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General Monotonicity

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This article employs techniques from convex analysis to present characterizations of (maximal) n-monotonicity, similar to the well-known characterizations of (maximal) monotonicity found in the existing literature. These characterizations are further illustrated through examples.

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1. Introduction and preliminaries

The primary objective of this paper is to provide methods for identifying (maximal) n-monotone operators using convex analysis, following the framework outlined in [7, 8, 10], where $n \in \mathbb{N} \cup \{\infty\}$, with $n \geq 1$, denoted as $n \in \overline{1,\infty} := \{1, 2, \ldots\} \cup \{\infty\}$.

In particular contexts, e.g., under reflexivity and when the operator's domain or range is convex, maximal cyclic monotonicity coincides with maximal monotonicity (see, e.g., [9]). There is a substantial body of literature addressing maximal monotonicity in reflexive spaces. Consequently, our focus is on the general context of dual systems that may not necessarily stem from reflexive spaces or dualities between a space and its topological dual. Our main contribution is a functional characterization of (maximal) n-monotonicity in this broad setting (see Theorems 2.1, 2.2, 2.7, 3.1, 3.5, 3.13 below).

This work intersects several previous lines of research (see, e.g., [1, 2, 4, 5]). What distinguishes our approach is not only the breadth of context and results, but also the introduction of new functions and operators closely linked to n-monotone operators. These innovations offer fresh insights and strengthen the link between convex analysis, n-monotonicity, and applications (see, e.g., Theorems 2.11, 3.9, Corollary 3.11, and Example 4.6).

Throughout this paper, we use standard logical abbreviations; in particular, "iff" means "if and only if".

In what follows, we adopt the following concepts and notations: for a multi-valued operator (or multi-function) $T : X \rightrightarrows Y$ we associate it with its

values: $T(x) \subset Y$, $x \in X$, *graph*: $\operatorname{Graph} T = \{(x, y) \in X \times Y \mid y \in T(x)\}$,
inverse: $T^{-1} : Y \rightrightarrows X$, $\operatorname{Graph}(T^{-1}) = \{(y, x) \mid (x, y) \in \operatorname{Graph} T\}$,
domain: $D(T) := \{x \in X \mid T(x) \neq \emptyset\} = \operatorname{Pr}_X(\operatorname{Graph} T)$,

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range: $R(T) := \{y \in Y \mid y \in T(x),\ \text{for some } x \in X\} = \Pr_Y(\operatorname{Graph} T)$,
direct image: for $A \subset X$, $T(A) = \cup_{x\in A} T(x) \subset Y$,
inverse image: for $B \subset Y$, $T^{-1}(B) = \{x \in X \mid T(x) \cap B \neq \emptyset\} \subset X$.
Here $\Pr_X(x, y) := x$ and $\Pr_Y(x, y) := y$, $(x, y) \in X \times Y$ are the canonical projections.
We do not identify an operator with its graph. A subset $S \subset X \times Y$ is said to have a certain property if the associated operator $T : X \rightrightarrows Y$ with $\operatorname{Graph} T = S$ has that property, and conversely.

For a topological vector space (E, μ), $A \subset E$, and $f, g : E \to \overline{\mathbb{R}}$ we denote by:

E^* – the topological dual of E;

$\operatorname{cl}_\mu A$ – the μ-*closure* of A; $x \in \operatorname{cl}_\mu A$ iff $x_i \xrightarrow{\mu} x$, for some net $(x_i)_i \subset A$. Here "$\xrightarrow{\mu}$" denotes the convergence of nets in the μ topology;

$\operatorname{conv} A$ – the *convex hull* of A; $x \in \operatorname{conv} A$ iff $(x, 1) = \sum_{k=1}^{n} \lambda_k (x_k, 1)$ for some $(\lambda_k)_{k\in\overline{1,n}} \subset \mathbb{R}^+ := [0, +\infty) = \{r \in \mathbb{R} \mid r \geq 0\}$, $(x_k)_{k\in\overline{1,n}} \subset A$;

$\operatorname{Epi} f := \{(x, t) \in E \times \mathbb{R} \mid f(x) \leq t\} \subset E \times \mathbb{R}$ – the *epigraph* of f;

$\operatorname{cl}_\mu f$ – the μ-*lower semicontinuous hull* of f, which is the greatest μ-lower semicontinuous function majorized by f;

$\operatorname{Epi}(\operatorname{cl}_\mu f) = \operatorname{cl}_{\mu\times\tau_0}(\operatorname{Epi} f)$, where τ_0 is the usual topology of $\mathbb{R}$;

$\operatorname{conv} f$ – the *convex hull* of f, that is, the greatest convex function majorized by f,

$(\operatorname{conv} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{conv}(\operatorname{Epi} f)\}$;

$\operatorname{cl}_\mu \operatorname{conv} f$ – the μ-*lower semicontinuous convex hull* of f, which is the greatest μ-lower semicontinuous convex function majorized by f;

$\operatorname{Epi}(\operatorname{cl}_\mu \operatorname{conv} f) := \operatorname{cl}_{\mu\times\tau_0} \operatorname{conv}(\operatorname{Epi} f)$.

The set $[f \leq g] := \{x \in E \mid f(x) \leq g(x)\}$; the sets $[f = g]$, $[f < g]$, $[f > g]$, $[f \geq g]$ are similarly defined; $f \leq (<, >, \geq) g$ means $[f \leq (<, >, \geq) g] = E$, or, equivalently, for every $x \in E$, $f(x) \leq (<, >, \geq) g(x)$.

$\Lambda(E)$ – the class of proper convex functions $f : E \to \overline{\mathbb{R}}$. Recall that f is *proper* if $\operatorname{dom} f := \{x \in E \mid f(x) < \infty\}$ is nonempty and f does not take the value $-\infty$; f is *convex* if, for every $x, y \in E$, $t \in [0, 1]$, we have $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ while observing the conventions: $\infty - \infty = \infty$, $0 \cdot \pm\infty = 0$.

$\Gamma_\mu(E)$ – the class of functions $f \in \Lambda(E)$ that are μ-lower semicontinuous.

We avoid using the μ-notation when the topology is implicitly understood.

A *dual system* is a triple (X, Y, c), where X, Y are vector spaces, and c is a bilinear map $c = \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. The weakest topology on X, for which all linear forms $\{X \ni x \to \langle x, y \rangle \mid y \in Y\}$ are continuous, is denoted by $\sigma(X, Y)$ and it is called *the weak topology of* X with respect to the duality (X, Y, c). The weak topology $\sigma(Y, X)$ on Y is defined in a similar manner. The spaces X and Y are regarded as locally convex spaces, endowed with their respective weak topologies.

Given a dual system $(X, Y, \langle \cdot, \cdot \rangle)$, $S \subset X$, and $f : X \to \overline{\mathbb{R}}$. Then

- $f^* : Y \to \overline{\mathbb{R}}$ is the *convex conjugate* of $f : X \to \overline{\mathbb{R}}$ with respect to the dual system $(X, Y, \langle \cdot, \cdot \rangle)$, $f^*(y) := \sup\{\langle x, y \rangle - f(x) \mid x \in X\}$ for $y \in Y$; similarly, the convex conjugate of $g : Y \to \overline{\mathbb{R}}$ is $g^*(x) := \sup\{\langle x, y\rangle - g(y) \mid y \in Y\}$ for $x \in X$;

- $\partial f(x)$ is the *subdifferential* of $f : X \to \overline{\mathbb{R}}$ at $x \in X$;

$$\partial f(x) := \{y \in Y \mid \forall w \in X,\ \langle w - x, y\rangle + f(x) \le f(w)\}, \tag{1}$$

for $x \in X$ with $f(x) \in \mathbb{R}$; $\partial f(x) := \emptyset$ whenever $f(x) \notin \mathbb{R}$. Note that for an improper f we have $\operatorname{Graph}(\partial f) = \emptyset$.

- $Z := X \times Y$ forms naturally a dual system $(Z, Z, \cdot)$ where

$$z \cdot w := \langle x, v\rangle + \langle u, y\rangle,\ z = (x, y) \in Z,\ w = (u, v) \in Z. \tag{2}$$

- $S^\perp := \{y \in Y \mid \forall x \in S,\ \langle x, y\rangle = 0\}$ is the *orthogonal of* S, $\sigma_S(y) := \iota_S^*(y) = \sup_{x \in S}\langle x, y\rangle$, $y \in Y$ is the *support functional* of S; note that $\sigma_S = \iota_{S^\perp}$ whenever S is a linear subspace of X. Here $\iota_S(z) = 0$, for $z \in S$, and $\iota_S(z) = +\infty$ otherwise.

The space Z is endowed with a locally convex topology μ compatible with the natural duality $(Z, Z, \cdot)$, meaning that $(Z, \mu)^* = Z$. For example $\mu = \sigma(Z, Z) = \sigma(X, Y) \times \sigma(Y, X)$. All notions associated with a proper function $f : Z \to \overline{\mathbb{R}}$ are similarly defined as above. Furthermore, with respect to the natural dual system $(Z, Z, \cdot)$, the conjugate of f is

$$f^\square : Z \to \overline{\mathbb{R}}, \quad f^\square(z) = \sup\{z \cdot z' - f(z') \mid z' \in Z\}. \tag{3}$$

By the biconjugate formula, $f^{\square\square} = \operatorname{cl}\operatorname{conv} f$, whenever $f^\square$ (or $\operatorname{cl}\operatorname{conv} f$) is proper (see e.g. [6]).

We use distinct notations for conjugates to reflect both the domain and the underlying duality: f^* denotes the conjugate with respect to a general duality, while $f^\square$ denotes the conjugate with respect to the natural dual system associated with that duality.

To $A \subset Z$ we associate $c_A : Z \to \overline{\mathbb{R}}$, $c_A := c + \iota_A$, where $\iota_A(z) = 0$, for $z \in A$, $\iota_A(z) = +\infty$, otherwise. The *Fitzpatrick function* of A is

$$\varphi_A : Z \to \overline{\mathbb{R}},\ \varphi_A(z) := c_A^\square(z) = \sup\{z \cdot w - c(w) \mid w \in A\},$$

and $\psi_A : Z \to \overline{\mathbb{R}}$, $\psi_A := \varphi_A^\square = c_A^{\square\square}$. The functions φ_A, ψ_A were first introduced by Fitzpatrick in [7].

In particular when φ_A is proper (or $\operatorname{cl}\operatorname{conv} c_A$ is proper),

$$\psi_A = \operatorname{cl}\operatorname{conv} c_A,\ \varphi_A = \psi_A^\square.$$

Similarly, for a multifunction $T : X \rightrightarrows Y$ we define $c_T := c_{\operatorname{Graph} T}$, $\psi_T := \psi_{\operatorname{Graph} T}$, and the *Fitzpatrick function* of T, $\varphi_T := \varphi_{\operatorname{Graph} T}$. By convention we have $\varphi_\emptyset = -\infty$, $c_\emptyset = \operatorname{conv} c_\emptyset = \psi_\emptyset = +\infty$ in agreement with the usual conventions of $\inf_\emptyset = +\infty$, $\sup_\emptyset = -\infty$.

In expanded form, for $T : X \rightrightarrows Y$ and $z = (x, y) \in Z$,

$$\varphi_T(z) = \sup\{z \cdot w - c(w) \mid w \in \operatorname{Graph} T\},$$
$$\varphi_T(x, y) = \sup\{\langle x - u, v\rangle + \langle u, y\rangle \mid (u, v) \in \operatorname{Graph} T\}. \tag{4}$$

Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $n \in \mathbb{N}$, $n \ge 1$.

An operator $T : X \rightrightarrows Y$ is

- *n-monotone* (in $(X, Y, \langle\cdot,\cdot\rangle)$) if, for every $\{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$ with $x_{n+1} = x_1$, we have

$$\sum_{i=1}^{n}\langle x_i - x_{i+1}, y_i\rangle = \langle x_1 - x_2, y_1\rangle + \ldots + \langle x_{n-1} - x_n, y_{n-1}\rangle + \langle x_n - x_1, y_n\rangle \geq 0. \quad (5)$$

  Equivalently, reversing the order of the pairs, T is n-monotone iff, for every $\{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$ with $x_0 = x_n$,

$$\begin{aligned}\langle x_n - x_{n-1}, y_n\rangle + \ldots + \langle x_2 - x_1, y_2\rangle + \langle x_1 - x_n, y_1\rangle&\\ = \sum_{k=1}^{n}\langle x_{n+1-k} - x_{n-k}, y_{n+1-k}\rangle = \sum_{i=1}^{n}\langle x_i - x_{i-1}, y_i\rangle \geq 0.&\end{aligned} \quad (6)$$

- *cyclically (or ∞-)monotone* if, for every $k \in \mathbb{N}$, $k \geq 2$, T is k-monotone.
- *maximal n-monotone* if T is n-monotone and it does not allow for a proper n-monotone extension in terms of graph inclusion. Here $n \in \overline{1,\infty} := \mathbb{N}^* \cup \{\infty\}$.

Every operator is inherently 1-monotone, and $Z = X \times Y$ is the only maximal 1-monotone set. The term "monotone" typically refers to 2-monotone operators. For $n, m \in \overline{2,\infty}$, with $n \leq m$, every m-monotone operator is also n-monotone. Moreover, an operator that is both m-monotone and maximal n-monotone is also maximal m-monotone. The prototype for cyclically monotone operators is given by the convex subdifferential, as described in [8, Theorem 1, p. 500] or Theorem 2.10 below.

Definition 1.1. For $n \in \overline{1,\infty}$, the *$n$-th Fitzpatrick function* associated with the operator $T : X \rightrightarrows Y$ is denoted as $\overset{\langle n\rangle}{\varphi}_T : X \times Y \to \overline{\mathbb{R}}$ and is defined as follows:

$$\overset{\langle 1\rangle}{\varphi}_T(x, y) := \varphi_T(x, y) := \sup\{\langle x - u, v\rangle + \langle u, y\rangle \mid (u, v) \in \operatorname{Graph} T\}; \quad (7)$$

for $2 \leq n < \infty$:

$$\begin{aligned}\overset{\langle n\rangle}{\varphi}_T(x, y) :&= \sup_{\{(x_i,y_i)\}_{i\in\overline{1,n}}\subset \operatorname{Graph} T} \{\langle x - x_1, y_1\rangle + \langle x_1 - x_2, y_2\rangle + \ldots + \langle x_{n-1} - x_n, y_n\rangle + \langle x_n, y\rangle\};\\ &= \sup_{\{(x_i,y_i)\}_{i\in\overline{1,n}}\subset \operatorname{Graph} T} \{\langle x - x_1, y_1\rangle + \sum_{i=2}^{n}\langle x_{i-1} - x_i, y_i\rangle + \langle x_n, y\rangle\}; \end{aligned}\quad (8)$$

and $$\overset{\langle \infty\rangle}{\varphi}_T = \sup_{n\geq 1} \overset{\langle n\rangle}{\varphi}_T. \quad (9)$$

Note that $\overset{\langle 1\rangle}{\varphi}_T = c_T^{\square}$. For $n \in \mathbb{N}$, $n \geq 2$, after using a reversed pair order,

$$\begin{aligned}\overset{\langle n\rangle}{\varphi}_T(x, y) :&= \sup_{\{(x_k,y_k)\}_{k\in\overline{1,n}}\subset \operatorname{Graph} T} \{\langle x - x_n, y_n\rangle + \sum_{k=2}^{n}\langle x_{n+2-k} - x_{n+1-k}, y_{n+1-k}\rangle + \langle x_1, y\rangle\}\\ &= \sup_{\{(x_i,y_i)\}_{i\in\overline{1,n}}\subset \operatorname{Graph} T} \{\langle x - x_n, y_n\rangle + \sum_{i=1}^{n-1}\langle x_{i+1} - x_i, y_i\rangle + \langle x_1, y\rangle\}.\end{aligned} \quad (10)$$

For $n, m \in \overline{1,\infty}$, with $n \le m$, it follows that $\overset{\langle n\rangle}{\varphi}_T \le \overset{\langle m\rangle}{\varphi}_T$. As a result we obtain $\overset{\langle\infty\rangle}{\varphi}_T = \lim_{n\to\infty} \overset{\langle n\rangle}{\varphi}_T$.

In general, for $n \in \mathbb{N}$, $n \ge 1$, and $x \in D(T)$, $y \in Y$, if we pick $(x_i, y_i) = (x, v)$, $i \in \overline{1,n}$, for some $v \in T(x)$, then, from (7), (8), $\overset{\langle n\rangle}{\varphi}_T(x, y) \ge \langle x, y\rangle$. Therefore

$$\operatorname{Graph} T \subset (D(T) \times Y) \cup (X \times R(T)) \subset [\overset{\langle n\rangle}{\varphi}_T \ge c] \subset [\overset{\langle\infty\rangle}{\varphi}_T \ge c]. \tag{11}$$

for all $T : X \rightrightarrows Y$ and for all $n \in \overline{1,\infty}$. Note also that for all $n \in \overline{1,\infty}$,

$$\overset{\langle n\rangle}{\varphi}_T(x, y) = \overset{\langle n\rangle}{\varphi}_{T^{-1}}(y, x),\ x \in X,\ y \in Y. \tag{12}$$

Definition 1.2. For $n \in \overline{1,\infty}$, $\overset{\langle n\rangle}{\psi}_T : X \times Y \to \overline{\mathbb{R}}$ is the convex conjugate of the n-th Fitzpatrick function, defined as

$$\overset{\langle n\rangle}{\psi}_T = (\overset{\langle n\rangle}{\varphi}_T)^{\square}, \tag{13}$$

relative to the natural dual system $(Z, Z, \cdot)$.

