}_k(t).$$

Here for each k the set-valued mapping $t \to \mathcal{U}_k(t)$ is measurable, uniformly bounded and has convex compact values. Moreover, there exists a set $\mathcal{S} \subset [0, +\infty)$ such that for any $t > 0$ the Lebesgue measure of the set $\mathcal{S} \cap [0, t]$ is equal to t. For all $s, t \in \mathcal{S}$, $s \leq t$, and all $k = 1, \ldots, m$, the inclusion $\mathcal{U}_k(s) \subset \mathcal{U}_k(t)$ holds. The numerical functions $\lambda_k(t)$ are measurable, have strictly positive values for all k, and there exists a number $\lambda_0 > 0$ such that $\lambda_0 \leq \frac{\lambda_k(t)}{\lambda_k(s)}$ for all $t, s \in \mathcal{S}$.

Consider a linear autonomous system with constant matrices $A(t) \equiv A$ and $B(t) \equiv I$. For it, the following assertion follows from Lemma 2.3 and Corollary 2.4.

Theorem 3.1. *Let the mapping $t \to \mathcal{U}(t)$ be simple and $0 \in \mathcal{U}(t)$ for all $t \geq 0$. Consider the autonomous system $x'(t) = Ax(t) + u(t)$, $u(t) \in \mathcal{U}(t)$.*

In order for the Pontryagin maximum principle to be a sufficient condition for optimality in the time-optimal control problem for any initial condition $x(0) = x_0 \in \mathbb{R}^n$ with $T(x_0) < \infty$ in the system under consideration, it is necessary and sufficient that for any $t > 0$ the inclusion $0 \in \operatorname{int} \mathcal{X}(t)$ holds.

Proof. Necessity follows from Corollary 2.4. Let us prove sufficiency.

Suppose that for every $t > 0$ the inclusion $0 \in \operatorname{int} \mathcal{X}(t)$ holds. Let us fix arbitrary $t \geq 0$ and $\Delta t > 0$.

Due to the autonomy of the system, we have $\Phi^{-1}(t) \equiv e^{-At}$. Therefore, replacing s with $s + t$ in the integral, we obtain

$$\int_t^{t+\Delta t} e^{-As}\mathcal{U}(s)\,ds = \int_0^{\Delta t} e^{-A(s+t)}\mathcal{U}(s+t)\,ds = e^{-At}\int_0^{\Delta t} e^{-As}\sum_{k=1}^m \lambda_k(s+t)\mathcal{U}_k(s+t)\,ds.$$

Since for all k for a.a. $s \in [0, \Delta t]$ the inclusion $\mathcal{U}_k(s+t) \supset \mathcal{U}_k(s)$ holds, then

$$\sum_{k=1}^m \lambda_k(s+t)\mathcal{U}_k(s+t) \supset \sum_{k=1}^m \lambda_k(s+t)\mathcal{U}_k(s)$$

and hence

$$\int_t^{t+\Delta t} e^{-As}\mathcal{U}(s)\,ds \supset e^{-At}\int_0^{\Delta t} e^{-As}\sum_{k=1}^m \lambda_k(s+t)\mathcal{U}_k(s)\,ds$$

$$= e^{-At}\int_0^{\Delta t} e^{-As}\sum_{k=1}^m \frac{\lambda_k(s+t)}{\lambda_k(s)}\lambda_k(s)\mathcal{U}_k(s)\,ds.$$

Since for each term in the last integrand the inclusion holds

$$\frac{\lambda_k(s+t)}{\lambda_k(s)}\lambda_k(s)\mathcal{U}_k(s) \supset \lambda_0\lambda_k(s)\mathcal{U}_k(s),$$

for a.a. $s \in [0, \Delta t]$, then

$$\int_t^{t+\Delta t} e^{-As}\mathcal{U}(s)\,ds \supset \lambda_0 e^{-At}\int_0^{\Delta t} e^{-As}\sum_{k=1}^m \lambda_k(s)\mathcal{U}_k(s)\,ds$$

$$= \lambda_0 e^{-At}\int_0^{\Delta t} e^{-As}\mathcal{U}(s)\,ds = \lambda_0 e^{-At}(-\mathcal{X}(\Delta t)).$$