Definition 1.3. For $n \in \overline{1,\infty}$, consider $\overset{\langle n\rangle}{\asymp}_T : X \times Y \to \overline{\mathbb{R}}$ given by

$$\overset{\langle 1\rangle}{\asymp}_T = c_T,\ \overset{\langle 2\rangle}{\asymp}_T(x, y) = \inf\{\langle x_1, y\rangle + \langle x - x_1, y_2\rangle \mid x_1 \in T^{-1}(y),\ y_2 \in T(x)\}, \tag{14}$$

for $3 \le n < \infty$:

$$\overset{\langle n\rangle}{\asymp}_T(x, y) = \inf \left\{ \begin{array}{l} \langle x_1, y\rangle + \sum_{i=2}^{n-1} \langle x_i - x_{i-1}, y_i\rangle + \langle x - x_{n-1}, y_n\rangle \mid \\ x_1 \in T^{-1}(y),\ y_n \in T(x),\ \{(x_i, y_i)\}_{i\in\overline{2,n-1}} \subset \operatorname{Graph} T \end{array} \right\}, \tag{15}$$

and

$$\overset{\langle\infty\rangle}{\asymp}_T(x, y) = \inf_{n\ge 1} \overset{\langle n\rangle}{\asymp}_T(x, y). \tag{16}$$

Note that $\operatorname{dom} \overset{\langle 1\rangle}{\asymp}_T = \operatorname{Graph} T$ and, for $n \in \overline{2,\infty}$,

$$\operatorname{dom} \overset{\langle n\rangle}{\asymp}_T = [\overset{\langle n\rangle}{\asymp}_T < \infty] = D(T) \times R(T).$$

Lemma 1.4. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system, let $n \in \overline{1,\infty}$, and let $T : X \rightrightarrows Y$. Then*

$$\forall n \in \overline{1,\infty},\ \overset{\langle n\rangle}{\varphi}_T = (\overset{\langle n\rangle}{\asymp}_T)^{\square},\ \overset{\langle n\rangle}{\psi}_T = (\overset{\langle n\rangle}{\asymp}_T)^{\square\square} \tag{17}$$

and, whenever $\overset{\langle n\rangle}{\varphi}_T$ or $\operatorname{cl}\operatorname{conv}(\overset{\langle n\rangle}{\asymp}_T)$ is proper, $\overset{\langle n\rangle}{\psi}_T = \operatorname{cl}\operatorname{conv}(\overset{\langle n\rangle}{\asymp}_T)$.

For $n, m \in \overline{1,\infty}$, with $n \le m$, $\overset{\langle m\rangle}{\asymp}_T \le \overset{\langle n\rangle}{\asymp}_T$; in particular

$$\forall n \in \overline{1,\infty},\ \overset{\langle n\rangle}{\psi}_T \le \overset{\langle n\rangle}{\asymp}_T \le \overset{\langle 1\rangle}{\asymp}_T = c_T,\ \operatorname{Graph} T \subset [\overset{\langle n\rangle}{\asymp}_T \le c] \subset [\overset{\langle n\rangle}{\psi}_T \le c]. \tag{18}$$

Proof. For (17) we use (7), (8), (9), (14), (15), (16) followed by the biconjugate formula.

For every $(x, y) \in \operatorname{Graph} T$, pick $x_1 = x$ or $y_2 = y$ in the definition of $\overset{\langle 2\rangle}{\between}_T$ to get $\overset{\langle 2\rangle}{\between}_T \le c_T = \overset{\langle 1\rangle}{\between}_T$.

Let $n, m \in \mathbb{N}$ be such that $2 \le n < m$. For every $(x, y) \in D(T) \times R(T)$ and every $x_1 \in T^{-1}(y)$, $y_m \in T(x)$, $\{(x_i, y_i)\}_{i \in \overline{2,m-1}} \subset \operatorname{Graph} T$, we have

$$\overset{\langle m\rangle}{\between}_T(x, y) \le \langle x_1, y\rangle + \sum_{i=2}^{m-1} \langle x_i - x_{i-1}, y_i\rangle + \langle x - x_{m-1}, y_m\rangle.$$

For $k \in \overline{n, m-1}$ take $(x_k, y_k) = (x_{n-1}, y_{n-1})$ to get

$$\forall (x, y) \in D(T) \times R(T),\ \forall x_1 \in T^{-1}(y),\ \forall y_n \in T(x),\ \forall \{(x_i, y_i)\}_{i \in \overline{2,n-1}} \subset \operatorname{Graph} T,$$

$$\overset{\langle m\rangle}{\between}_T(x, y) \le \langle x_1, y\rangle + \sum_{i=2}^{n-1} \langle x_i - x_{i-1}, y_i\rangle + \langle x - x_{n-1}, y_n\rangle.$$

Pass to infimum over $x_1 \in T^{-1}(y)$, $y_n \in T(x)$, $\{(x_i, y_i)\}_{i \in \overline{2,n-1}} \subset \operatorname{Graph} T$ to find

$$\forall (x, y) \in D(T) \times R(T),\ \overset{\langle m\rangle}{\between}_T(x, y) \le \overset{\langle n\rangle}{\between}_T(x, y).$$

Since elsewhere $\overset{\langle m\rangle}{\between}_T(x, y) = \overset{\langle n\rangle}{\between}_T(x, y) = +\infty$ the proof is complete. □

Lemma 1.5. *Let (X, Y, c) be a dual system, and let $f \in \Lambda(X \times Y)$ be such that $f \ge c$. Then*

$$[f = c] \subset [f^{\square} = c]. \tag{19}$$

Proof. For every $z, w \in Z = X \times Y$ and $t > 0$, $z \cdot w = \frac{1}{t}(c(z + tw) - c(z)) - tc(w)$. Let $z \in [f = c]$. Since $f \ge c$ and f is convex, for every $w \in Z$,

$$z \cdot w = \lim_{t \downarrow 0} \tfrac{1}{t}(c(z + tw) - c(z)) \le \lim_{t \downarrow 0} \tfrac{1}{t}(f(z + tw) - f(z))$$

$$= \inf_{t > 0} \tfrac{1}{t}(f(z + tw) - f(z)) \le f(z + w) - f(z),$$

which shows that $z \in \partial f(z)$. Hence $f(z) + f^{\square}(z) = z \cdot z = 2c(z)$ and so,

$$z \in [f^{\square} = c].$$ □

2. n-Monotonicity

Theorem 2.1. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $n \in \overline{1, \infty}$.*

The operator $T : X \rightrightarrows Y$ is n-monotone iff $\overset{\langle n\rangle}{\between}_T \ge c$.

In this case, $\operatorname{Graph} T \subset [\overset{\langle n\rangle}{\between}_T = c]$ *and, whenever* $\operatorname{Graph} T \ne \emptyset$, $\overset{\langle n\rangle}{\between}_T$ *is proper.*

Proof. If $n = 1$ or $\operatorname{Graph} T = \emptyset$, then the conclusion is straightforward, as $\overset{\langle 1\rangle}{\asymp}_T = c_T$. Otherwise, suppose that $n \in \overline{2,\infty}$ and $T : X \rightrightarrows Y$ has $\operatorname{Graph} T \neq \emptyset$.

Assume first that n is finite. If T is n-monotone then, for every $(x, y) \in D(T)\times R(T)$ and every $x_1 \in T^{-1}(y),\ y_n \in T(x),\ \{(x_i, y_i)\}_{i\in\overline{2,n-1}} \subset \operatorname{Graph} T$, the n-monotonicity of T on the pairs (x_1, y), (x, y_n), $(x_{n-1}, y_{n-1}), \ldots, (x_2, y_2)$ provides

$$\langle x_1 - x, y\rangle + \langle x - x_{n-1}, y_n\rangle + \sum_{i=2}^{n-1}\langle x_i - x_{i-1}, y_i\rangle \geq 0.$$

Pass to infimum over $x_1 \in T^{-1}(y),\ y_n \in T(x),\ \{(x_i, y_i)\}_{i\in\overline{2,n-1}} \subset \operatorname{Graph} T$ to obtain $(\overset{\langle n\rangle}{\asymp}_T - c)(x, y) \geq 0$. Therefore $\overset{\langle n\rangle}{\asymp}_T \geq c$.

Conversely, suppose that $\overset{\langle n\rangle}{\asymp}_T \geq c$. For every $\{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$,

$$\sum_{i=2}^{n}\langle x_i - x_{i-1}, y_i\rangle + \langle x_1 - x_n, y_1\rangle \geq (\overset{\langle n\rangle}{\asymp}_T - c)(x_n, y_1) \geq 0.$$

According to (6), T is n-monotone.

The case $n = \infty$ follows directly as a consequence of the cases where $1 \leq n < \infty$.

In general, for every $n\in\overline{1,\infty}$, $\overset{\langle n\rangle}{\asymp}_T \leq \overset{\langle 1\rangle}{\asymp}_T = c_T$ which implies $\operatorname{Graph} T \subset [\overset{\langle n\rangle}{\asymp}_T \leq c]$.

When T is n-monotone, meaning that $\overset{\langle n\rangle}{\asymp}_T \geq c$, we obtain $\operatorname{Graph} T \subset [\overset{\langle n\rangle}{\asymp}_T = c]$. □

Theorem 2.2. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $n, m \in \overline{1,\infty}$. The operator $T : X \rightrightarrows Y$ is $(n+m)$-monotone iff one of the following conditions holds:*

(i) $\overset{\langle n\rangle}{\varphi}_T \leq \overset{\langle m\rangle}{\asymp}_T$, (ii) $\overset{\langle n\rangle}{\varphi}_T \leq \operatorname{conv} \overset{\langle m\rangle}{\asymp}_T$, (iii) $\overset{\langle n\rangle}{\varphi}_T \leq \overset{\langle m\rangle}{\psi}_T$.

In this case,

$$\forall k \in \overline{1, n+m-1},\ \operatorname{Graph} T \subset [\overset{\langle k\rangle}{\varphi}_T = c]\cap[\overset{\langle k\rangle}{\asymp}_T = c]\cap[\operatorname{conv} \overset{\langle k\rangle}{\asymp}_T = c]\cap[\overset{\langle k\rangle}{\psi}_T = c], \quad (20)$$

and, whenever $\operatorname{Graph} T \neq \emptyset$, $\overset{\langle k\rangle}{\varphi}_T$, $\overset{\langle k\rangle}{\psi}_T$, $\operatorname{conv} \overset{\langle k\rangle}{\asymp}_T$, $\overset{\langle k\rangle}{\asymp}_T$ *are proper,* $k \in \overline{1, n+m-1}$.

Proof. If $\operatorname{Graph} T = \emptyset$, then $\overset{\langle n\rangle}{\varphi}_T = -\infty$, $\overset{\langle m\rangle}{\asymp}_T = \operatorname{conv} \overset{\langle m\rangle}{\asymp}_T = \overset{\langle m\rangle}{\psi}_T = +\infty$, and the conclusion follows. Otherwise, suppose that $\operatorname{Graph} T \neq \emptyset$.

Assume that T is $(n + m)$-monotone, where $m \geq 1$, $n \geq 1$ are finite integers.

For every $(x, y) \in D(T) \times R(T)$, $x'_1 \in T^{-1}(y)$, $y'_m \in T(x)$ and every

$$\{(x_i, y_i)\}_{i\in\overline{1,n}} \cup \{(x'_j, y'_j)\}_{j\in\overline{2,m-1}} \subset \operatorname{Graph} T,$$

the $(n + m)$-monotonicity condition (5) for the pairs

$$(x_n, y_n), (x_{n-1}, y_{n-1}), \ldots, (x_1, y_1),\quad (x, y'_m), (x'_{m-1}, y'_{m-1}), \ldots, (x'_2, y'_2), (x'_1, y)$$

provides

$$\sum_{i=2}^{n}\langle x_i - x_{i-1}, y_i\rangle + \langle x_1 - x, y_1\rangle + \langle x - x'_{m-1}, y'_m\rangle + \sum_{j=2}^{m-1}\langle x'_j - x'_{j-1}, y'_j\rangle + \langle x'_1 - x_n, y\rangle \geq 0,$$

and

$$\langle x-x_1,y_1\rangle+\sum_{i=2}^{n}\langle x_{i-1}-x_i,y_i\rangle+\langle x_n,y\rangle \le \langle x_1',y\rangle+\sum_{j=2}^{m-1}\langle x_j'-x_{j-1}',y_j'\rangle+\langle x-x_{m-1}',y_m'\rangle.$$

Pass to supremum over $\{(x_k,y_k)\}_{k\in\overline{1,n}}\subset \operatorname{Graph}T$ on the left hand side and to infimum over $x_1'\in T^{-1}(y)$, $y_m'\in T(x)$, $\{(x_j',y_j')\}_{j\in\overline{2,m-1}}\subset\operatorname{Graph}T$ on the right hand side to get $\overset{\langle n\rangle}{\varphi}_T(x,y)\le \overset{\langle m\rangle}{\asymp}_T(x,y)$. Since $\overset{\langle m\rangle}{\asymp}_T=+\infty$ outside $D(T)\times R(T)$, we infer that (i) is true. In particular, $\operatorname{Graph}T\subset[\overset{\langle n\rangle}{\varphi}_T=c]\cap[\overset{\langle m\rangle}{\asymp}_T=c]$, since, according to (11), (18), $\operatorname{Graph}T\subset[\overset{\langle n\rangle}{\varphi}_T\ge c]\cap[\overset{\langle m\rangle}{\asymp}_T\le c]$, and $\overset{\langle n\rangle}{\varphi}_T$ is proper.

Assume that (i) holds. For every $\{(x_i,y_i)\}_{i\in\overline{1,n}}\cup\{(x_j',y_j')\}_{j\in\overline{1,m}}\subset\operatorname{Graph}T$

$$\langle x_m'-x_1,y_1\rangle+\sum_{i=2}^{n}\langle x_{i-1}-x_i,y_i\rangle+\langle x_n,y_1'\rangle\le\overset{\langle n\rangle}{\varphi}_T(x_m',y_1')\le\overset{\langle m\rangle}{\asymp}_T(x_m',y_1')$$

$$\le\langle x_1',y_1'\rangle+\sum_{j=2}^{m-1}\langle x_j'-x_{j-1}',y_j'\rangle+\langle x_m'-x_{m-1}',y_m'\rangle=\langle x_1',y_1'\rangle+\sum_{j=2}^{m}\langle x_j'-x_{j-1}',y_j'\rangle,$$

which leads to

$$\sum_{i=2}^{n}\langle x_i-x_{i-1},y_i\rangle+\langle x_1-x_m',y_1\rangle+\sum_{j=2}^{m}\langle x_j'-x_{j-1}',y_j'\rangle+\langle x_1'-x_n,y_1'\rangle\ge 0,$$

i.e., the $(n+m)$-monotonicity condition for the pairs

$$(x_n,y_n),(x_{n-1},y_{n-1}),\ldots,(x_1,y_1),\quad (x_m',y_m'),(x_{m-1}',y_{m-1}'),\ldots,(x_1',y_1'),$$

holds. Therefore T is $(n+m)$-monotone.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are plain, $\overset{\langle m\rangle}{\psi}_T$, $\operatorname{conv}\overset{\langle m\rangle}{\asymp}_T$, $\overset{\langle m\rangle}{\asymp}_T$ are proper, and (20) is true in this case, since $\overset{\langle n\rangle}{\varphi}_T\in\Gamma(Z)$ and $\overset{\langle n\rangle}{\varphi}_T\le\overset{\langle m\rangle}{\psi}_T=\operatorname{cl}\operatorname{conv}(\overset{\langle m\rangle}{\asymp}_T)\le\operatorname{conv}\overset{\langle m\rangle}{\asymp}_T\le\overset{\langle m\rangle}{\asymp}_T$.

In case $m=\infty$ (or $n=\infty$) we use the above argument for any finite integer $1\le m<\infty$ ($1\le n<\infty$) and pass to infimum (supremum) over $m\ge1$ ($n\ge1$) to conclude. $\square$

Corollary 2.3. *Let $(X,Y,\langle\cdot,\cdot\rangle)$ be a dual system and let $n\in\overline{1,\infty}$. The operator $T:X\rightrightarrows Y$ is $(n+1)$-monotone iff one of the following conditions holds:*

(i) $\overset{\langle n\rangle}{\varphi}_T\le c_T$, (ii) $\operatorname{Graph}T\subset[\overset{\langle n\rangle}{\varphi}_T\le c]$, (iii) $\operatorname{Graph}T\subset[\overset{\langle n\rangle}{\varphi}_T=c]$.

Proof. We use Theorem 2.2 for $m=1$ to infer that T is $(n+1)$-monotone iff $\overset{\langle n\rangle}{\varphi}_T\le\overset{\langle 1\rangle}{\asymp}_T=c_T$, that is, (i) holds; (i) $\Leftrightarrow$ (ii) is plain; and (ii) $\Leftrightarrow$ (iii) follows with the aid of (11). $\square$

Definition 2.4. Given $(X, Y, \langle\cdot,\cdot\rangle)$ a dual system, $n \in \mathbb{N}$, $n \geq 2$, and $T : X \rightrightarrows Y$, the pair $(x, y) \in X \times Y$ is *n-monotonically related* (m.r. for short) to T if, for every $\{(x_i, y_i)\}_{i \in \overline{1,n-1}} \subset \operatorname{Graph} T$,

$$\langle x - x_{n-1}, y\rangle + \langle x_{n-1} - x_{n-2}, y_{n-1}\rangle + \ldots + \langle x_2 - x_1, y_2\rangle + \langle x_1 - x, y_1\rangle \geq 0. \quad (21)$$

The pair $(x, y) \in X \times Y$ is *cyclically* (∞)*-monotonically related* (m.r. for short) to T if, for every $n \geq 2$, (x, y) is n-m.r. to T.

For $n \in \overline{2,\infty}$, provided that T is n-monotone, (x, y) is n-m.r. to T if and only if $\operatorname{Graph} T \cup \{(x, y)\}$ is n-monotone.

Theorem 2.5. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system, let $n \in \overline{2,\infty}$, and let $T : X \rightrightarrows Y$. Then*

(i) $[\overset{\langle n-1\rangle}{\varphi}_T \leq c] = \{(x, y) \in X \times Y \mid (x, y)$ *is n-m.r. to T*$\}$*;*

(ii) *T is n-monotone iff* $\operatorname{Graph} T \subset [\overset{\langle n-1\rangle}{\varphi}_T \leq c]$ *iff* $\operatorname{Graph} T \subset [\overset{\langle n-1\rangle}{\varphi}_T = c]$.

Proof. (i) From (8), (21), it is easily checked that, for $n \in \mathbb{N}$, $n \geq 2$, (x, y) is n-m.r. to T iff $(x, y) \in [\overset{\langle n-1\rangle}{\varphi}_T \leq c]$. As a consequence the same holds for $n = \infty$.

(ii) This fact has already been observed in Corollary 2.3. Alternatively, it follows from (i) and (11). □

The concept of (2-)monotonicity has been extensively studied in [10, 11, 12, 13, 14, 16, 17, 18] with the key findings summarized in the following theorem.

Theorem 2.6. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $T : X \rightrightarrows Y$. The operator $T : X \rightrightarrows Y$ is (2-)monotone iff one of the following conditions holds:*

(i) $\operatorname{Graph} T \subset [\varphi_T = c]$,	(v) $\operatorname{conv} c_T \geq c$,
(ii) $\overset{\langle 1\rangle}{\varphi}_T \leq c_T$,	(vi) *there is* $h \in \Lambda(Z)$ *with* $h \geq c$ *and* $\operatorname{Graph} T \subset [h = c]$,
(iii) $\overset{\langle 1\rangle}{\varphi}_T \leq \overset{\langle 1\rangle}{\psi}_T$,	(vii) $\overset{\langle 1\rangle}{\varphi}_{T^+} \geq c$,
(iv) $\overset{\langle 1\rangle}{\psi}_T \geq c$,	(viii) *T^{++} is monotone.*

Here $\operatorname{Graph}(T^+) = [\overset{\langle 1\rangle}{\varphi}_T \leq c]$ *and* $T^{++} = (T^+)^+$.

Theorem 2.7. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $T : X \rightrightarrows Y$. The operator $T : X \rightrightarrows Y$ is cyclically monotone iff one of the following conditions holds:*

(i) $\operatorname{Graph} T \subset [\overset{\langle\infty\rangle}{\varphi}_T(\leq) = c]$,	(iv) $\overset{\langle\infty\rangle}{\psi}_T \geq c$,
(ii) $\overset{\langle\infty\rangle}{\varphi}_T \leq \overset{\langle\infty\rangle}{{)\!\!-\!\!(}}_T$,	(v) $\operatorname{conv} \overset{\langle\infty\rangle}{{)\!\!-\!\!(}}_T \geq c$,
(iii) $\overset{\langle\infty\rangle}{\varphi}_T \leq \overset{\langle\infty\rangle}{\psi}_T$,	(vi) *there is* $h \in \Lambda(Z)$ *such that* $c \leq h \leq \overset{\langle\infty\rangle}{{)\!\!-\!\!(}}_T$.

Proof. We may assume that $\operatorname{Graph} T \neq \emptyset$, otherwise the argument is plain.

In this case and under any of the given conditions, $\overset{\langle\infty\rangle}{\varphi}_T$, $\overset{\langle\infty\rangle}{\psi}_T$, $\overset{\langle\infty\rangle}{{)\!\!-\!\!(}}_T$ are proper.

The equivalences of T being cyclically monotone with (i), (ii), (iii) are consequences of Theorems 2.5, 2.2 for $n = m = \infty$.

(iii) $\Rightarrow$ (iv) For every $z \in Z$,

$$\overset{\langle\infty\rangle}{\psi}_T(z) \geq \tfrac{1}{2}(\overset{\langle\infty\rangle}{\psi}_T(z) + \overset{\langle\infty\rangle}{\varphi}_T(z)) = \tfrac{1}{2}((\overset{\langle\infty\rangle}{\varphi}_T)^{\square}(z) + \overset{\langle\infty\rangle}{\varphi}_T(z)) \geq \tfrac{1}{2}z \cdot z = c(z).$$

(iv) $\Rightarrow$ (v) follows from $\overset{\langle\infty\rangle}{\psi}_T = \operatorname{cl}\operatorname{conv} \overset{\langle\infty\rangle}{\bowtie}_T \leq \operatorname{conv} \overset{\langle\infty\rangle}{\bowtie}_T$.

(v) $\Rightarrow$ (vi) is straightforward with $h = \operatorname{conv} \overset{\langle\infty\rangle}{\bowtie}_T$.

If (vi) holds then T is cyclically monotone due to Theorem 2.1. $\square$

Notice that condition (vi) in both Theorems 2.6, 2.7 shares the same structure, because $h \geq c$ and $\operatorname{Graph} T \subset [h = c]$ iff $c \leq h \leq c_T = \overset{\langle 1\rangle}{\bowtie}_T$.

Remark 2.8. For $n \in \overline{2,\infty}$, let $T_n^+ : Z \rightrightarrows Z$ be defined by

$$\operatorname{Graph}(T_n^+) = [\overset{\langle n-1\rangle}{\varphi}_T \leq c],$$

i.e., $z \in \operatorname{Graph}(T_n^+)$ iff z is n-m.r. to T, and let $T_n^{++} = (T_n^+)_n^+$.

For any operator T, we have $\operatorname{Graph} T \subset \operatorname{Graph}(T_{(2)}^{++})$. However, for $n \in \overline{3,\infty}$, it is generally not true that $\operatorname{Graph} T \subset \operatorname{Graph}(T_n^{++})$. More precisely,

$$\exists\, T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2,\ \forall n \in \overline{3,\infty},\ \operatorname{Graph} T \not\subset \operatorname{Graph}(T_n^{++}). \tag{22}$$

Take $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by $\operatorname{Graph} T = \{(0,0),(5,5)\}$. Then

$$\{(1,1),(2,0.32)\} \subset \operatorname{Graph}(T_\infty^+) = \cap_{n=2}^{\infty} \operatorname{Graph}(T_n^+),$$

because

$$\operatorname{Graph} T \cup \{(1,1)\} \subset \operatorname{Graph}(\partial(\tfrac{1}{2}x^2)) = \{(x,x) \mid x \in \mathbb{R}\},$$

$$\operatorname{Graph} T \cup \{(2,0.32)\} \subset \operatorname{Graph}(\partial(\tfrac{1}{100}x^4)) = \{(x, \tfrac{1}{25}x^3) \mid x \in \mathbb{R}\}.$$

However, for every $n \in \overline{3,\infty}$, we have $(0,0) \in \operatorname{Graph} T \setminus \operatorname{Graph}(T_n^{++})$, since

$$\operatorname{Graph}(T_n^{++}) \subset \operatorname{Graph}((T_\infty^+)_n^+) \subset \operatorname{Graph}((T_\infty^+)_3^+),$$

and $(0,0) \notin \operatorname{Graph}((T_\infty^+)_3^+)$, i.e., $(0,0)$ is not 3-m.r. to T_∞^+, as evidenced when we verify (21) for $(x,y) = (0,0)$, $(x_1,y_1) = (2,0.32)$, $(x_2,y_2) = (1,1)$. Indeed, in this case

$$(x - x_2)y + (x_2 - x_1)y_2 + (x_1 - x)y_1 = -0.36 < 0.$$

Definition 2.9. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system. Then, for $T : X \rightrightarrows Y$ and $w = (x_0, y_0) \in \operatorname{Graph} T$ define *Rockafellar's antiderivative* $r_T^w : X \to \overline{\mathbb{R}}$ as

$$r_T^w(x) := \sup\{\langle x - x_n, y_n\rangle + \sum_{i=0}^{n-1}\langle x_{i+1} - x_i, y_i\rangle \mid n \geq 1, \{(x_i,y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T\}. \tag{23}$$

It is worthwhile to recall Rockafellar's result [8, Theorem 1, p. 500], which characterizes the convex subdifferential as a template for cyclically monotone operators.

We provide a proof for the sake of completeness and in light of the broader context, particularly with regard to equation (25) below.

Theorem 2.10. *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual system. Then $T : X \rightrightarrows Y$ is cyclically monotone iff there is $h : X \to \overline{\mathbb{R}}$ ($h \in \Gamma_{\sigma(X,Y)}(X)$) such that* $\operatorname{Graph} T \subset \operatorname{Graph}(\partial h)$. *In this case,*

$$\forall w = (x_0, y_0) \in \operatorname{Graph} T,\ r_T^w(x_0) = 0,\ \operatorname{Graph} T \subset \operatorname{Graph}(\partial r_T^w), \tag{24}$$

and, for every $x \in X$,

$$\begin{aligned} r_T^w(x) &= \min\{h(x) \mid h : X \to \overline{\mathbb{R}},\ h(x_0) = 0,\ \operatorname{Graph} T \subset \operatorname{Graph}(\partial h)\} \\ &= \min\{h(x) \mid h \in \Gamma_{\sigma(X,Y)}(X),\ h(x_0) = 0,\ \operatorname{Graph} T \subset \operatorname{Graph}(\partial h)\}. \end{aligned} \tag{25}$$

Here "∂" represents the convex subdifferential taken with respect to $(X, Y, \langle \cdot, \cdot \rangle)$.

Proof. Assume that $\operatorname{Graph} T \subset \operatorname{Graph}(\partial h)$ for some $h : X \to \overline{\mathbb{R}}$. For every integer $n \geq 2$ and $\{(x_i, y_i)\}_{i \in \overline{1,n}} \subset \operatorname{Graph} T$, with $x_{n+1} = x_1$, we have

$$\sum_{i=1}^{n} \langle x_i - x_{i+1}, y_i \rangle \geq \sum_{i=1}^{n} h(x_i) - h(x_{i+1}) = 0,$$

that is, T is cyclically monotone.

Let T be non-empty cyclically monotone and let $w = (x_0, y_0) \in \operatorname{Graph} T$. For every $(x, y) \in \operatorname{Graph} T$, $n \geq 1$, $\{(x_i, y_i)\}_{i \in \overline{1,n}} \subset \operatorname{Graph} T$ we have from the cyclic monotonicity of T on the family $\{(x_i, y_i)\}_{i \in \overline{0,n+1}} \subset \operatorname{Graph} T$, where $(x_{n+1}, y_{n+1}) = (x, y)$

$$\sum_{i=0}^{n-1} \langle x_i - x_{i+1}, y_i \rangle + \langle x_n - x, y_n \rangle + \langle x - x_0, y \rangle \geq 0,$$

$$\langle x - x_n, y_n \rangle + \sum_{i=0}^{n-1} \langle x_{i+1} - x_i, y_i \rangle \leq \langle x - x_0, y \rangle. \tag{26}$$

After passing to the supremum in (26) over $n \geq 1$, $\{(x_i, y_i)\}_{i \in \overline{1,n}} \subset \operatorname{Graph} T$, we get that, for every $(x, y) \in \operatorname{Graph} T$, $r_T^w(x) \leq \langle x - x_0, y \rangle < \infty$. This shows that $D(T) \subset \operatorname{dom}(r_T^w)$ and $r_T^w(x_0) \leq 0$.

Take $n = 1$ and $(x_1, y_1) = (x_0, y_0)$ in the definition of r_T^w to find $r_T^w(x) \geq \langle x - x_0, y_0 \rangle$, from which $r_T^w(x_0) \geq 0$, and so, $r_T^w(x_0) = 0$. Note that $r_T^w \in \Gamma_{\sigma(X,Y)}(X)$ since it is a supremum of continuous affine functions.

Arbitrarily fix $x \in X$ and $(\overline{x}, \overline{y}) \in \operatorname{Graph} T$. For every $n \geq 1$, $\{(x_i, y_i)\}_{i \in \overline{1,n}} \subset \operatorname{Graph} T$, from the definition of r_T^w used for $\{(x_i, y_i)\}_{i \in \overline{1,n+1}} \subset \operatorname{Graph} T$, where $(x_{n+1}, y_{n+1}) = (\overline{x}, \overline{y})$, we find

$$r_T^w(x) \geq \langle x - \overline{x}, \overline{y} \rangle + \langle \overline{x} - x_n, y_n \rangle + \sum_{i=0}^{n-1} \langle x_{i+1} - x_i, y_i \rangle.$$

Pass to supremum over $n \geq 1$, $\{(x_i, y_i)\}_{i \in \overline{1,n}} \subset \operatorname{Graph} T$ to get

$$r_T^w(x) \geq \langle x - \overline{x}, \overline{y} \rangle + r_T^w(\overline{x}),$$

that is, $(\overline{x}, \overline{y}) \in \operatorname{Graph}(\partial r_T^w)$. Therefore $\operatorname{Graph} T \subset \operatorname{Graph}(\partial r_T^w)$.

Let $h : X \to \overline{\mathbb{R}}$ be such that $h(x_0) = 0$ and $\operatorname{Graph} T \subset \operatorname{Graph}(\partial h)$.

For every $\{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$ we have $\langle x_{i+1} - x_i, y_i\rangle \le h(x_{i+1}) - h(x_i)$, $i \in \overline{1,n}$ and so

$$\sum_{i=0}^{n-1} \langle x_{i+1} - x_i, y_i\rangle \le \sum_{i=0}^{n-1} [h(x_{i+1}) - h(x_i)] = h(x_n).$$

Together with $\langle x - x_n, y_n\rangle \le h(x) - h(x_n)$ one gets that for every $x \in X$

$$\langle x - x_n, y_n\rangle + \sum_{i=0}^{n-1} \langle x_{i+1} - x_i, y_i\rangle \le h(x).$$

Hence $r_T^w \le h$. □

Theorem 2.11. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system, let $T : X \rightrightarrows Y$, and let $w = (x_0, y_0) \in \operatorname{Graph} T$. Then, for every $x \in X$,*

$$r_T^{(x_0,y_0)}(x) = \overset{\langle\infty\rangle}{\varphi}_T(x, y_0) - \langle x_0, y_0\rangle. \tag{27}$$

Proof. Relation (27) is an immediate consequence of (9), (10), and (23). □

Corollary 2.12. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $T : X \rightrightarrows Y$ be cyclically monotone. Then,*

$$\forall x_0 \in D(T),\ \forall y_0, y_0' \in T(x_0),\ r_T^{(x_0,y_0)} = r_T^{(x_0,y_0')}. \tag{28}$$

Equivalently, for all $x \in X$, for all $x_0 \in D(T)$, and for all $y_0, y_0' \in T(x_0)$,

$$\overset{\langle\infty\rangle}{\varphi}_T(x, y_0) - \langle x_0, y_0\rangle = \overset{\langle\infty\rangle}{\varphi}_T(x, y_0') - \langle x_0, y_0'\rangle. \tag{29}$$

For every $h : X \to \overline{\mathbb{R}}$ such that $\operatorname{Graph} T \subset \operatorname{Graph}(\partial h)$ we have

$$\forall x \in X,\ \forall (x_0, y_0) \in \operatorname{Graph} T,\ h(x) \ge h(x_0) + r_T^{(x_0,y_0)}(x). \tag{30}$$

Proof. Relations (28), (30) are direct consequences of (25). According to (27), (29) is equivalent to (28). □

Theorem 2.13. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $T : X \rightrightarrows Y$. Define the operator*

$$K_T : D(K_T) = D(T) \times R(T) \subset Z = X \times Y \rightrightarrows Z$$

as $K_T(x, y) := T^{-1}(y) \times T(x)$, $(x, y) \in Z$, or, equivalently

$$((x, y), (u, v)) \in \operatorname{Graph} K_T \ \Leftrightarrow\ (x, v), (u, y) \in \operatorname{Graph} T. \tag{31}$$

Then T is cyclically monotone in $(X, Y, \langle\cdot,\cdot\rangle)$ iff K_T is cyclically monotone in $(Z, Z, \cdot)$ iff $D(T) \times R(T) \subset \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T$ iff

$$\operatorname{Graph} K_T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T) \tag{32}$$

iff

$$\operatorname{Graph} K_T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\psi}_T). \tag{33}$$

Here "∂" represents the convex subdifferential taken with respect to $(Z, Z, \cdot)$.

Proof. We may assume that $\operatorname{Graph} T \neq \emptyset$, otherwise the argument is straightforward. Fix $((x, y), (u, v)) \in \operatorname{Graph} K_T$, that is, fix $(x, v), (u, y) \in \operatorname{Graph} T$. For every $n \geq 1$, $\{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$, use (8), the definition of $\overset{\langle n+2\rangle}{\varphi}_T$, on the pairs

$$(x, v), (x_n, y_n), \dots, (x_1, y_1), (u, y)$$

to get that, for every $(a, b) \in Z$,

$$\overset{\langle n+2\rangle}{\varphi}_T(a, b) \geq \langle a - x, v\rangle + \langle x - x_n, y_n\rangle + \sum_{i=1}^{n-1} \langle x_{i+1} - x_i, y_i\rangle + \langle x_1 - u, y\rangle + \langle u, b\rangle.$$

Pass to the supremum over $\{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$ to find for all $(a, b) \in Z$,

$$\overset{\langle n+2\rangle}{\varphi}_T(a, b) \geq \langle a - x, v\rangle + \overset{\langle n\rangle}{\varphi}_T(x, y) + \langle u, b - y\rangle.$$

Therefore, for all $(x, v), (u, y) \in \operatorname{Graph} T$ and all $(a, b) \in Z$,

$$\overset{\langle n+2\rangle}{\varphi}_T(a, b) \geq (u, v) \cdot ((a, b) - (x, y)) + \overset{\langle n\rangle}{\varphi}_T(x, y). \tag{34}$$

Pass to the supremum over $n \geq 1$ to obtain for all $(x, v), (u, y) \in \operatorname{Graph} T$ and all $(a, b) \in Z$,

$$\overset{\langle\infty\rangle}{\varphi}_T(a, b) \geq (u, v) \cdot ((a, b) - (x, y)) + \overset{\langle\infty\rangle}{\varphi}_T(x, y). \tag{35}$$

From (35),

$$\begin{aligned} D(T) \times R(T) \subset \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T \ &\Leftrightarrow\ \forall (x, v), (u, y) \in \operatorname{Graph} T,\ (u, v) \in \partial \overset{\langle\infty\rangle}{\varphi}_T(x, y) \\ &\Leftrightarrow\ \operatorname{Graph} K_T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T). \end{aligned} \tag{36}$$

Conditions (32) and (33) are equivalent because

$$(\partial \overset{\langle\infty\rangle}{\varphi}_T)^{-1} = \partial \overset{\langle\infty\rangle}{\psi}_T \text{ and } (K_T)^{-1} = K_T.$$

Assume that T is cyclically monotone. According to Theorem 2.2, $\overset{\langle\infty\rangle}{\varphi}_T \leq \overset{\langle\infty\rangle}{\asymp}_T$. In particular we have $D(T) \times R(T) = \operatorname{dom} \overset{\langle\infty\rangle}{\asymp}_T \subset \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T$. From (36) we conclude $\operatorname{Graph} K_T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T)$.