According to the condition of the theorem, $0 \in \operatorname{int} \mathcal{X}(\Delta t)$ for any $\Delta t > 0$, then, due to the inequality $\lambda_0 > 0$ and the non-degeneracy of the matrix e^{-At}, we have $0 \in \operatorname{int}\big(\lambda_0 e^{-At}(-\mathcal{X}(\Delta t))\big)$. The sufficiency of Pontryagin's maximum principle follows from Lemma 2.3. $\square$

3.2. System with analytic matrices

Let us consider the system (1) in the case of analytical matrices $A(t)$ and $B(t)$.

For the function $\Psi : [0,+\infty) \to \mathbb{R}^N$, by *analyticity* we will mean the following property: for any number $s_0 \in [0,+\infty)$ there is a radius $R(s_0) > 0$ such that

$$\Psi(s) = \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(s_0)}{k!}(s-s_0)^k$$

in the circle $(s_0 - R(s_0), s_0 + R(s_0)) \cap [0,+\infty)$, i.e. the function $\Psi(s)$ can be represented by a Taylor series. Any analytic vector function has analytic components. The product of analytic functions is an analytic function.

Recall that the identity theorem [10] holds for analytic functions. Let the function $\Psi : [0,+\infty) \to \mathbb{R}^N$ be analytic and there exist a countable set of points

$$\{s_i\}_{i=1}^{\infty} \subset [0,+\infty), \quad \lim_{i\to\infty} s_i = s_0 \in (0,+\infty),$$

such that $\Psi(s_i) = 0$. Then $\Psi(s) \equiv 0$ for all $s \in [0,+\infty)$. Note that the statement of the identity theorem is certainly true if there is a non-trivial interval on the semi-axis $[0,+\infty)$ on which $\Psi \equiv 0$.

It is known that in the case of analyticity of the matrix $A(t)$, the fundamental matrix $\Phi(t)$ of the system $x'(t) = A(t)x(t)$ is also analytic. This follows from the corresponding results for differential equations with analytic right-hand side, see [2, Ch. 6, §11]. In this case, the matrix $\Phi^{-1}(t)$ is analytic too.

Theorem 3.2. *Let in the system* (1) *matrices* $A(t)$ *and* $B(t)$ *be analytic, the set-valued mapping* $t \to \mathcal{U}(t)$ *be measurable, uniformly bounded and have convex compact values,* $0 \in \mathcal{U}(t)$ *for a.a.* $t \geq 0$. *Suppose that the normal cone* $\mathcal{N}(\mathcal{U}(t), 0)$ *for almost all* $t \geq 0$ *is a nontrivial subspace and is given by the equation*

$$\mathcal{N}(\mathcal{U}(t), 0) = \{x \in \mathbb{R}^n \ : \ L(t)x = 0\},$$

where $L(t)$ *is an analytic matrix.*

If for some $t_0 > 0$ *the inclusion* $0 \in \operatorname{int} \mathcal{X}(t_0)$ *holds for the controllability set, then for any* $t \geq 0$, $\Delta t > 0$

$$0 \in \operatorname{int} \mathcal{I}(t, \Delta t), \qquad \mathcal{I}(t, \Delta t) = \int_t^{t+\Delta t} \Phi^{-1}(s)B(s)\mathcal{U}(s)\, ds.$$

Proof. Put $\Psi(s) = (\Phi^{-1}(s)B(s))^T$.

Reasoning by contradiction, assume that $0 \in \operatorname{bd} \mathcal{I}(t, \Delta t)$ for some $t \geq 0$, $\Delta t > 0$.