Assume that $\operatorname{Graph} K_T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T)$. Then K_T is cyclically monotone, since so is $\partial \overset{\langle\infty\rangle}{\varphi}_T$. Also, $D(T) \times R(T) = D(K_T) \subset \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T$.

Pick $b = y$ in (35) to get $v \in \partial \overset{\langle\infty\rangle}{\varphi}_T(\cdot, y)(x)$; whence, according to (27),

$$\operatorname{Graph} T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T(\cdot, y)) = \operatorname{Graph}(\partial r_T^{(u,y)}); \tag{37}$$

in particular T is cyclically monotone.

Assume that K_T is cyclically monotone. For every finite integer $n \geq 2$ and for every $\{(x_i, v_i)\}_{i\in\overline{1,n}} \cup \{(u_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$, the n-monotonicity condition of K_T on the set $\{((x_i, y_i), (u_i, v_i))\}_{i\in\overline{1,n}} \subset \operatorname{Graph} K_T$ translates into

$$\sum_{i=1}^{n} ((x_i, y_i) - (x_{i+1}, y_{i+1})) \cdot (u_i, v_i) = \sum_{i=1}^{n} \langle x_i - x_{i+1}, v_i\rangle + \sum_{i=1}^{n} \langle u_i, y_i - y_{i+1}\rangle \geq 0, \tag{38}$$

where $(x_{n+1}, y_{n+1}, u_{n+1}, v_{n+1}) = (x_1, y_1, u_1, v_1)$.

For some $i \in \overline{1,n}$ we take $(u_i, y_i) = (u, y)$ a fixed element of $\operatorname{Graph} T$ to obtain $\sum_{i=1}^{n} \langle x_i - x_{i+1}, v_i \rangle \geq 0$, i.e., T is n-monotone. Therefore T is cyclically monotone. $\square$

Remark 2.14. As we have seen in (37), we note that (24) is a consequence of (32) via (35). More precisely, given T cyclically monotone and $w = (x_0, y_0) \in \operatorname{Graph} T$,

$$r_T^w(x_0) = \overset{\langle\infty\rangle}{\varphi}_T(x_0, y_0) - \langle x_0, y_0 \rangle = 0,$$

because, according to Theorem 2.7 we have $\operatorname{Graph} T \subset [\overset{\langle\infty\rangle}{\varphi}_T = c]$. $\square$

Remark 2.15. It has been observed in several places (see e.g. [7, Theorem 3.4] or [3, Theorem 3.3]) that $T : X \rightrightarrows Y$ is monotone in $(X, Y, \langle \cdot, \cdot \rangle)$ if and only if $\Delta_T := \{(z, z) \mid z \in \operatorname{Graph} T\}$ is (cyclically) monotone in $(Z, Z, \cdot)$, mainly, because, for every $n \geq 1$ and $\{z_1, \ldots, z_n\} \subset Z$, $z_{n+1} = z_1$, we have

$$\begin{aligned} \sum_{i=1}^{n} c(z_i - z_{i+1}) &= \sum_{i=1}^{n} [c(z_i) + c(z_{i+1}) - z_i \cdot z_{i+1}] \\ &= \sum_{i=1}^{n} [2c(z_i) - z_i \cdot z_{i+1}] = \sum_{i=1}^{n} (z_i - z_{i+1}) \cdot z_i \end{aligned} \tag{39}$$

Although Δ_T cannot differentiate between various types of monotonicities, its extension K_T can, namely, for $n \in \overline{2, \infty}$

$$T \text{ is } n\text{-monotone in } (X, Y, \langle \cdot, \cdot \rangle) \ \Leftrightarrow \ K_T \text{ is } n\text{-monotone in } (Z, Z, \cdot). \tag{40}$$

Indeed, the converse implication in (40) is established within the proof of Theorem 2.13.

Directly, if T is n-monotone then so is T^{-1}.

For every $\{(x_i, v_i)\}_{i \in \overline{1,n}} \cup \{(u_i, y_i)\}_{i \in \overline{1,n}} \subset \operatorname{Graph} T$, with

$$(x_{n+1}, y_{n+1}, u_{n+1}, v_{n+1}) = (x_1, y_1, u_1, v_1),$$

$\sum_{i=1}^{n} \langle x_i - x_{i+1}, v_i \rangle \geq 0$ because T is n-monotone, $\sum_{i=1}^{n} \langle u_i, y_i - y_{i+1} \rangle \geq 0$, since T^{-1} is n-monotone, and so, (38) holds, that is, K_T is n-monotone. $\square$

Analogous to Theorem 2.13, the following variant of Theorem 2.10 can be established.

Theorem 2.16. *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual system. Then $T : X \rightrightarrows Y$ is cyclically monotone iff one of the following conditions holds*

$$\forall(\exists) y_0 \in R(T),\ D(T) \subset \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T(\cdot, y_0), \tag{41}$$

$$\forall(\exists) y_0 \in R(T),\ \operatorname{Graph} T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T(\cdot, y_0)), \tag{42}$$

$$\forall(\exists) w = (x_0, y_0) \in \operatorname{Graph} T,\ \operatorname{Graph} T \subset \operatorname{Graph}(\partial r_T^w). \tag{43}$$

Here "∂" represents the convex subdifferential taken with respect to $(X, Y, \langle \cdot, \cdot \rangle)$.

Theorem 2.17. *Let* $(X, Y, \langle\cdot,\cdot\rangle)$ *be a dual system, let* $n \in \overline{1,\infty}$, *let* $T : X \rightrightarrows Y$, *and let* $K_T : Z \rightrightarrows Z$ *be defined as in* (31). *Then*

$$\overset{\langle n\rangle}{\varphi}_{K_T}(x,y,u,v) = \overset{\langle n\rangle}{\varphi}_T(x,v) + \overset{\langle n\rangle}{\varphi}_T(u,y),\ \ x,u \in X,\ \ y,v \in Y, \tag{44}$$

$$\overset{\langle n\rangle}{\psi}_{K_T}(x,y,u,v) = \overset{\langle n\rangle}{\psi}_T(x,v) + \overset{\langle n\rangle}{\psi}_T(u,y),\ \ x,u \in X,\ \ y,v \in Y, \tag{45}$$

$$\begin{cases} r_{K_T}^{((x_0,y_0),(u_0,v_0))}(x,y) = r_T^{(x_0,v_0)}(x) + r_{T^{-1}}^{(y_0,u_0)}(y), \\ (x,y) \in X \times Y,\ \ (x_0,v_0),(u_0,y_0) \in \operatorname{Graph} T, \end{cases} \tag{46}$$

and T *is (maximal)* n*-monotone (in* $(X,Y,\langle\cdot,\cdot\rangle)$*) iff* K_T *is (maximal)* n*-monotone (in* $(Z,Z,\cdot)$*).*

If T *is cyclically monotone in* $(X,Y,\langle\cdot,\cdot\rangle)$ *then, for every* $(x,y) \in X \times Y$, *every* $(x_0,v_0) \in \operatorname{Graph} T$, *and every* $(u_0,y_0) \in \operatorname{Graph} T$,

$$r_{T^{-1}}^{(y_0,u_0)}(y) \le (r_T^{(x_0,v_0)})^*(y) - (r_T^{(x_0,v_0)})^*(y_0), \tag{47}$$

$$\overset{\langle\infty\rangle}{\varphi}_T(x,v_0) + \overset{\langle\infty\rangle}{\varphi}_T(u_0,y) - \langle x_0,v_0\rangle - \langle u_0,y_0\rangle \le \overset{\langle\infty\rangle}{\varphi}_T(x,y) - \overset{\langle\infty\rangle}{\varphi}_T(x_0,y_0). \tag{48}$$

Proof. For every finite integer $n \ge 1$, for every $(x,y),(u,v) \in X \times Y$, and for every $\{((x_i,y_i),(u_i,v_i))\}_{i\in\overline{1,n}} \subset \operatorname{Graph} K_T$, i.e., $\{(x_i,v_i)\}_{i\in\overline{1,n}} \cup \{(u_i,y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$ we have

$$((x,y)-(x_n,y_n))\cdot(u_n,v_n) + \sum_{i=1}^{n-1}((x_{i+1},y_{i+1})-(x_i,y_i))\cdot(u_i,v_i) + (x_1,y_1)\cdot(u,v)$$
$$= \langle x-x_n,v_n\rangle + \sum_{i=1}^{n-1}\langle x_{i+1}-x_i,v_i\rangle + \langle x_1,v\rangle + \langle u_n,y-y_n\rangle + \sum_{i=1}^{n-1}\langle u_i,y_{i+1}-y_i\rangle + \langle u,y_1\rangle.$$

After we pass to the supremum over $\{(x_i,v_i)\}_{i\in\overline{1,n}} \cup \{(u_i,y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph} T$, we find that

$$\overset{\langle n\rangle}{\varphi}_{K_T}(x,y,u,v) = \overset{\langle n\rangle}{\varphi}_T(x,v) + \overset{\langle n\rangle}{\varphi}_{T^{-1}}(y,u) = \overset{\langle n\rangle}{\varphi}_T(x,v) + \overset{\langle n\rangle}{\varphi}_T(u,y).$$

Let $n \to \infty$ to complete (44). Relation (45) is obtained from (44) by convex conjugation.

The equality in (46) is similarly verified or it is yielded by (44), (27), (12).

If $n = 1$ then T is maximal 1-monotone in $(X,Y,\langle\cdot,\cdot\rangle)$ iff $\operatorname{Graph} T = X \times Y$ iff $\operatorname{Graph} K_T = Z \times Z$ iff K_T is maximal 1-monotone in $(Z,Z,\cdot)$.

Assume that T is maximal n-monotone in $(X,Y,\langle\cdot,\cdot\rangle)$, for some $n \in \overline{2,\infty}$. According to Theorem 3.1 below, $[\overset{\langle n-1\rangle}{\varphi}_T \le c] \subset \operatorname{Graph} T$ and $\overset{\langle n-1\rangle}{\varphi}_T \ge c$. Similarly, since T^{-1} is maximal n-monotone, $[\overset{\langle n-1\rangle}{\varphi}_{T^{-1}} \le c] \subset \operatorname{Graph}(T^{-1})$ and $\overset{\langle n-1\rangle}{\varphi}_{T^{-1}} \ge c$. After using (44), (12), and $\overset{\langle n-1\rangle}{\varphi}_T \ge c$, $\overset{\langle n-1\rangle}{\varphi}_{T^{-1}} \ge c$, one finds that $[\overset{\langle n-1\rangle}{\varphi}_{K_T} \le C] \subset \operatorname{Graph} K_T$. Here $C(z,w) = z\cdot w$, $z,w \in Z$. Therefore, again from Theorem 3.1, K_T is maximal n-monotone in $(Z,Z,\cdot)$.

Conversely, if K_T is maximal n-monotone then $[\overset{\langle n-1\rangle}{\varphi}_{K_T} \le C] \subset \operatorname{Graph} K_T$, which yields, after considering (44), that $[\overset{\langle n-1\rangle}{\varphi}_T \le c] \subset \operatorname{Graph} T$, i.e., T is maximal n-monotone.

If T is non-empty cyclically monotone then, for every $(x_0, v_0) \in \operatorname{Graph} T$, we have $\operatorname{Graph} T \subset \operatorname{Graph}(\partial r_T^{(x_0,v_0)})$. Therefore $\operatorname{Graph} K_T \subset \operatorname{Graph}(\partial f)$, where

$$f(x, y) = r_T^{(x_0,v_0)}(x) + (r_T^{(x_0,v_0)})^*(y),\ x \in X,\ y \in Y.$$

Using (25) for K_T yields that, for every $x \in X$, $y \in Y$, $(x_0, v_0), (u_0, y_0) \in \operatorname{Graph} T$,

$$\begin{aligned} r_{K_T}^{((x_0,y_0),(u_0,v_0))}(x, y) &= r_T^{(x_0,v_0)}(x) + r_{T^{-1}}^{(y_0,u_0)}(y) \leq f(x, y) - f(x_0, y_0) \\ &= r_T^{(x_0,v_0)}(x) + (r_T^{(x_0,v_0)})^*(y) - (r_T^{(x_0,v_0)})^*(y_0), \end{aligned}$$

from which (47) follows.

Similarly, from $\operatorname{Graph} K_T \subset \operatorname{Graph}(\partial \overset{\langle\infty\rangle}{\varphi}_T)$ we get

$$r_{K_T}^{((x_0,y_0),(u_0,v_0))}(x, y) = r_T^{(x_0,v_0)}(x) + r_{T^{-1}}^{(y_0,u_0)}(y) \leq \overset{\langle\infty\rangle}{\varphi}_T(x, y) - \overset{\langle\infty\rangle}{\varphi}_T(x_0, y_0)$$

which provides (48) via (27). □

Proposition 2.18. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h \in \Gamma_{\sigma(X,Y)}(X)$. Then $K_{\partial h} = \partial f$, where $f(x, y) := h(x) + h^*(y)$, $x \in X$, $y \in Y$.*

Proof. It suffices to note that $\operatorname{Graph}(\partial h^*) = \operatorname{Graph}(\partial h)^{-1}$. □

3. Maximal n-monotonicity

Theorem 3.1. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $n \in \overline{2,\infty}$. The operator $T : X \rightrightarrows Y$ is maximal n-monotone iff one of the following conditions holds:*

(i) $\operatorname{Graph} T = (\supset)[\overset{\langle n-1\rangle}{\varphi}_T \leq c]$;

(ii) $\overset{\langle n-1\rangle}{\varphi}_T \geq c$ *and* $\operatorname{Graph} T = [\overset{\langle n-1\rangle}{\varphi}_T = c]$;

(iii) $\overset{\langle n-1\rangle}{\varphi}_T \geq c$ *and T is n-representable, which is defined as T is n-monotone and*

$$\exists f \in \Gamma(Z),\ c \leq f \leq \max\{\overset{\langle n-1\rangle}{\varphi}_T, \overset{\langle n-1\rangle}{\psi}_T\},\ \operatorname{Graph} T = [f = c]; \tag{49}$$

(iv) *T is n-representable and, for every $(x, y) \in X \times Y$, condition $C_n(x, y)$ holds, where*

$$C_2(x, y) :\ \exists (x_1, y_1) \in \operatorname{Graph} T,\ \langle x - x_1, y_1\rangle + \langle x_1 - x, y\rangle \geq 0, \tag{50}$$

$$\text{For } 3 \leq n < \infty,\ C_n(x, y) : \begin{cases} \exists\, \{(x_i, y_i)\}_{i \in \overline{1,n-1}} \subset \operatorname{Graph} T, \\ \langle x - x_1, y_1\rangle + \sum_{i=2}^{n-1} \langle x_{i-1} - x_i, y_i\rangle + \langle x_{n-1} - x, y\rangle \geq 0, \end{cases} \tag{51}$$

$$C_\infty(x, y) :\ \exists m \in \mathbb{N},\ m \geq 2,\ C_m(x, y) \text{ holds.} \tag{52}$$

Proof. Assume that T is maximal n-monotone. Every $(x, y) \in [\overset{\langle n-1\rangle}{\varphi}_T \leq c]$ is n-m.r. to T which is n-monotone. Hence $\operatorname{Graph} T \cup \{(x, y)\}$ is n-monotone and contains $\operatorname{Graph} T$, from which $(x, y) \in \operatorname{Graph} T$. We proved $[\overset{\langle n-1\rangle}{\varphi}_T \leq c] \subset \operatorname{Graph} T$. Using Corollary 2.3 or Theorem 2.5, we deduce the converse inclusion from the n-monotonicity of T, thus completing the proof of (i).

Suppose that (i) holds. Let M be any n-monotone extension of T. Then every $(x, y) \in \operatorname{Graph} M$ is n-m.r. to T, because M is n-monotone and contains T. From Theorem 2.5 (i) we know that $(x, y) \in [\overset{\langle n-1\rangle}{\varphi}_T \leq c] \subset \operatorname{Graph} T$. We proved that $\operatorname{Graph} M \subset \operatorname{Graph} T$. Therefore T is maximal n-monotone.

(ii) $\Rightarrow$ (i) is plain, since $\overset{\langle n-1\rangle}{\varphi}_T \geq c$ provides $[\overset{\langle n-1\rangle}{\varphi}_T \leq c] = [\overset{\langle n-1\rangle}{\varphi}_T = c]$.

(i) $\Rightarrow$ (ii) According to (11), $[\overset{\langle n-1\rangle}{\varphi}_T \leq c] \subset \operatorname{Graph} T \subset [\overset{\langle n-1\rangle}{\varphi}_T \geq c]$. Hence we have $[\overset{\langle n-1\rangle}{\varphi}_T < c] = \emptyset$, i.e, $\overset{\langle n-1\rangle}{\varphi}_T \geq c$, and $\operatorname{Graph} T = [\overset{\langle n-1\rangle}{\varphi}_T = c]$.

(ii) $\Rightarrow$ (iii) is straightforward from Theorem 2.5 (ii) with $f = \overset{\langle n-1\rangle}{\varphi}_T$.

(iii) $\Rightarrow$ (ii) Let $\overset{\langle n-1\rangle}{\gamma}_T := \max\{\overset{\langle n-1\rangle}{\varphi}_T, \overset{\langle n-1\rangle}{\psi}_T\}$.

Since T is n-monotone we know from Theorem 2.5 (ii) that $\operatorname{Graph} T \subset [\overset{\langle n-1\rangle}{\varphi}_T = c]$.

Conversely, let $z \in [\overset{\langle n-1\rangle}{\varphi}_T = c]$. Then T is non-empty, $\overset{\langle n-1\rangle}{\varphi}_T \in \Gamma(Z)$, and, according to Lemma 1.5, $z \in [\overset{\langle n-1\rangle}{\psi}_T = c]$; whence $z \in [\overset{\langle n-1\rangle}{\gamma}_T = c] \subset [f = c] \subset \operatorname{Graph} T$.

(iii) $\Rightarrow$ (iv) Assume that n is finite. If $(x, y) \in \operatorname{Graph} T$ then we take $(x_i, y_i) = (x, y)$, $i \in \overline{1, n-1}$, for which

$$\langle x - x_1, y_1\rangle + \langle x_1 - x, y\rangle = \langle x - x_1, y_1\rangle + \sum_{i=2}^{n-1} \langle x_{i-1} - x_i, y_i\rangle + \langle x_{n-1} - x, y\rangle = 0.$$

If $(x, y) \notin \operatorname{Graph} T$ then, since T is maximal n-monotone, $\{(x, y)\} \cup \operatorname{Graph} T$ is not n-monotone. Hence, there is $\{(x_i, y_i)\}_{i \in \overline{1,n-1}} \subset \operatorname{Graph} T$ such that

$$\langle x - x_{n-1}, y\rangle + \langle x_{n-1} - x_{n-2}, y_{n-1}\rangle + \ldots + \langle x_2 - x_1, y_1\rangle + \langle x_1 - x, y_1\rangle < 0.$$

In both cases we get $\langle x - x_1, y_1\rangle + \sum_{i=2}^{n-1} \langle x_{i-1} - x_i, y_i\rangle + \langle x_{n-1} - x, y\rangle \geq 0.$

When $n = \infty$ and $(x, y) \notin \operatorname{Graph} T$, $\{(x, y)\} \cup \operatorname{Graph} T$ is not ∞-monotone. Therefore, for some $m \in \mathbb{N}$, $m \geq 2$, $\{(x, y)\} \cup \operatorname{Graph} T$ is not m-monotone and we proceed as above.

(iv) $\Rightarrow$ (iii) It suffices to prove that $\overset{\langle n-1\rangle}{\varphi}_T \geq c$.

For every $(x, y) \in X \times Y$ let $m \in \mathbb{N}, m \geq 2$ and $\{(x_i, y_i)\}_{i \in \overline{1,m-1}} \subset \operatorname{Graph} T$ be such that (52) holds, with $m = n$ when n is finite. According to $C_n(x, y)$, we have

$$\overset{\langle n-1\rangle}{\varphi}_T(x, y) \geq \overset{\langle m-1\rangle}{\varphi}_T(x, y) \geq \langle x - x_1, y_1\rangle + \sum_{i=2}^{m-1} \langle x_{i-1} - x_i, y_i\rangle + \langle x_{m-1}, y\rangle \geq \langle x, y\rangle. \quad \square$$

Remark 3.2. In the specific cases where $n \in \{2, \infty\}$,

$$T \text{ is } n\text{-representable iff } \exists f \in \Gamma(Z),\ c \leq f \leq \overset{\langle n-1\rangle}{\psi}_T,\ \operatorname{Graph} T = [f = c]; \tag{53}$$

because, in this case, according to Theorems 2.6 and Theorem 2.7, T is n-monotone iff $\overset{\langle n-1\rangle}{\psi}_T \geq \overset{\langle n-1\rangle}{\varphi}_T$ iff $\overset{\langle n-1\rangle}{\psi}_T \geq c$. In particular

$$\forall n \in \{2, \infty\},\ (T \text{ is } n\text{-representable } \Rightarrow\ T \text{ is } n\text{-monotone}). \tag{54}$$

Because $f \in \Gamma(Z)$, we can relax $f \leq \overset{\langle n-1\rangle}{\psi}_T$ in (53) to $f \leq \overset{\langle n-1\rangle}{)\!(}_T$, and $\operatorname{Graph} T = [f = c]$ to $\operatorname{Graph} T \supset [f = c]$ because from $c \leq f \leq \overset{\langle n-1\rangle}{\psi}_T \leq \overset{\langle n-1\rangle}{)\!(}_T \leq \overset{\langle 1\rangle}{)\!(}_T = c_T$ we know that $\operatorname{Graph} T \subset [f = c]$. More precisely, for all $n \in \{2, \infty\}$,

$$T \text{ is } n\text{-representable } \textit{iff } \exists f \in \Gamma(Z),\ c \leq f \leq \overset{\langle n-1\rangle}{)\!(}_T,\ \operatorname{Graph} T \supset [f = c]. \tag{55}$$

The 2-representability definition aligns with that in [10], i.e.,

$$T \text{ is 2-representable iff } \exists f \in \Gamma(Z),\ f \geq c,\ \operatorname{Graph} T = [f = c], \tag{56}$$

since, from $\operatorname{Graph} T \subset [f = c]$, we obtain $f \leq c_T = \overset{\langle 1\rangle}{)\!(}_T$. Consequently, the condition $f \leq \overset{\langle 1\rangle}{\psi}_T$ is redundant.

Note also that, for $n \in \overline{2, \infty}$

$$T \text{ is } \infty\text{-representable} \Rightarrow T \text{ is } n\text{-representable} \Rightarrow (49) \Rightarrow T \text{ is 2-representable.} \tag{57}$$

Indeed, the penultimate implication is plain, and the final implication follows from (56). For the first implication we assume that T is ∞-representable, i.e., we assume $\operatorname{Graph} T = [f = c]$, for some $f \in \Gamma(Z)$ with $c \leq f \leq \overset{\langle\infty\rangle}{\psi}_T$. Since T is ∞-monotone, for every $n \in \overline{2, \infty}$, we know that T is n-monotone and

$$\overset{\langle n-1\rangle}{\varphi}_T \leq \overset{\langle\infty\rangle}{\varphi}_T \leq \overset{\langle\infty\rangle}{\psi}_T \leq \overset{\langle n-1\rangle}{\psi}_T.$$

Therefore $f \leq \overset{\langle n-1\rangle}{\psi}_T = \max\{\overset{\langle n-1\rangle}{\varphi}_T, \overset{\langle n-1\rangle}{\psi}_T\}$ and so T is n-representable. □

Theorem 3.3. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system, let $n \in \{2, \infty\}$, and let $T : X \rightrightarrows Y$ be such that $\operatorname{Graph} T \neq \emptyset$.*
Then T is n-representable iff $\overset{\langle n-1\rangle}{\psi}_T \geq c$ and $\operatorname{Graph} T(\supset) = [\overset{\langle n-1\rangle}{\psi}_T = c]$ iff T is n-monotone and $\operatorname{Graph} T(\supset) = [\overset{\langle n-1\rangle}{\psi}_T = c]$.

Proof. The second equivalence is plain due to Theorems 2.6 and 2.7.
The converse implication in the first equivalence is plain; take $f = \overset{\langle n-1\rangle}{\psi}_T$ (see also (18)). Assume that T is n-representable, that is, $\operatorname{Graph} T \supset [f = c]$, for some $f \in \Gamma(Z)$ with $c \leq f \leq \overset{\langle n-1\rangle}{\psi}_T$.

Together with (18) we conclude that $\operatorname{Graph} T = [\overset{\langle n-1\rangle}{\psi}_T = c]$, since

$$[\overset{\langle n-1\rangle}{\psi}_T = c] \subset [f = c](\subset) = \operatorname{Graph} T \subset [\overset{\langle n-1\rangle}{\psi}_T \le c] = [\overset{\langle n-1\rangle}{\psi}_T = c]. \qquad \square$$

Proposition 3.4. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h : X \to \overline{\mathbb{R}}$. Then*

$$\forall (x, y) \in X \times Y,\ \overset{\langle\infty\rangle}{\between}_{\partial h}(x, y) \ge (h + \iota_{D(\partial h)})(x) + (h^* + \iota_{R(\partial h)})(y). \tag{58}$$

$$\forall (x, y) \in X \times Y,\ \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) \le (h^* + \iota_{R(\partial h)})^*(x) + (h + \iota_{D(\partial h)})^*(y). \tag{59}$$

If, in addition, $h \in \Gamma_{\sigma(X,Y)}(X)$ then ∂h is cyclically representable.

Proof. If $\operatorname{Graph}(\partial h) = \emptyset$ then $\overset{\langle\infty\rangle}{\between}_{\partial h} = +\infty$, $\overset{\langle\infty\rangle}{\varphi}_{\partial h} = -\infty$, and so (58), (59) are true.

Assume that $\operatorname{Graph}(\partial h) \ne \emptyset$. For every $n \in \mathbb{N}$, $n \ge 3$, every $(x, y) \in D(\partial h) \times R(\partial h)$, and every

$$x_1 \in (\partial h)^{-1}(y),\ y_n \in \partial h(x),\ \{(x_i, y_i)\}_{\in \overline{2,n-1}} \subset \operatorname{Graph}(\partial h),$$

we have
$$\begin{aligned}
&\langle x_1, y\rangle + \sum_{i=2}^{n-1} \langle x_i - x_{i-1}, y_i\rangle + \langle x - x_{n-1}, y_n\rangle \\
&\ge h(x_1) + h^*(y) + \sum_{i=2}^{n-1} [h(x_i) - h(x_{i-1})] + h(x) + h^*(y_n) - \langle x_{n-1}, y_n\rangle \\
&= h(x_1) + h^*(y) + [h(x_{n-1}) - h(x_1)] + h(x) + h^*(y_n) - \langle x_{n-1}, y_n\rangle \\
&= h(x) + h^*(y) + h(x_{n-1}) + h^*(y_n) - \langle x_{n-1}, y_n\rangle \ge h(x) + h^*(y).
\end{aligned}$$

Pass to the infimum over $x_1 \in (\partial h)^{-1}(y)$, $y_n \in \partial h(x)$, $\{(x_i, y_i)\}_{\in \overline{2,n-1}} \subset \operatorname{Graph}(\partial h)$ to get

$$\forall (x, y) \in D(\partial h) \times R(\partial h),\ \forall n \in \mathbb{N},\ n \ge 3,\ h(x) + h^*(y) \le \overset{\langle n\rangle}{\between}_{\partial h}(x, y),$$

$$\forall (x, y) \in D(\partial h) \times R(\partial h),\ h(x) + h^*(y) \le \inf_{n \ge 3} \overset{\langle n\rangle}{\between}_{\partial h}(x, y) = \overset{\langle\infty\rangle}{\between}_{\partial h}(x, y).$$

Since $\operatorname{dom}(\overset{\langle\infty\rangle}{\between}_{\partial h}) = D(\partial h) \times R(\partial h)$, (58) holds; (59) is obtained by convex conjugation.

If, in addition, $h \in \Gamma_{\sigma(X,Y)}(X)$, then let $f(x, y) := h(x) + h^*(y)$, $x \in X$, $y \in Y$. Clearly, $f \in \Gamma(Z)$, $\operatorname{Graph}(\partial h) = [f = c]$, and, according to (58), $c \le f \le \overset{\langle\infty\rangle}{\between}_{\partial h}$. Hence ∂h is ∞-representable (see (55)). $\square$

Theorem 3.5. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $T : X \rightrightarrows Y$ be cyclically monotone. The following statements are equivalent:*

(i) *T is maximal cyclically monotone,*

(ii) $T = \partial h$, *for some* $h \in \Gamma_{\sigma(X,Y)}(X)$, *and, for every* $(x, y) \in X \times Y$, *there are* $n \in \mathbb{N}$, $n \geq 2$, *and* $\{(x_i, y_i)\}_{i \in \overline{1,n-1}} \subset \operatorname{Graph} T$ *such that*

$$\langle x - x_1, y_1 \rangle + \sum_{i=2}^{n-1} \langle x_{i-1} - x_i, y_i \rangle + \langle x_{n-1} - x, y \rangle \geq 0, \tag{60}$$

where for $n = 2$ *the previous inequality reads* $\langle x - x_1, y_1 \rangle + \langle x_1 - x, y \rangle \geq 0$.

(iii) $T = \partial h$, *for some* $h \in \Gamma_{\sigma(X,Y)}(X)$, *and* $\overset{\langle\infty\rangle}{\varphi}_T \geq c$.

(iv) $T = \partial h$, *for some* $h \in \Gamma_{\sigma(X,Y)}(X)$, *and* T *admits a unique maximal cyclically monotone extension.*

Proof. (i) $\Rightarrow$ (iv) is plain; (i) $\Rightarrow$ (ii) is a consequence of Theorems 2.10, 3.1.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow from Proposition 3.4, Theorem 3.1.

(iv) $\Rightarrow$ (iii) It suffices to prove that $\overset{\langle\infty\rangle}{\varphi}_T \geq c$. Assume by contradiction that there exists $z_0 \in [\overset{\langle\infty\rangle}{\varphi}_T < c]$. Let $f(x, y) = h(x) + h^*(y)$, $(x, y) \in Z$. Then $f \in \Gamma(Z)$, $f = f^\square \geq c$, and $f \geq \overset{\langle\infty\rangle}{\varphi}_T$, since (58) provides $f = f^\square \leq \overset{\langle\infty\rangle}{)\!(}_T$.
Assume that M is the unique maximal ∞-monotone extension of $\operatorname{Graph} T$. According to Theorem 2.5 (i), every $z \in [\overset{\langle\infty\rangle}{\varphi}_T \leq c]$ is ∞-m.r. to T which is ∞-monotone. Hence $\operatorname{Graph} T \cup \{z\}$ is ∞-monotone. From Zorn's Lemma, $\operatorname{Graph} T \cup \{z\}$ admits a maximal ∞-monotone extension which is also a ∞-monotone extension of $\operatorname{Graph} T$, so, every such extension coincides with M. In particular $z \in M$. We proved that $[\overset{\langle\infty\rangle}{\varphi}_T \leq c] \subset M$. In particular $[\overset{\langle\infty\rangle}{\varphi}_T \leq c]$ is (2-)monotone.
We claim that $\operatorname{dom} f \subset \{z_0\}^+ := \{z \in Z \mid c(z - z_0) \geq 0\}$.

Indeed, if there is a $z \in \operatorname{dom} f$ ($\subset \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T$) such that $c(z - z_0) < 0$ then $z \notin [\overset{\langle\infty\rangle}{\varphi}_T \leq c]$, since $[\overset{\langle\infty\rangle}{\varphi}_T \leq c]$ is monotone and $z_0 \in [\overset{\langle\infty\rangle}{\varphi}_T \leq c]$. Hence $z \in [\overset{\langle\infty\rangle}{\varphi}_T > c] \cap \operatorname{dom} \overset{\langle\infty\rangle}{\varphi}_T$. From the continuity of $\alpha : [0, 1] \to \mathbb{R}$, $\alpha(t) := (\overset{\langle\infty\rangle}{\varphi}_T - c)(tz + (1 - t)z_0)$ together with $\alpha(0)\alpha(1) < 0$, we infer that there exists $s \in (0, 1) \cap [\alpha = 0]$, that means, $w := sz + (1 - s)z_0 \in [\overset{\langle\infty\rangle}{\varphi}_T = c]$. It follows that $c(w - z_0) = s^2 c(z - z_0) < 0$, which contradicts the monotonicity of $[\overset{\langle\infty\rangle}{\varphi}_T \leq c]$. Our claim is proved.
Therefore, for every $z \in \operatorname{dom} f$

$$f(z) \geq c(z) \geq c(z) - c(z - z_0) = z \cdot z_0 - c(z_0) \;\Rightarrow\; c(z_0) \geq z \cdot z_0 - f(z).$$

We find that $z_0 \in [f^\square = c] = \operatorname{Graph} T \subset [\overset{\langle\infty\rangle}{\varphi}_T = c]$ a clear contradiction. Hence $\overset{\langle\infty\rangle}{\varphi}_T \geq c$. $\square$

Remark 3.6. Note the power of the condition $T = \partial h$, for some $h \in \Gamma_{\sigma(X,Y)}(X)$. In its presence the argument in the proof of Theorem 3.5 actually shows that

$$\begin{aligned} &[\overset{\langle\infty\rangle}{\varphi}_T \leq c] \text{ monotone} \\ &\Rightarrow \operatorname{Graph} T = [\overset{\langle\infty\rangle}{\varphi}_T \leq c] = [\overset{\langle\infty\rangle}{\varphi}_T = c] \text{ is maximal } \infty - \text{monotone}. \end{aligned} \tag{61}$$

Example 3.7. Let $T : D(T) = [0, +\infty) \subset \mathbb{R} \to \mathbb{R}$, $T(x) = 0$. For every $n \in \overline{2, \infty}$, $x, y \in \mathbb{R}$

$$\overset{\langle n\rangle}{\varphi}_T(x,y) = \sup_{u\geq 0} uy = \iota_{(-\infty,0]}(y) := \begin{cases} 0, & y \leq 0 \\ +\infty, & y > 0 \end{cases},$$

$$\overset{\langle n\rangle}{\psi}_T(x,y) = (\overset{\langle n\rangle}{\varphi}_T)^{\Box}(x,y) = \sup_{v\leq 0, u\in\mathbb{R}} (xv + uy) = \begin{cases} \iota_{[0,+\infty)}(x), & y = 0 \\ +\infty, & y \neq 0 \end{cases},$$

T is ∞-representable, since $\overset{\langle\infty\rangle}{\psi}_T \geq c$ and $\operatorname{Graph} T = [\overset{\langle\infty\rangle}{\psi}_T = c]$, and T is not maximal ∞-monotone, because $\overset{\langle\infty\rangle}{\varphi}_T \not\geq c$. Consequently, T is neither a maximal monotone operator nor the subdifferential of any proper, convex, and lower semicontinuous function. □

Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h : X \to \overline{\mathbb{R}}$. To h we associate

$$\begin{aligned} h^{\cup} : X \to \overline{\mathbb{R}},\ h^{\cup}(x) :&= (h^* + \iota_{R(\partial h)})^*(x) = \sup\{\langle x, y\rangle - h^*(y) \mid y \in R(\partial h)\} \\ &= \sup\{\langle x - u, y\rangle + h(u) \mid (u, y) \in \operatorname{Graph}(\partial h)\}, \end{aligned} \tag{62}$$

$$\begin{aligned} h^{\circledast} : Y \to \overline{\mathbb{R}},\ h^{\circledast}(y) :&= (h + \iota_{D(\partial h)})^*(y) = \sup\{\langle x, y\rangle - h(x) \mid x \in D(\partial h)\} \\ &= \sup\{\langle x, y - v\rangle + h^*(v) \mid (x, v) \in \operatorname{Graph}(\partial h)\}, \end{aligned} \tag{63}$$

Proposition 3.8. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h : X \to \overline{\mathbb{R}}$. Then*

(i) $h^{\cup} \leq h^{**}$, $h^{\circledast} \leq h^*$; *for $x \in D(\partial h)$, $h^{\cup}(x) = h^{**}(x) = h(x) = h^{\circledast *}(x)$, and, for $y \in R(\partial h)$, $h^{\cup *}(y) = h^*(y) = h^{\circledast}(y)$,*

(ii) $h^{\cup}$ *is proper iff* $\operatorname{Graph}(\partial h) \neq \emptyset$ *iff* $h^{\circledast}$ *is proper. In this case $h^{\cup} \in \Gamma_{\sigma(X,Y)}(X)$, $h^{\circledast} \in \Gamma_{\sigma(Y,X)}(Y)$.*

(iii) *For every $n \in \mathbb{N}$, $n \geq 1$, $h^{n\cup} = h^{\cup}$, where*

$$\begin{aligned} h^{n\cup}(x) := \sup\{\langle x - x_1, y_1\rangle + \langle x_1 - x_2, y_2\rangle + \ldots + \langle x_{n-1} - x_n, y_n\rangle + h(x_n) \mid \\ \{(x_i, y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph}(\partial h)\}. \end{aligned} \tag{64}$$

(iv) $\operatorname{Graph}(\partial h) \subset \operatorname{Graph}(\partial h^{\cup}) \cap \operatorname{Graph}(\partial h^{\circledast})^{-1}$.