Since the set $\mathcal{I}(t, \Delta t)$ is convex and compact, then by the finite-dimensional theorem on the separation of the boundary point 0 of the set $\mathcal{I}(t, \Delta t)$ from the set $\mathcal{I}(t, \Delta t)$ there exists a unit vector $p \in \mathbb{R}^n$ such that $p \in \mathcal{N}(\mathcal{I}(t, \Delta t), 0)$ and

$$0 = \langle p, 0 \rangle = c(p, \mathcal{I}(t, \Delta t)) = \int_t^{t+\Delta t} c\left(p, \Phi^{-1}(s)B(s)\mathcal{U}(s)\right)\, ds = \int_t^{t+\Delta t} c\left(\Psi(s)p, \mathcal{U}(s)\right)\, ds.$$

From here, as well as from the integrability of the function $s \to c\,(\Psi(s)p, \mathcal{U}(s))$ and the non-negativity of its values, we have the equality $c\,(\Psi(s)p, \mathcal{U}(s)) = 0$ for a.a. $s \in [t, t + \Delta t]$, i.e. $\Psi(s)p \in \mathcal{N}(\mathcal{U}(s), 0)$.

Then $L(s)\Psi(s)p = 0$ for a.a. $s \in [t, t+\Delta t]$, and therefore, from the continuity of the function $s \to L(s)\Psi(s)p$ we obtain the equality $L(s)\Psi(s)p = 0$ for all $s \in [t, t + \Delta t]$.

Due to the analyticity of the function $s \to L(s)\Psi(s)p$, by the identity theorem $L(s)\Psi(s)p = 0$ for all $s \geq 0$. his means that $\Psi(s)p \in \mathcal{N}(\mathcal{U}(s), 0)$ for almost all $s \geq 0$, so

$$c(p, -\mathcal{X}(t_0)) = \int_0^{t_0} c\,(\Psi(s)p, \mathcal{U}(s))\; ds = 0.$$

The latter contradicts the inclusion $0 \in \operatorname{int} \mathcal{X}(t_0)$.

The contradiction shows that the assumption $0 \in \operatorname{bd} \mathcal{I}(t, \Delta t)$ is false. Hence we have $0 \in \operatorname{int} \mathcal{I}(t, \Delta t)$ for all $t \geq 0$, $\Delta t > 0$. □

Theorem 3.3. *Let in system* (1) *matrices* $A(t)$ *and* $B(t)$ *be analytic and* $B(t)$ *is non-degenerate for all* $t \geq 0$*, the set-valued mapping* $t \to \mathcal{U}(t)$ *be measurable, uniformly bounded and have convex compact values,* $0 \in \mathcal{U}(t)$ *for a.a.* $t \geq 0$*. Suppose that the normal cone* $\mathcal{N}(\mathcal{U}(t), 0)$ *is a ray for a.a.* $t \geq 0$*: there exists a unit vector* $q(t) \in \mathbb{R}^n$ *such that* $\mathcal{N}(\mathcal{U}(t), 0) = \{\lambda q(t) \;:\; \lambda \geq 0\}$*. Assume that the function* $t \to q(t)$ *is analytic. If for some* $t_0 > 0$ *the inclusion* $0 \in \operatorname{int} \mathcal{X}(t_0)$ *holds for the controllability set, then for any* $t \geq 0$*,* $\Delta t > 0$

$$0 \in \operatorname{int} \int_t^{t+\Delta t} \Phi^{-1}(s) B(s) \mathcal{U}(s)\, ds.$$

Proof. We use the notations from the proof of Theorem 3.2. The matrix $\Psi(s) = \left(\Phi^{-1}(s)B(s)\right)^T$ is analytic and non-singular for all $s \geq 0$.

Suppose that $0 \in \operatorname{bd} \mathcal{I}(t, \Delta t)$ for some $t \geq 0$, $\Delta t > 0$. Then, by virtue of the finite-dimensional theorem on the separation of the boundary point 0 of the set $\mathcal{I}(t, \Delta t)$ from the set $\mathcal{I}(t, \Delta t)$, there exists a unit vector $p \in \mathcal{N}(\mathcal{I}(t, \Delta t), 0)$ such that

$$0 = c(p, \mathcal{I}(t, \Delta t)) = \int_t^{t+\Delta t} c\left(p, \Phi^{-1}(s)B(s)\mathcal{U}(s)\right)\, ds = \int_t^{t+\Delta t} c\,(\Psi(s)p, \mathcal{U}(s))\; ds.$$

Hence, as in Theorem 3.2, $c\,(\Psi(s)p, \mathcal{U}(s)) = 0$ and $\Psi(s)p \in \mathcal{N}(\mathcal{U}(s), 0)$ for a.a. $s \in [t, t + \Delta t]$.