Here $h^{\circledast *} := (h^{\circledast})^*$, $h^{\cup *} := (h^{\cup})^*$.

Proof. (i) $h^{\cup} \leq h^{**}$ follows directly from $h^* + \iota_{R(\partial h)} \geq h^*$ or the definition of ∂h. Similarly, $h + \iota_{D(\partial h)} \geq h$ implies that $h^{\circledast} \leq h^*$.

If $x \in D(\partial h)$ then pick $u = x$ in the definition of $h^{\cup}(x)$ to get $h^{\cup}(x) = h(x)$. Together with $h^{\cup} \leq h^{**} \leq h^{\circledast *} = (h + \iota_{D(\partial h)})^{**} \leq h + \iota_{D(\partial h)}$ that yields $h^{\cup}(x) = h^{**}(x) = h(x) = h^{\circledast *}(x)$. In a similar fashion, for $y \in R(\partial h)$, $h^* \in \Gamma(Y)$, because $\operatorname{Graph}(\partial h)^{-1} \subset \operatorname{Graph}(\partial h^*)$, $h^{\circledast}(y) = h^*(y)$, after we take $v = y$ in the definition of $h^{\circledast}$, and $h^{\circledast} \leq h^* = h^{***} \leq h^{\cup *} = (h^* + \iota_{R(\partial h)})^{**} \leq h^* + \iota_{R(\partial h)}$ provides $h^{\cup *}(y) = h^*(y) = h^{\circledast}(y)$.

(ii) If $\operatorname{Graph}(\partial h) = \emptyset$ then $h^{\cup} = -\infty$, $h^{\circledast} = -\infty$. Conversely, if $\operatorname{Graph}(\partial h) \neq \emptyset$ then h is proper, $h^* \in \Gamma(Y)$. Using $h^{\cup} \leq h$ we see that $h^{\cup}$ is proper. Similarly, from $h^{\circledast} \leq h^*$ we get that $h^{\circledast}$ is proper.

(iii) Let $(u,y) \in \operatorname{Graph}(\partial h)$. If we pick $(x_i,y_i)=(u,y)$, $i \in \overline{1,n}$, in the definition of $h^{n\cup}$ and pass to supremum over $(u,y) \in \operatorname{Graph}(\partial h)$, then we obtain $h^{n\cup} \geq h^{\cup}$. Conversely, for every $\{(x_i,y_i)\}_{i\in\overline{1,n}} \subset \operatorname{Graph}(\partial h)$

$$\sum_{i=1}^{n-1}\langle x_i - x_{i+1}, y_{i+1}\rangle \leq \sum_{i=1}^{n-1} h(x_i)-h(x_{i+1}) = h(x_1)-h(x_n),$$

from which $h^{n\cup} \leq h^{\cup}$.

(iv) For every $(x,y) \in \operatorname{Graph}(\partial h)$, $w \in X$, we have $h^{\cup}(x)=h(x)$ and, from the definition of $h^{\cup}(w)$, $\langle w-x,y\rangle + h(x) \leq h^{\cup}(w)$. Hence $(x,y) \in \operatorname{Graph}(\partial h^{\cup})$.

Analogously, for every $(x,y) \in \operatorname{Graph}(\partial h)$, $\zeta \in Y$, we have $h^{\circledast}(\zeta)=h^*(\zeta)$ and, from the definition of $h^{\circledast}(\zeta)$, $\langle x, \zeta - y\rangle + h^*(y) \leq h^{\circledast}(\zeta)$. Hence $(y,x) \in \operatorname{Graph}(\partial h^{\circledast})$. □

Theorem 3.9. *Let $(X,Y,\langle\cdot,\cdot\rangle)$ be a dual system and let $h : X \to \overline{\mathbb{R}}$. For every $(x,y) \in X \times Y$, $(x_0,y_0) \in \operatorname{Graph}(\partial h)$*

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,y) = h^{\cup}(x) + h^{\circledast}(y),\ \ \overset{\langle\infty\rangle}{\psi}_{\partial h}(x,y) = h^{\circledast *}(x) + h^{\cup *}(y) \tag{65}$$

$$r_{\partial h}^{(x_0,y_0)}(x) = h^{\cup}(x) - h(x_0),\ \ r_{(\partial h)^{-1}}^{(y_0,x_0)}(x) = h^{\circledast}(y) - h^*(y_0). \tag{66}$$

Proof. If $\operatorname{Graph}(\partial h)=\emptyset$ then (65), (66) are trivially verified with

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h} = h^{\cup} = h^{\circledast} = r_{\partial h}^{(x_0,y_0)} = -\infty, \ \text{ and } \ \overset{\langle\infty\rangle}{\psi}_{\partial h} = h^{\circledast *} = h^{\cup *} = +\infty.$$

Assume that $\operatorname{Graph}(\partial h) \neq \emptyset$. According to Proposition 3.8 (ii), we have $h^{\cup} \in \Gamma(X)$ and $h^{\circledast} \in \Gamma(Y)$. Let us now prove that, for every $x \in X$, $y \in Y$,

$$\sup_{v_0 \in R(\partial h)} \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,v_0) - h^*(v_0) = h^{\cup}(x),\ \ \sup_{u_0 \in D(\partial h)} \overset{\langle\infty\rangle}{\varphi}_{\partial h}(u_0,y) - h(u_0) = h^{\circledast}(y). \tag{67}$$

According to (11), for every $x \in X$, $v_0 \in R(\partial h)$, $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,v_0) \geq \langle x, v_0\rangle$, and so $\sup_{v_0\in R(\partial h)} \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,v_0) - h^*(v_0) \geq \sup_{v_0 \in R(\partial h)}\langle x,v_0\rangle - h^*(v_0) = h^{\cup}(x)$. The converse inequality is a consequence of (59) and $h^* = h^{\circledast}$ on $R(\partial h)$, or $h^{\circledast} \leq h^*$. We proceed similarly for the second part in (67).

We use (48) for $x \in X$, $(x_0,v_0),(u_0,y_0) \in \operatorname{Graph}(\partial h) \subset [\overset{\langle\infty\rangle}{\varphi}_{\partial h} = c]$, and $y = y_0$ to get that, for every $x \in X$, $y_0 \in R(\partial h)$, $(x_0,v_0) \in \operatorname{Graph}(\partial h)$

$$\begin{aligned}\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,y_0) &\geq \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0,y_0) + \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,v_0) - \langle x_0,v_0\rangle \\ &= [\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,v_0) - h^*(v_0)] + [\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0,y_0) - h(x_0)].\end{aligned}$$

Pass to the supremum over $(x_0,v_0) \in \operatorname{Graph}(\partial h)$ to obtain, according to (67), (59), that, for every $x \in X$, $y_0 \in R(\partial h)$,

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,y_0) = h^{\cup}(x) + h^{\circledast}(y_0) = h^{\cup}(x) + h^*(y_0). \tag{68}$$

Similarly, for every $y \in Y$, $x_0 \in D(\partial h)$,

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0,y) = h(x_0) + h^{\circledast}(y). \tag{69}$$

For every $(x, y) \in X \times Y$, we use (48) for $(x_0, v_0) = (u_0, y_0) \in \mathrm{Graph}(\partial h)$ to get

$$\begin{aligned}
\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) - \langle x_0, y_0\rangle &= \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) - \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0, y_0) \\
&\geq \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, v_0) + \overset{\langle\infty\rangle}{\varphi}_{\partial h}(u_0, y) - \langle x_0, v_0\rangle - \langle u_0, y_0\rangle \\
&= h^{\cup}(x) + h^*(v_0) + h(u_0) + h^{\circledast}(y) - \langle x_0, v_0\rangle - \langle u_0, y_0\rangle \\
&= h^{\cup}(x) + h^{\circledast}(y) - h(x_0) - h^*(y_0),
\end{aligned}$$

so $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) \geq h^{\cup}(x) + h^{\circledast}(y)$.

The first equality in (65) follows from (59). The second equality in (65) is obtained by convex conjugation from the first part. For (66) we use (27), (65), (12), and $h^{\cup} = h$ on $D(\partial h)$, $h^{\circledast} = h^*$ on $R(\partial h)$. □

Corollary 3.10. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h : X \to \overline{\mathbb{R}}$. Then*

$$\forall x_0 \in D(\partial h),\ \ h^{\cup}(x) = \min\left\{\alpha(x) \,\middle|\, \begin{array}{c} \alpha : X \to \overline{\mathbb{R}},\ \alpha(x_0) = h(x_0), \\ \mathrm{Graph}(\partial\alpha) \subset \mathrm{Graph}(\partial h) \end{array}\right\}, \tag{70}$$

$$\forall y_0 \in R(\partial h),\ \ h^{\circledast}(y) = \min\left\{\beta(y) \,\middle|\, \begin{array}{c} \beta : Y \to \overline{\mathbb{R}},\ \beta(y_0) = h^*(y_0), \\ \mathrm{Graph}(\partial\beta) \subset \mathrm{Graph}(\partial h)^{-1} \end{array}\right\}. \tag{71}$$

Proof. Relations (70), (71) are translations of (25) with the aid of (66). □

Corollary 3.11. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h, g \in \Gamma_{\sigma(X,Y)}(X)$. Then $\partial h = \partial g$ iff $\overset{\langle\infty\rangle}{\varphi}_{\partial h} = \overset{\langle\infty\rangle}{\varphi}_{\partial g}$ iff $h^{\cup} - g^{\cup} = h^{\circledast *} - g^{\circledast *}$ is a constant function.*

Proof. The conclusion is straightforward if $\mathrm{Graph}(\partial h) = \emptyset$ or $\mathrm{Graph}(\partial g) = \emptyset$; in which case $\overset{\langle\infty\rangle}{\varphi}_{\partial h} = \overset{\langle\infty\rangle}{\varphi}_{\partial g} = -\infty$ and $h^{\cup} - g^{\cup} = h^{\circledast *} - g^{\circledast *} = +\infty$.

Assume that $\mathrm{Graph}(\partial h) \neq \emptyset$ and $\mathrm{Graph}(\partial g) \neq \emptyset$; this implies $h^{\cup}, g^{\cup} \in \Gamma(X)$, $h^{\circledast}, g^{\circledast} \in \Gamma(Y)$.

If $\partial h = \partial g$ then $\overset{\langle\infty\rangle}{\varphi}_{\partial h} = \overset{\langle\infty\rangle}{\varphi}_{\partial g}$. From (65) we get that $h^{\cup} = g^{\cup} + k$, where $k = g^{\circledast}(y_0) - h^{\circledast}(y_0) = g^*(y_0) - h^*(y_0) \in \mathbb{R}$, for any fixed $y_0 \in R(\partial h) = R(\partial g)$. Similarly, $h^{\circledast} = g^{\circledast} + h$, for some $h \in \mathbb{R}$. Again $\overset{\langle\infty\rangle}{\varphi}_{\partial h} = \overset{\langle\infty\rangle}{\varphi}_{\partial g}$ shows that $h = -k$, and so, $h^{\circledast *} - g^{\circledast *} = k$.

Conversely, if $h^{\cup} - g^{\cup} = h^{\circledast *} - g^{\circledast *} = k \in \mathbb{R}$ then $h^{\circledast} = g^{\circledast} - k$, $\overset{\langle\infty\rangle}{\varphi}_{\partial h} = \overset{\langle\infty\rangle}{\varphi}_{\partial g}$, followed by $\overset{\langle\infty\rangle}{\psi}_{\partial h} = \overset{\langle\infty\rangle}{\psi}_{\partial g}$. According to Proposition 3.4, ∂h, ∂g are ∞-representable. From Theorem 3.3, we find that

$$\mathrm{Graph}(\partial h) = [\overset{\langle\infty\rangle}{\psi}_{\partial h} = c] = [\overset{\langle\infty\rangle}{\psi}_{\partial g} = c] = \mathrm{Graph}(\partial g).$$

□

Remark 3.12. Whenever $h : X \to \overline{\mathbb{R}}$ has $\mathrm{Graph}(\partial h) \neq \emptyset$ we have

$$h^{**} = h^{\cup},\ h^{*} = h^{\circledast} \ \Leftrightarrow\ h^{\cup} = h^{\circledast *} \ \Leftrightarrow\ h^{\cup *} = h^{\circledast}. \tag{72}$$

Indeed, for the first equivalence, if $h^{**} = h^{\cup}$ and $h^{*} = h^{\circledast}$ then, by conjugation, $h^{\cup} = h^{\circledast *}$. Conversely, if $h^{\cup} = h^{\circledast *}$, we know from $h^{\cup} \le h^{**} \le h^{\circledast *}$ that $h^{**} = h^{\cup} = h^{\circledast *}$ and $h^{*} = h^{\circledast}$, since $h^{*}, h^{\circledast} \in \Gamma(Y)$. The second equivalence follows again from the biconjugate formula applied for $h^{\cup} \in \Gamma(X)$ and $h^{\circledast} \in \Gamma(Y)$.

Theorem 3.13. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h \in \Gamma_{\sigma(X,Y)}(X)$. The following statements are equivalent:*

(i) ∂h *is maximal cyclically monotone,*

(ii) $h^{\cup} = h^{\circledast *}$,

(iii) $\forall (x, y) \in X \times Y$, $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) = h(x) + h^{*}(y)$,

(iv) $\forall (x, y) \in X \times Y$, $\forall (x_0, v_0) \in \mathrm{Graph}(\partial h)$,

$$r_{\partial h}^{(x_0,y_0)}(x) = h(x) - h(x_0),\ r_{\partial h^{*}}^{(y_0,x_0)}(y) = h^{*}(y) - h^{*}(y_0),$$

(v) $\forall (x, y) \in X \times Y$, $\forall x_0 \in D(\partial h)$, $\forall y_0 \in R(\partial h)$,

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y_0) = h(x) + h^{*}(y_0),\ \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0, y) = h(x_0) + h^{*}(y),$$

(vi) $\forall (x, y) \in X \times Y$, $\exists (x_0, y_0) \in \mathrm{Graph}(\partial h)$

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y_0) = h(x) + h^{*}(y_0),\ \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0, y) = h(x_0) + h^{*}(y).$$

Proof. (i) $\Rightarrow$ (ii) Since ∂h is maximal ∞-monotone, we know that $\mathrm{Graph}(\partial h) \neq \emptyset$, and so, $h^{\cup} \in \Gamma(X)$, $h^{\circledast} \in \Gamma(Y)$.

For every $x \in X$ and $y = 0$ we use Theorem 3.5 to obtain $n \in \mathbb{N}$, $n \ge 2$ and $\{(x_i, y_i)\}_{i \in \overline{1,n-1}} \subset \mathrm{Graph}(\partial h)$ such that

$$\langle x - x_1, y_1\rangle + \sum_{i=2}^{n-1} \langle x_{i-1} - x_i, y_i\rangle \ge 0,$$

where for $n = 2$ the previous inequality reads $\langle x - x_1, y_1\rangle \ge 0$.

Therefore, for every $x \in X$

$$\begin{aligned} h(x) \ge h^{\cup}(x) = h^{(n-1)\cup}(x) &\ge \langle x - x_1, y_1\rangle + \sum_{i=2}^{n-1} \langle x_{i-1} - x_i, y_i\rangle + h(x_{n-1}) \\ &\ge h(x_{n-1}) \ge \inf_{D(\partial h)} h \ge \inf_X h, \end{aligned}$$

and so, $\inf_X h = \inf_{D(\partial h)} h = \inf_X h^{\cup}$ or $h^{\circledast}(0) = h^{\cup *}(0)$.

But ∂h maximal ∞-monotone implies (is equivalent to) that $\partial(h - y)$ is maximal ∞-monotone, for every (some) $y \in Y$; where $(h - y)(x) := h(x) - \langle x, y\rangle$, $x \in X$, because $\mathrm{Graph}(\partial(h - y)) = \mathrm{Graph}(\partial h) - (0, y)$. Hence we have for every $y \in Y$, $(h - y)^{\cup *}(0) = (h - y)^{\circledast}(0)$.

Note that, for $x \in X$, $y, v \in Y$, $(h-y)^{\cup}(x) = h^{\cup}(x) - \langle x, y\rangle$, $(h-y)^{\cup *}(v) = h^{\cup *}(y+v)$, $(h - y)^{\circledast}(v) = h^{\circledast}(y + v)$. Set $v = 0$ to get $h^{\cup *} = h^{\circledast}$ and, by conjugation, $h^{\cup} = h^{\circledast *}$.

For an alternative argument for this implication, we use Theorem 3.5 and (73) below.

(ii) $\Rightarrow$ (iii) follows from (65) (see also Remark 3.12).

(iii) $\Rightarrow$ (i) is a direct consequence of Proposition 3.4, and Theorem 3.1 or Theorem 3.5.

(iv) and (v) are equivalent due to (27), $\partial h^* = (\partial h)^{-1}$, and (12).

(iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) are plain.

(vi) $\Rightarrow$ (iii) For every $(x, y) \in X \times Y$, let $(x_0, y_0) \in \operatorname{Graph}(\partial h)$ be such that $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y_0) = h(x) + h^*(y_0)$, $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0, y) = h(x_0) + h^*(y)$.

We use (48) for $(x_0, v_0) = (u_0, y_0) \in \operatorname{Graph}(\partial h)$ to get

$$\begin{aligned}\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) - \langle x_0, y_0\rangle &= \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) - \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x_0, y_0)\\ &\geq \overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, v_0) + \overset{\langle\infty\rangle}{\varphi}_{\partial h}(u_0, y) - \langle x_0, v_0\rangle - \langle u_0, y_0\rangle\\ &= h(x) + h^*(v_0) + h(u_0) + h^*(y) - \langle x_0, v_0\rangle - \langle u_0, y_0\rangle\\ &= h(x) + h^*(y) - h(x_0) - h^*(y_0),\end{aligned}$$

so $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) \geq h(x) + h^*(y)$.

Together with (65) and $h^{\cup} \leq h$, $h^{\circledast} \leq h^*$ we find $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x, y) = h(x) + h^*(y)$ and (ii). $\square$

Corollary 3.14. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $h, g \in \Gamma_{\sigma(X,Y)}(X)$ be such that ∂h is maximal cyclically monotone. Then $\partial h (\subset) = \partial g$ iff $h - g$ is a constant function.*

Proof. The converse is plain. For the direct implication, given that $\partial h = \partial g$ are both maximal cyclically monotone, from Theorem 3.13 and Proposition 3.8 (i) or Remark 3.12, we infer that $h^{\cup} = h^{\circledast *} = h$, $g^{\cup} = g^{\circledast *} = g$, and use Corollary 3.11 to conclude. $\square$

Remark 3.15. Given a dual system $(X, Y, \langle\cdot,\cdot\rangle)$, $T : X \rightrightarrows Y$ is maximal cyclically monotone iff $T = \partial h$, for some $h \in \Gamma_{\sigma(X,Y)}(X)$ such that $(h^* + \iota_{R(\partial h)})^* = (h + \iota_{D(\partial h)})^{**}$ (see Theorems 2.10, 3.13). $\square$

We end this section with some conclusive remarks concerning the maximal cyclic monotonicity of ∂h, where $h : X \to \overline{\mathbb{R}}$ is a general function.

If ∂h is maximal cyclically monotone then, since $\operatorname{Graph}(\partial h) \subset \operatorname{Graph}(\partial h^{**})$, we infer that $h^{**} \in \Gamma_{\sigma(X,Y)}(X)$, $h^* \in \Gamma_{\sigma(Y,X)}(Y)$, and $\partial h = \partial h^{**}$ is maximal cyclically monotone. According to Theorem 3.13, $h^{**\cup} = h^{**\circledast *} = h^{**}$. The converse is not true, since one can find $h : X \to \overline{\mathbb{R}}$ such that $\operatorname{Graph}(\partial h) \subsetneq \operatorname{Graph}(\partial h^{**})$ and ∂h^{**} is maximal (cyclically) monotone, e.g., when X is a Banach space and $h = \iota_C$, where $C \subsetneq X$ is open convex.

Note that, in all of the arguments above, the condition $h \in \Gamma_{\sigma(X,Y)}(X)$ is essential only in establishing the cyclical representability of ∂h.

Similar computations show that, in general, for $h : X \to \overline{\mathbb{R}}$

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h} \geq c \ \Leftrightarrow \ h^{\cup}(\geq) = h^{\circledast *}(= h^{**}) \ \Rightarrow \ \partial h^{**} \textit{ is maximal } \infty - \textit{monotone}. \tag{73}$$

Indeed, from (65), we know that $\overset{\langle\infty\rangle}{\varphi}_{\partial h}(x,y) = h^{\cup}(x) + h^{\circledast}(y)$, $x \in X$, $y \in Y$. Therefore $\overset{\langle\infty\rangle}{\varphi}_{\partial h} \geq c$ is equivalent to $h^{\cup} \geq h^{\circledast *}$. However, $h^{\cup} \leq h^{**} \leq h^{\circledast *}$.

If $\overset{\langle\infty\rangle}{\varphi}_{\partial h} \geq c$, then by Theorem 3.1, any cyclically representable extension of ∂h, such as ∂h^{**}, is maximal cyclically monotone.

Similarly, for $h, g : X \to \overline{\mathbb{R}}$:

$$\overset{\langle\infty\rangle}{\varphi}_{\partial h} = \overset{\langle\infty\rangle}{\varphi}_{\partial g} \tag{74}$$
$$\Leftrightarrow\ (h^* + \iota_{R(\partial h)})^* - (g^* + \iota_{R(\partial g)})^* = (h + \iota_{D(\partial h)})^{**} - (g + \iota_{D(\partial g)})^{**} \ \textit{is constant}.$$

4. Examples

Example 4.1. (Monotone subset of $\mathbb{R}^2$) If $T : \mathbb{R} \rightrightarrows \mathbb{R}$ is monotone, then T is cyclically monotone. Consequently, T is maximal monotone iff T is maximal cyclically monotone.

Proof. We prove by induction over $n \in \mathbb{N}$, $n \geq 2$ that T is n-monotone. Note that the base case is provided. Assume that T is n-monotone, for some $n \in \mathbb{N}$, $n \geq 2$.

Let $\{(x_i, y_i)\}_{i\in\overline{1,n+1}} \subset \operatorname{Graph} T$ be arbitrarily chosen with $x_{n+2} = x_1$. Pick $k \in \overline{1, n+1}$ such that $x_k = \min_{i\in\overline{1,n+1}} x_i$ and define $(x'_j, y'_j) = (x_{(j+k-1) \bmod n}, y_{(j+k-1) \bmod n})$ for $j \in \overline{1, n+1}$. Then $\{(x'_j, y'_j)\}_{j\in\overline{1,n+1}} \subset \operatorname{Graph} T$, $x'_1 = \min_{j\in\overline{1,n+1}} x'_j$, and, because T is monotone, $y'_1 = \min_{j\in\overline{1,n+1}} y'_j$. We have $\sum_{j=2}^{n}(x'_j - x'_{j+1})y'_j + (x'_{n+1} - x'_2)y'_{n+1} \geq 0$, since T is n-monotone, and so

$$\begin{aligned}\sum_{i=1}^{n+1}(x_i - x_{i+1})y_i &= \sum_{j=1}^{n+1}(x'_j - x'_{j+1})y'_j \\ &= (x'_1 - x'_2)(y'_1 - y'_{n+1}) + \sum_{j=2}^{n}(x'_j - x'_{j+1})y'_j + (x'_{n+1} - x'_2)y'_{n+1} \geq 0,\end{aligned}$$

because $x'_1 \leq x'_2$, $y'_1 \leq y'_{n+1}$. Therefore T is $(n+1)$-monotone, and so, T is ∞-monotone.

If T is maximal monotone then, since it is ∞-monotone, T is maximal ∞-monotone. Conversely, if T is maximal ∞-monotone then, for some $h \in \Gamma(\mathbb{R})$, $T = \partial h$ is maximal monotone, since $\mathbb{R}$ is a Banach space. □

Theorem 4.2. *Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system, let $T : X \rightrightarrows Y$, and let $n \in \overline{1, \infty}$. Then*

$$\begin{aligned}\overset{\langle n+1\rangle}{\curlyvee}_T(x,y) &= \inf_{u\in T^{-1}(y), v\in R(T)} \left\{\overset{\langle n\rangle}{\curlyvee}_T(x,v) + \langle u, y - v\rangle\right\} \\ &= \inf_{u\in D(T), v\in T(x)} \left\{\overset{\langle n\rangle}{\curlyvee}_T(u,y) + \langle x - u, v\rangle\right\};\end{aligned} \tag{75}$$

$$\begin{aligned}\overset{\langle n+1\rangle}{\varphi}_T(x,y) &= \sup_{(u,v)\in \operatorname{Graph} T} \left\{\overset{\langle n\rangle}{\varphi}_T(x,v) + \langle u, y - v\rangle\right\} \\ &= \sup_{(u,v)\in \operatorname{Graph} T} \left\{\overset{\langle n\rangle}{\varphi}_T(u,y) + \langle x - u, v\rangle\right\}.\end{aligned} \tag{76}$$

Proof. For $n = 1$, (75) is directly verified. For $n = 2$

$$\begin{aligned}
\overset{\langle 3\rangle}{\rotatebox{90}{ψ}}_T(x,y) &= \inf_{\substack{x_1\in T^{-1}(y),\ y_3\in T(x),\\ (x_2,y_2)\in \operatorname{Graph} T}} \{\langle x_1,y\rangle + \langle x_2 - x_1, y_2\rangle + \langle x - x_2, y_3\rangle\} \\
&= \inf_{\substack{x_2\in D(T),\\ y_3\in T(x)}} \Big(\inf_{\substack{x_1\in T^{-1}(y),\\ y_2\in T(x_2)}} \{\langle x_1,y\rangle + \langle x_2 - x_1, y_2\rangle\} + \langle x - x_2, y_3\rangle \Big) \\
&= \inf_{\substack{x_2\in D(T),\\ y_3\in T(x)}} \Big\{ \overset{\langle 2\rangle}{\rotatebox{90}{ψ}}_T(x_2,y) + \langle x - x_2, y_3\rangle \Big\}.
\end{aligned}$$

For $n \in \mathbb{N}$, $n \geq 3$ the second equality in (75) holds analogously, as follows

$$\overset{\langle n+1\rangle}{\rotatebox{90}{ψ}}_T(x,y) = \inf \left\{ \begin{array}{l} \langle x_1,y\rangle + \sum_{i=2}^{n} \langle x_i - x_{i-1}, y_i\rangle + \langle x - x_n, y_{n+1}\rangle \mid \\ x_1 \in T^{-1}(y),\ y_{n+1} \in T(x),\ (x_i,y_i)\}_{i\in\overline{2,n}} \subset \operatorname{Graph} T \end{array} \right\}$$

$$= \inf_{\substack{x_n\in D(T),\\ y_{n+1}\in T(x)}} \left[\inf \left\{ \begin{array}{l} \langle x_1,y\rangle + \sum_{i=2}^{n-1} \langle x_i - x_{i-1}, y_i\rangle + \langle x_n - x_{n-1}, y_n\rangle \mid \\ x_1 \in T^{-1}(y),\ y_n \in T(x_n),\ (x_i,y_i)\}_{i\in\overline{2,n-1}} \subset \operatorname{Graph} T \end{array} \right\} + \langle x - x_n, y_{n+1}\rangle \right]$$

$$= \inf_{\substack{x_n\in D(T),\\ y_{n+1}\in T(x)}} \Big\{ \overset{\langle n\rangle}{\rotatebox{90}{ψ}}_T(x_n,y) + \langle x - x_n, y_{n+1}\rangle \Big\}.$$

The equalities in (76) and the first equality in (75) follow similar reasoning. Alternatively, in both (75) and (76), we first prove one equality, and then use it for T replaced with T^{-1}, which then translates into the other equality for T. Also, (75) implies (76) via convex conjugation.

If we pass to infimum over $n \geq 1$ in (75) we obtain the corresponding (75) for $n = \infty$; similarly with supremum for (76). $\square$

In what follows, we apply mathematical induction and the recurrence relations in (75), (76) to derive closed forms for the n-th Fitzpatrick function $\overset{\langle n\rangle}{\varphi}_T$ and $\overset{\langle n\rangle}{\rotatebox{90}{ψ}}_T$. These closed forms are then employed to verify or refute (maximal) n-monotonicity with $\overset{\langle n\rangle}{\rotatebox{90}{ψ}}_T$ being more convenient for establishing monotonicity and $\overset{\langle n\rangle}{\varphi}_T$ for demonstrating maximality.

Example 4.3. (Normal cone) Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $C \subset X$. Then

$$\begin{aligned}
&\overset{\langle n\rangle}{\varphi}_{N_C}(x,y) = \iota_{C^\#}(x) + \sigma_C(y),\ x \in X,\ y \in Y,\ n \in \overline{1,\infty} \\
&\overset{\langle n\rangle}{\psi}_{N_C}(x,y) = \iota_{\operatorname{cl}\operatorname{conv} C}(x) + \sigma_{C^\#}(y),\ x \in X,\ y \in Y,\ n \in \overline{1,\infty}; \qquad (77)
\end{aligned}$$

$$\overset{\langle n\rangle}{\curlyvee}_{N_C}(x,y) = \iota_C(x) + \sigma_C(y) + \iota_{R(N_C)}(y)$$
$$= \begin{cases} \iota_C(x) + \sigma_C(y), & y \in R(N_C) \\ +\infty, & otherwise \end{cases}, \ x \in X,\ y \in Y,\ n \in \overline{2,\infty}, \tag{78}$$

where, for $A \subset X$, $\sigma_A(y) := \sup_{u\in A}\langle u, y\rangle$, and

$$C^{\#} := \{x \in X \mid \forall (u,v) \in \operatorname{Graph} N_C,\ \langle x-u, v\rangle \le 0\}$$

is the portable hull of C which is the intersection of all the supporting half-spaces that contain C and are supported at points in C (see [15]).

As a consequence, N_C is maximal cyclically monotone iff $C = C^{\#}$ iff N_C is maximal monotone.

Proof. If $C = \emptyset$ then
$\overset{\langle n\rangle}{\varphi}_{N_C} = \sigma_C = -\infty$, $C^{\#} = X$, $\iota_{C^{\#}} = 0$, $\overset{\langle n\rangle}{\psi}_{N_C} = \iota_{\operatorname{cl}\operatorname{conv} C} = \overset{\langle n\rangle}{\curlyvee}_{N_C} = +\infty$, $\sigma_{C^{\#}} = \iota_{\{0\}}$;
whence (77), (78) hold due to the convention $\infty - \infty = \infty$. Assume that $C \ne \emptyset$.

For $n \in \mathbb{N}$, $n \ge 2$, and $(x,y) \in X \times Y$

$$\begin{aligned}\overset{\langle n\rangle}{\varphi}_{N_C}(x,y) &= \sup_{\{(x_i,y_i)\}_{i\in\overline{1,n}}\subset \operatorname{Graph} N_C} \{\langle x-x_1, y_1\rangle + \sum_{i=2}^{n}\langle x_{i-1}-x_i, y_i\rangle + \langle x_n, y\rangle\} \\ &\le \sup_{(x_1,y_1)\in \operatorname{Graph} N_C}\{\langle x-x_1,y_1\rangle\} + \sup_{x_n\in C}\langle x_n, y\rangle = \iota_{C^{\#}}(x) + \sigma_C(y);\end{aligned}$$

the inequality is verified similarly for $n = 1$ and, as a consequence, holds for $n = \infty$.

For every $n \in \overline{1,\infty}$, $(x,y) \in \operatorname{dom} \overset{\langle n\rangle}{\varphi}_{N_C} \subset \operatorname{dom} \overset{\langle 1\rangle}{\varphi}_{N_C}$ we have

$$\forall a \in C,\ n^* \in N_C(a),\ \langle x-a, n^*\rangle + \langle a, y\rangle \le \overset{\langle 1\rangle}{\varphi}_{N_C}(x,y) < +\infty.$$

Since, for every $a \in C$, $N_C(a)$ is a cone, we find that $x \in C^{\#}$. Because, for every $a \in C$, $0 \in N_C(a)$ we infer that $\overset{\langle n\rangle}{\varphi}_{N_C}(x,y) \ge \overset{\langle 1\rangle}{\varphi}_{N_C}(x,y) \ge \sigma_C(y)$. Hence we obtain $\overset{\langle n\rangle}{\varphi}_{N_C}(x,y) \ge \iota_{C^{\#}}(x) + \sigma_C(y)$. Therefore the first part in (77) holds. The second part in (77) follows by convex conjugation.

To prove (78) we use induction over $n \ge 2$. According to (14),

$$\overset{\langle 2\rangle}{\curlyvee}_{N_C}(x,y) = \inf_{x_1\in (N_C)^{-1}(y),\ y_2\in N_C(x)}\{\langle x_1, y\rangle + \langle x-x_1, y_2\rangle\}.$$

To complete the basis step, it suffices to prove that $\overset{\langle 2\rangle}{\curlyvee}_{N_C}(x,y) = \sigma_C(y)$ when $x \in C$, $y \in R(N_C)$. To this end note that, when $x \in C$, for every $y_2 \in N_C(x)$, $\langle x-x_1, y_2\rangle \ge 0$, because $x_1 \in C$. Therefore, since $0 \in N_C(x)$, $\inf_{y_2\in N_C(x)}\langle x-x_1, y_2\rangle = 0$ and so $\overset{\langle 2\rangle}{\curlyvee}_{N_C}(x,y) = \inf_{x_1\in (N_C)^{-1}(y)}\langle x_1, y\rangle = \sigma_C(y)$.
Assume that $\overset{\langle n\rangle}{\curlyvee}_{N_C}(x,y) = \iota_C(x) + \sigma_C(y) + \iota_{R(N_C)}(y)$, for some integer $n \ge 2$. From (75)

$$\overset{\langle n+1\rangle}{\curlyvee}_{N_C}(x,y) = \sigma_C(y) + \iota_{R(N_C)}(y) + \inf_{u\in C, v\in N_C(x)}\langle x-u, v\rangle = \iota_C(x) + \sigma_C(y) + \iota_{R(N_C)}(y),$$

since, for $x \in C$, $\inf_{u\in C, v\in N_C(x)}\{\langle x-u, v\rangle\} = 0$. The inductive step is complete.

Hence (78) holds for every $n \in \mathbb{N}$, $n \geq 2$ and also for $n = \infty$.

Assume that N_C is maximal cyclically monotone.