This means that $\Psi(s)p = k(s)q(s)$, where $k(s) > 0$, for a.e. $s \in [t, t+\Delta t]$. Since the function $k(s) = \langle q(s), k(s)q(s)\rangle = \langle \Psi(s)p, q(s)\rangle$ is analytic for all $s \geq 0$, then from the equality to zero of the expression $\Psi(s)p - k(s)q(s) = 0$ for a.a. $s \in [t, t + \Delta t]$ from the continuity of $s \to \Psi(s)p - k(s)q(s)$ it follows that it is equal to zero for all $s \in [t, t + \Delta t]$.

Due to the analyticity of the function $s \to \Psi(s)p - k(s)q(s)$, the identity theorem implies that it is equal to zero for all $s \geq 0$.

Note that $k(s) > 0$ for all $s \geq 0$. If there exists a number $s_- \geq 0$ such that $k(s_-) \leq 0$, then from the condition $k(s) > 0$ for $t \leq s \leq t + \Delta t$, by virtue of the intermediate value theorem, there exists a point $s_0 \geq 0$ such that $k(s_0) = 0$.

This implies the equality $\Psi(s_0)p = 0$, which contradicts the non-degeneracy of the matrix $\Psi(s_0)$.

Thus, $\Psi(s)p \in \mathcal{N}(\mathcal{U}(s), 0)$ for almost all $s \geq 0$, so

$$c(p, -\mathcal{X}(t_0)) = \int_0^{t_0} c\left(\Psi(s)p, \mathcal{U}(s)\right)\, ds = 0.$$

This contradicts the inclusion $0 \in \operatorname{int} \mathcal{X}(t_0)$. The contradiction shows that the assumption $0 \in \operatorname{bd} \mathcal{I}(t, \Delta t)$ is false. Hence, $0 \in \operatorname{int} \mathcal{I}(t, \Delta t)$ for all $t \geq 0$, $\Delta t > 0$. □

Due to the structure of the normal cone $\mathcal{N}(\mathcal{U}(t), 0)$ in Theorems 3.2 and 3.3 the point 0 is a boundary point of $\mathcal{U}(t)$. Moreover, in Theorem 3.2 the condition that the cone $\mathcal{N}(\mathcal{U}(t), 0)$ is a subspace is equivalent to 0 being a relatively interior point of $\mathcal{U}(t)$. In Theorem 3.3, the point 0 is a point of smoothness of the boundary $\mathcal{U}(t)$ in the sense that there is a unique unit normal at zero to the set $\mathcal{U}(t)$.

Note that the results of Theorems 3.2 and 3.3 can be transferred to a set-valued integral of the form $\int_t^{t+\Delta t} \Psi(s)\mathcal{U}(s)\, ds$, where $\Psi(s)$ is an $n \times n$ analytic matrix, and $\mathcal{U}(s)$ is a convex compact subset depending on s.

Theorem 3.4. *Let $\Psi(t)$, $t \geq 0$, be analytic $n \times n$ matrix, the set-valued mapping $t \to \mathcal{U}(t)$ be measurable, uniformly bounded and have convex compact values, $0 \in \mathcal{U}(t)$ for a.a. $t \geq 0$.*

Let for some $t_0 > 0$ the inclusion $0 \in \operatorname{int} \int_0^{t_0} \Psi(s)\mathcal{U}(s)\, ds$ be satisfied.

1. Let the normal cone $\mathcal{N}(\mathcal{U}(t), 0)$ be a nontrivial subspace for all $t \geq 0$ and for a.a. $t \geq 0$ be given by the equation $\mathcal{N}(\mathcal{U}(t), 0) = \{x \in \mathbb{R}^n \,:\, L(t)x = 0\}$, where $L(t)$ is an analytic matrix. Then for any $t \geq 0$, $\Delta t > 0$

$$0 \in \operatorname{int} \int_t^{t+\Delta t} \Psi(s)\mathcal{U}(s)\, ds.$$

2. Let the matrix $\Psi(t)$ be non-degenerate for all t, and let the normal cone $\mathcal{N}(\mathcal{U}(t), 0)$ be a ray for a.a. $t \geq 0$: there exists a unit vector $q(t) \in \mathbb{R}^n$ such that $\mathcal{N}(\mathcal{U}(t), 0) = \{\lambda q(t) \,:\, \lambda \geq 0\}$. Suppose that the function $t \to q(t)$ is analytic. Then for all $t \geq 0$, $\Delta t > 0$

$$0 \in \operatorname{int} \int_t^{t+\Delta t} \Psi(s)\mathcal{U}(s)\, ds.$$

Proof. The proof repeats the proof of Theorems 3.2 and 3.3. □

Theorem 3.5. *Let the condition of Theorem* 3.2 *or Theorem* 3.3 *be satisfied. Then Pontryagin's maximum principle is a sufficient condition for optimality in the time-optimal control problem.*

Proof. By virtue of the corresponding theorem for any $t \geq 0$, $\Delta t > 0$

$$0 \in \operatorname{int} \int\limits_t^{t+\Delta t} \Phi^{-1}(s)\, B(s)\mathcal{U}(s)\, ds.$$

The assertion of the theorem follows from Lemma 2.3. □

4. Counterexamples

The first example demonstrates some general situation when the maximum principle is not a sufficient condition of optimality.

Example 4.1. In $\mathbb{R}^n$, consider the system $x'(t) = A(t)x(t) + u(t)$, where $A(t)$ is a given matrix with continuous coefficients and $x(0) = 0$. The control $u(t)$ belongs to $\mathcal{U}(t) \equiv \mathcal{U}$, where $\mathcal{U}$ is such a convex compact set such that $0 \in \operatorname{bd}\mathcal{U}$ and the normal cone $\mathcal{N}(\mathcal{U}, 0)$ has non-empty interior.

Since $x(0) = 0$, the optimal time $T(0) = 0$. Let us take the solution of the system $\psi'(t) = -A^T(t)\psi(t)$ with the initial condition $\psi(0) \in \operatorname{int}\mathcal{N}(\mathcal{U}, 0)$.

Then there exists a number $\delta > 0$ such that $\psi(t) \in \mathcal{N}(\mathcal{U}, 0)$ for any $t \in [0, \delta]$. By virtue of the maximum condition we have the equality

$$\max_{u \in \mathcal{U}} \langle \psi(t), u\rangle = \langle \psi(t), 0\rangle = 0 \quad \forall t \in [0, \delta].$$

This means that the zero control $u(t) \equiv 0$, $t \in [0, \delta]$, is extremal, but not optimal, since $\delta > T(x(0)) = 0$. □

Example 4.2 shows that the analyticity of the matrices in Theorems 3.2 and 3.4 is essential.

Example 4.2. Let $\lambda(\cdot) \in C^\infty$ be a function such that $\lambda(t) = 1$ for $t \in [0, \pi]$, $\lambda(t) = 0$ for $t > 2\pi$, $\lambda(t)$ decreases monotonically for $t \in [\pi, 2\pi]$. Note that the function $\lambda(\cdot)$ is not analytic.

Let us denote $\varphi_0 = \int\limits_0^{2\pi} \lambda(s)\, ds$ and $t_0 = 2\pi$.

In $\mathbb{R}^2$ we consider a system of the form (1) with matrix $A(t) = \lambda(t)A$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t) \equiv I \ \text{ and } \ \mathcal{U} = \operatorname{co}\left((-1, 0)^T, (1, 0)^T\right).$$

The set $\mathcal{N}(\mathcal{U}, 0) = \{\alpha(0, 1)^T \ : \ \alpha \in \mathbb{R}\}$ is a one-dimensional subspace.

Note that $\Phi(t) = \exp\left\{ \int\limits_0^t \lambda(s)\, ds \cdot A \right\}$ is the fundamental matrix $\Phi'(t) \equiv A(t)\Phi(t)$.

Any solution $\psi'(t) = -A^T(t)\psi(t)$ is $\psi(t) = \left(\Phi^{-1}(t)\right)^T p$ for some vector $p \in \mathbb{R}^n$.

Let $t \in (0, \pi]$. Using the equality $\Phi^{-1}(s) = e^{-As}$ for $s \in [0, \pi]$, we obtain

$$-\mathcal{X}(t) = \int_0^t \Phi^{-1}(s)\mathcal{U}\,ds = \int_0^t e^{-As}\mathcal{U}\,ds.$$

For $p = (\cos\varphi, \sin\varphi)^T$, $\varphi \in [0, 2\pi)$, the equalities

$$e^{-A^T s}p = (\cos(s+\varphi), \sin(s+\varphi))^T$$

and
$$c(p, -\mathcal{X}(t)) = \int_0^t c\left(e^{-A^T s}p, \mathcal{U}\right)\,ds = \int_0^t |\cos(s+\varphi)|\,ds > 0$$

are satisfied for all $t \in (0, \pi]$, $\varphi \in [0, 2\pi)$. Due to the continuity of the last integral with respect to φ, we have $\min\limits_{\varphi\in[0,2\pi)} \int_0^t |\cos(s+\varphi)|\,ds > 0$. Therefore, $0 \in \operatorname{int}\mathcal{X}(t)$ for all $t \in (0, \pi]$. Due to the monotonicity with respect to inclusion of sets $\{\mathcal{X}(t)\}_{t\geq 0}$, we have $0 \in \operatorname{int}\mathcal{X}(t)$ for all $t > 0$.

Suppose that $t \geq t_0$ and $\Delta t > 0$. Then

$$A(s) = 0 \text{ for } s \geq t_0, \quad \Phi^{-1}(s) = e^{-\varphi_0 A} = \begin{pmatrix} \cos\varphi_0 & \sin\varphi_0 \\ -\sin\varphi_0 & \cos\varphi_0 \end{pmatrix}$$

and
$$\begin{aligned}\mathcal{I}(t, \Delta t) &= \int_t^{t+\Delta t} \Phi^{-1}(s)\mathcal{U}\,ds = \int_t^{t+\Delta t} e^{-\varphi_0 A}\mathcal{U}\,ds \\ &= \Delta t e^{-\varphi_0 A}\cdot\mathcal{U} = \Delta t \cdot \operatorname{co}\left(\pm(\cos\varphi_0, -\sin\varphi_0)^T\right).\end{aligned}$$

Thus, the set $\mathcal{I}(t, \Delta t)$ has empty interior.

Since $\mathcal{U}$ is a centrally symmetric set with respect to zero, then $-\mathcal{X}(t)$ is the same, and therefore $\mathcal{X}(t) = -\mathcal{X}(t)$.

Put $p = (\sin\varphi_0, \cos\varphi_0)^T$. Fix $x_0 \in \operatorname{bd}\mathcal{X}(t_0)$: $\langle p, x_0\rangle = c(p, \mathcal{X}(t_0))$. Then for any $t > t_0$ the equality holds $\mathcal{X}(t) = \mathcal{X}(t_0) - \int_{t_0}^t e^{-\varphi_0 A}\mathcal{U}\,ds = \mathcal{X}(t_0) - (t - t_0)e^{-\varphi_0 A}\cdot\mathcal{U}$. Since $e^{-\varphi_0 A}\mathcal{U}$ is a segment centered at zero and normal p, then $\langle p, x_0\rangle = c(p, \mathcal{X}(t))$ and $x_0 \in \operatorname{bd}\mathcal{X}(t)$. As follows from Lemma 2.1, the maximum principle is not a sufficient condition of optimality for $t > t_0$. □

Example 4.3 shows that the property of the mapping $t \to \mathcal{U}(t)$ to be simple in Theorem 3.1 is essential.

Example 4.3. In $\mathbb{R}^2$, consider a system of the form (1) with matrices

$$A(t) = A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t) \equiv I$$

and
$$\mathcal{U}(t) = \begin{cases} \operatorname{co}(-(1,0)^T, (1,0)^T), & t \in [0, \pi], \\ \operatorname{co}((0,0)^T, (1,0)^T), & t > \pi. \end{cases}$$

Obviously, $t \to \mathcal{U}(t)$ is not a simple mapping.

From Example 2 we obtain that $0 \in \operatorname{int} \mathcal{X}(t)$ for all $t > 0$. For $t = \pi$, $\Delta t > 0$ and $p = (\cos\varphi, \sin\varphi)^T$ we have

$$c\left(p, \int\limits_{\pi}^{\pi+\Delta t} e^{-As}\mathcal{U}(s)\,ds\right) = \int\limits_{\pi}^{\pi+\Delta t} c\left(e^{-A^T s}p, \mathcal{U}(s)\right)\,ds = \int\limits_{\pi}^{\pi+\Delta t} \max\{0, \cos(s+\varphi)\}\,ds$$

$$= \int\limits_{0}^{\Delta t} \max\{0, -\cos(s+\varphi)\}\,ds.$$

If $\varphi = 0$ and $\Delta t \in (0, \frac{1}{2}\pi)$, then

$$c\left(p, \int\limits_{\pi}^{\pi+\Delta t} e^{-As}\mathcal{U}(s)\,ds\right) = 0 \quad \Rightarrow \quad 0 \in \operatorname{bd} \int\limits_{\pi}^{\pi+\Delta t} e^{-As}\mathcal{U}(s)\,ds$$

and by Lemma 2.3 the maximum principle is not a sufficient condition of optimality.

Notice that for $e_1 = (1,0)^T$ we have $\operatorname{rank}[e_1 \vdots Ae_1] = 2$ and the generality condition [14, Ch. 3, §23, (89)] is fulfilled. But for any $t > \pi$ zero is a relatively boundary point for the set $\mathcal{U}(t) = \operatorname{co}((0,0)^T, (1,0)^T)$. $\square$

References

[1] J.-P. Aubin, H. Frankowska: *Set-Valued Analysis*, Birkhäuser, Boston (2009).

[2] G. Birkhoff, G.-C. Rota: *Ordinary Differential Equations*, 4th ed., John Wiley and Sons, New York (1989).

[3] V. G. Boltyanskii: *Mathematical Methods of Optimal Control*, Holt, Rinehart and Winston, New York (1971).

[4] P. Cannarsa, C. Sinestrari: *Convexity properties of the minimum time function*, Calc. Var. 3 (1995) 273–298.

[5] L. C. Evans, M. R. Janaes: *The Hamilton-Jacobi-Bellman equation for time optimal control*, SIAM J. Control Optim. 27 (1989) 1477–1489.

[6] A. F. Filippov: *On some questions of the theory of optimal control*, Moscow Univ. Phys. Bull. 2 (1959) 25–32.

[7] C. A. Harvey, E. B. Lee: *On the uniqueness of time-optimal control for linear processes*, J. Math. Anal. Appl. 5 (1962) 258–268.

[8] H. Hermes, J. P. LaSalle: *Functional Analysis and Time Optimal Control*, Academic Press, New York (1969).

[9] A. D. Ioffe, V. M. Tikhomirov: *Theory of Extremal Problems*, Elsevier North-Holland, Amsterdam (1979).

[10] S. G. Krantz, H. R. Parks: *A Primer of Real Analytic Functions*, 2nd ed., Birkhäuser, Boston (2002).

[11] E. B. Lee, L. Markus: *Foundations of Optimal Control Theory*, John Wiley and Sons, New York (1967).

[12] N. N. Petrov: *Controllability of autonomous systems*, Diff. Equations 4 (1968) 311–317.

[13] N. N. Petrov: *On the Bellman function for the time-optimal process problem*, J. Appl. Math. Mech. 34 (1970) 785–791.

[14] L. S. Pontryagin, V. G. Boltyanskij, R. V. Gamkrelidze, E. F. Mishchenko: *The Mathematical Theory of Optimal Processes*, Interscience Publishers, John Wiley and Sons, New York (1962).

[15] P. Soravia: *Holder continuity of the minimum-time function for C^1-manifold targets*, J. Optim. Theory Appl. 75 (1992) 401–421.