Since $\operatorname{Graph}(N_C) \subset \operatorname{Graph}(N_{\operatorname{cl}\operatorname{conv} C})$, we know that C is closed convex. According to Theorem 3.13,

$$\iota_{C^\#} = \iota_C^{\cup} = \iota_C^{\circledast *} = \iota_C^{**} = \iota_C,$$

that is, $C^\# = C$. Hence, for every $x \in X$, $y \in Y$, we have

$$\overset{\langle 1\rangle}{\varphi}_{N_C}(x,y) = \iota_C(x) + \sigma_C(y) \geq \langle x, y\rangle.$$

We conclude that N_C is maximal monotone with the aid of Theorem 3.1 (ii). $\square$

Lemma 4.4. *Let $(H,\cdot)$ be a Hilbert space with induced norm $\|\cdot\|$. For every $v \in H$,*

$$\sup_{u\in H}\{\alpha\|u\|^2 \pm u\cdot v\} = \begin{cases} -\frac{1}{4\alpha}\|v\|^2, & \alpha < 0 \\ \iota_{\{0\}}(v), & \alpha = 0 \\ \infty, & \alpha > 0 \end{cases},$$

$$\inf_{u\in H}\{\alpha\|u\|^2 \pm u\cdot v\} = \begin{cases} -\frac{1}{4\alpha}\|v\|^2, & \alpha > 0 \\ -\iota_{\{0\}}(v), & \alpha = 0 \\ -\infty, & \alpha < 0 \end{cases}. \tag{79}$$

Proof. Note that, for $a \neq 0$, $u, v \in H$, we have

$$a\|u\|^2 \pm u\cdot v = a\|u \pm \tfrac{1}{2a}v\|^2 - \tfrac{1}{4a}\|v\|^2. \qquad \square$$

Example 4.5. (Identity) Let $(H,\cdot)$ be a Hilbert space with induced norm $\|\cdot\|$ and let $I : H \to H$, $I(x) = x$. For every $x, y \in H$

$$\overset{\langle n\rangle}{\varphi}_I(x,y) = \tfrac{n}{2(n+1)}\|x-y\|^2 + x\cdot y,\ n \geq 1; \tag{80}$$

$$\overset{\langle 1\rangle}{\psi}_I(x,y) = \overset{\langle 1\rangle}{)\!\!-\!\!(}_I = c_I,\ \overset{\langle n\rangle}{\psi}_I(x,y) = \overset{\langle n\rangle}{)\!\!-\!\!(}_I(x,y) = \tfrac{n}{2(n-1)}\|x-y\|^2 + x\cdot y,\ n \geq 2; \tag{81}$$

$$\overset{\langle \infty\rangle}{\varphi}_I(x,y) = \overset{\langle \infty\rangle}{)\!\!-\!\!(}_I(x,y) = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2; \tag{82}$$

and I is cyclically and maximal monotone.

Proof. For (80) we use induction on $n \geq 1$. From (79), for every $x, y \in H$

$$\overset{\langle 1\rangle}{\varphi}_I(x,y) = \sup_{u\in H}\{(x-u)\cdot u + u\cdot y\} = \sup_{u\in H}\{-\|u\|^2 + u\cdot(x+y)\} = \tfrac{1}{4}\|x+y\|^2 = \tfrac{1}{4}\|x-y\|^2 + x\cdot y.$$

Assume that, for every $x, y \in H$, $\overset{\langle n\rangle}{\varphi}_I(x,y) = \frac{n}{2(n+1)}\|x-y\|^2 + x\cdot y$, for some finite $n \geq 1$. From (76), (79) we get

$$\begin{aligned}
\overset{\langle n+1\rangle}{\varphi}_I(x,y) &= \sup_{w\in H}\left\{\frac{n}{2(n+1)}\|w-y\|^2 + w\cdot y + (x-w)\cdot w\right\} \\
&= \sup_{u\in H}\left\{\frac{n}{2(n+1)}\|u\|^2 + (y+u)\cdot y + (x-y-u)\cdot(y+u)\right\} \\
&= \sup_{u\in H}\left\{-\frac{n+2}{2(n+1)}\|u\|^2 + u\cdot(x-y)\right\} + x\cdot y = \frac{n+1}{2(n+2)}\|x-y\|^2 + x\cdot y.
\end{aligned}$$

This completes the inductive step and proves (80).

We proceed similarly to prove, by induction on $n \geq 2$, that

$$\overset{\langle n\rangle}{\asymp}_I(x,y) = \frac{n}{2(n-1)}\|x-y\|^2 + x\cdot y. \tag{83}$$

By a direct computation $\overset{\langle 2\rangle}{\asymp}_I(x,y) = \|x\|^2 + \|y\|^2 - x\cdot y = \|x-y\|^2 + x\cdot y$.

Assume that (83) is true for some finite integer $n \geq 2$. Use (75), (79) to get

$$\begin{aligned}\overset{\langle n+1\rangle}{\asymp}_I(x,y) &= \inf_{w\in H}\left\{\frac{n}{2(n-1)}\|w-y\|^2 + w\cdot(y-x) + \|x\|^2\right\}\\ &= \inf_{u\in H}\left\{\frac{n}{2(n-1)}\|u\|^2 + u\cdot(y-x)\right\} + \|x-y\|^2 + x\cdot y\\ &= -\frac{n-1}{2n}\|x-y\|^2 + \|x-y\|^2 + x\cdot y = \frac{n+1}{2n}\|x-y\|^2 + x\cdot y.\end{aligned}$$

Note that $\overset{\langle n\rangle}{\asymp}_I \in \Gamma(H)$, e.g. because its Gâteaux derivative

$$\nabla\overset{\langle n\rangle}{\asymp}_I(x,y) = \frac{1}{(n-1)}(nx-y, ny-x),\ x,y\in H,$$

is linear and monotone with

$$\langle \nabla\overset{\langle n\rangle}{\asymp}_I(x,y),(x,y)\rangle_{H\times H} = \|x\|^2 + \|y\|^2 + \frac{1}{n-1}\|x-y\|^2 \geq 0,\ \ x,y\in H,$$

Hence $\overset{\langle n\rangle}{\psi}_I = \overset{\langle n\rangle}{\asymp}_I$ and the proof of (81) is complete. □

Example 4.6. (Rotation) Let R_θ be the counterclockwise vector rotation in $\mathbb{R}^2$ by an angle of $\theta \in [-\pi,\pi]$, i.e., $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$,

$$R_\theta(x) = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}x_1\cos\theta - x_2\sin\theta\\ x_1\sin\theta + x_2\cos\theta\end{pmatrix},\ \ x = \begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

For $x,y\in\mathbb{R}^2$, $n\in\mathbb{N}$, $n\geq 1$,

$$\overset{\langle n\rangle}{\varphi}_{R_\theta}(x,y) = \begin{cases}\frac{n}{2(n+1)}\|y-R_\theta(x)\|^2 + x\cdot y, & \theta = 0\\ \frac{\sin(n\theta)}{2\sin((n+1)\theta)}\|y-R_\theta(x)\|^2 + x\cdot y, & \theta\in(-\frac{\pi}{n+1},\frac{\pi}{n+1})\setminus\{0\}\\ c_{R_\theta}(x,y), & \theta = \pm\frac{\pi}{n+1}\\ +\infty, & \theta\in[-\pi,-\frac{\pi}{n+1})\cup(\frac{\pi}{n+1},\pi]\end{cases}; \tag{84}$$

$$\overset{\langle\infty\rangle}{\varphi}_{R_\theta}(x,y) = \begin{cases}\frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2, & \theta = 0\\ +\infty, & \theta\in[-\pi,\pi]\setminus\{0\}\end{cases}; \tag{85}$$

$$\overset{\langle 2\rangle}{\asymp}_{R_\theta}(x,y) = \cos\theta\|y-R_\theta(x)\|^2 + x\cdot y,\ \ \theta\in[-\pi,\pi]; \tag{86}$$

For $n \in \mathbb{N}$, $n \geq 3$,

$$\overset{\langle n\rangle}{\Join}_{R_\theta}(x,y) = \begin{cases} \frac{n}{2(n-1)}\|y-R_\theta(x)\|^2 + x\cdot y, & \theta = 0 \\ \frac{\sin(n\theta)}{2\sin((n-1)\theta)}\|y-R_\theta(x)\|^2 + x\cdot y, & \theta \in (-\frac{\pi}{n-1}, \frac{\pi}{n-1}) \smallsetminus \{0\} \\ c - \iota_{\{y-R_\theta(x)\}}, & \theta = \pm\frac{\pi}{n-1} \\ -\infty, & \theta \in [-\pi, -\frac{\pi}{n-1}) \cup (\frac{\pi}{n-1}, \pi] \end{cases}, \quad (87)$$

$$\overset{\langle \infty\rangle}{\Join}_{R_\theta}(x,y) = \begin{cases} \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2, & \theta = 0 \\ -\infty, & \theta \in [-\pi, \pi] \smallsetminus \{0\} \end{cases}. \quad (88)$$

For $n \in \mathbb{N}$, $n \geq 2$, R_θ is (maximal) n-monotone iff $\theta \in [-\frac{\pi}{n}, \frac{\pi}{n}]$; R_θ is n-monotone and not $(n+1)$-monotone iff $\theta \in [-\frac{\pi}{n}, -\frac{\pi}{n+1}) \cup (\frac{\pi}{n+1}, \frac{\pi}{n}]$; and R_θ is (maximal) cyclically monotone iff $\theta = 0$.

Proof. We repeatedly use the fact that both the norm and inner product in $\mathbb{R}^2$ are invariant under any rotation, namely, $\|R_\theta(x)\| = \|x\|$ and $R_\theta(x)\cdot R_\theta(y) = x\cdot y$, $x, y \in \mathbb{R}^2$. By direct computation we get $x \cdot R_\theta(x) = \cos\theta\|x\|^2$ and so, together with $R_\theta^* = R_\theta^{-1} = R_{-\theta}$,

$$\begin{aligned} \overset{\langle 2\rangle}{\Join}_{R_\theta}(x,y) &= R_{-\theta}(y)\cdot y + (x - R_{-\theta}(y))\cdot R_\theta(x) \\ &= (y - R_\theta(x))\cdot R_\theta(y - R_\theta(x)) + x\cdot y = \cos\theta\|y - R_\theta(x)\|^2 + x\cdot y. \end{aligned}$$

Therefore, according to Theorem 2.1, R_θ is (2-)monotone iff $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$; in this case, it is maximal (2-)monotone because it is monotone and defined on $\mathbb{R}^2$.

Alternatively, R_θ is maximal monotone for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, since $c_{R_\theta} \in \Gamma(\mathbb{R}^2)$, which implies that $\overset{\langle 1\rangle}{\psi}_{R_\theta} = \overset{\langle 1\rangle}{\Join}_{R_\theta} = c_{R_\theta} \geq c$. Hence R_θ is 2-representable, due to the relations $\operatorname{Graph} R_\theta = [\overset{\langle 1\rangle}{\psi}_{R_\theta} = c]$ and $\overset{\langle 1\rangle}{\psi}_{R_\theta} \geq c$ (see Theorem 3.3). In addition, from (84), $\overset{\langle 1\rangle}{\varphi}_{R_\theta} \geq c$. Theorem 3.1 completes the argument.

We restrict ourselves to $\theta \in (0, \pi]$, since $R_0 = I$ is covered in Example 4.5 and we can extend our conclusions to $\theta \in [-\pi, 0)$ via $R_{-\theta} = R_\theta^{-1} = R_\theta^*$ and the equalities $\overset{\langle n\rangle}{\varphi}_{R_\theta}(x,y) = \overset{\langle n\rangle}{\varphi}_{R_{-\theta}}(y,x)$, $\overset{\langle n\rangle}{\Join}_{R_\theta}(x,y) = \overset{\langle n\rangle}{\Join}_{R_{-\theta}}(y,x)$.

According to (75), (79)

$$\begin{aligned} \overset{\langle 3\rangle}{\Join}_{R_\theta}(x,y) &= \inf_{v\in\mathbb{R}^2} \left\{\overset{\langle 2\rangle}{\Join}_{R_\theta}(x,v) + R_\theta^{-1}(y)\cdot(y-v)\right\} \\ &= \inf_{v\in\mathbb{R}^2} \left\{\cos\theta\|v - R_\theta(x)\|^2 + v\cdot(x - R_\theta^{-1}(y)) + y\cdot R_\theta(y)\right\} \\ &= \inf_{u\in\mathbb{R}^2} \left\{\cos\theta\|u\|^2 + u\cdot(x - R_\theta^{-1}(y))\right\} + \overset{\langle 2\rangle}{\Join}_{R_\theta}(x,y) \\ &= \inf_{u\in\mathbb{R}^2} \left\{\cos\theta\|u\|^2 + u\cdot(x - R_\theta^{-1}(y))\right\} + \cos\theta\|y - R_\theta(x)\|^2 + x\cdot y \\ &= \begin{cases} (\cos\theta - \frac{1}{4\cos\theta})\|y - R_\theta(x)\|^2 + x\cdot y, & \cos\theta > 0 \\ c - \iota_{\{y - R_\theta(x)\}}, & \cos\theta = 0 \\ -\infty & \cos\theta < 0 \end{cases} \end{aligned}$$

$$
= \begin{cases} \frac{\sin(3\theta)}{2\sin(2\theta)}\|y - R_\theta(x)\|^2 + x\cdot y, & \theta \in (0, \frac{\pi}{2}) \\ c - \iota_{\{y - R_\theta(x)\}}, & \theta = \frac{\pi}{2} \\ -\infty & \theta \in (\frac{\pi}{2}, \pi] \end{cases},
$$

after we used the identity $\cos\theta - \frac{1}{4\cos\theta} = \frac{\sin(3\theta)}{2\sin(2\theta)}$. Hence the base case for an induction argument on $n \ge 3$ used for (87) is complete.

Assume that (87) holds for some finite integer $n \ge 3$. From (75)

$$
\overset{\langle n+1\rangle}{)\!(}_{R_\theta}(x,y) = \inf_{v\in\mathbb{R}^2}\left\{ \overset{\langle n\rangle}{)\!(}_T(x,v) + R_\theta^{-1}(y)\cdot(y - v)\right\};
$$

from which $\overset{\langle n+1\rangle}{)\!(}_{R_\theta}(x,y) = -\infty$, whenever $\theta \in [\frac{\pi}{n-1}, \pi]$. Therefore, for $\theta \in (0, \frac{\pi}{n-1})$

$$
\begin{aligned}
\overset{\langle n+1\rangle}{)\!(}_{R_\theta}(x,y) &= \inf_{v\in\mathbb{R}^2}\left\{ \tfrac{\sin(n\theta)}{2\sin((n-1)\theta)}\|v - R_\theta(x)\|^2 + v\cdot(x - R_\theta^{-1}(y)) + y\cdot R_\theta(y)\right\} \\
&= \inf_{u\in\mathbb{R}^2}\left\{ \tfrac{\sin(n\theta)}{2\sin((n-1)\theta)}\|u\|^2 + u\cdot(x - R_\theta^{-1}(y))\right\} + \overset{\langle 2\rangle}{)\!(}_{R_\theta}(x,y) \\
&= \begin{cases} (\cos\theta - \frac{\sin((n-1)\theta)}{2\sin(n\theta)})\|y - R_\theta(x)\|^2 + x\cdot y, & \frac{\sin(n\theta)}{\sin((n-1)\theta)} > 0 \\ c - \iota_{\{y - R_\theta(x)\}}, & \sin(n\theta) = 0 \\ -\infty & \frac{\sin(n\theta)}{\sin((n-1)\theta)} < 0 \end{cases} \\
&= \begin{cases} \frac{\sin((n+1)\theta)}{2\sin(n\theta)}\|y - R_\theta(x)\|^2 + x\cdot y, & \theta \in (0, \frac{\pi}{n}) \\ c - \iota_{\{y - R_\theta(x)\}}, & \theta = \frac{\pi}{n} \\ -\infty & \theta \in (\frac{\pi}{n}, \frac{\pi}{n-1}) \end{cases}
\end{aligned}
$$

where the identity $\cos\theta - \frac{\sin((n-1)\theta)}{2\sin(n\theta)} = \frac{\sin((n+1)\theta)}{2\sin(n\theta)}$ is used and, since $\theta \in (0, \frac{\pi}{n-1})$, i.e., $\sin((n-1)\theta) > 0$, we know that the sign of $\frac{\sin(n\theta)}{\sin((n-1)\theta)}$ is given by the sign of $\sin(n\theta)$.

Hence (87) holds for every finite integer $n \ge 3$.

We establish (84) by applying a similar argument. For $n = 1$

$$
\begin{aligned}
\overset{\langle 1\rangle}{\varphi}_{R_\theta}(x,y) &= \sup_{u\in\mathbb{R}^2}\{(x-u)\cdot R_\theta(u) + u\cdot y\} = \sup_{u\in\mathbb{R}^2}\{-\cos\theta\|u\|^2 + u\cdot(y + R_\theta^*(x))\} \\
&= \begin{cases} \frac{1}{4\cos\theta}\|y + R_\theta^*(x)\|^2, & \cos\theta > 0 \\ \iota_{\{0\}}(y + R_\theta^*(x)), & \cos\theta = 0 \\ +\infty, & \cos\theta < 0 \end{cases} \\
&= \begin{cases} \frac{1}{4\cos\theta}\|y - R_\theta(x)\|^2 + x\cdot y, & \theta \in (0, \frac{\pi}{2}) \\ c_{R_{\frac{\pi}{2}}}(x,y), & \theta = \frac{\pi}{2} \\ +\infty, & \theta \in (\frac{\pi}{2}, \pi] \end{cases}
\end{aligned}
$$

since $R_\theta(x) + R_\theta^*(x) = (2\cos\theta)x$, $R_{\frac{\pi}{2}}^* = R_{-\frac{\pi}{2}} = -R_{\frac{\pi}{2}}$, and $\frac{1}{4\cos\theta} = \frac{\sin\theta}{2\sin(2\theta)}$.

Assume that (84) holds for a finite integer $n \geq 1$. From (75)

$$\overset{\langle n+1\rangle}{\varphi}_{R_\theta}(x,y) = \sup_{u\in\mathbb{R}^2}\left\{\overset{\langle n\rangle}{\varphi}_{R_\theta}(u,y) + (x-u)\cdot R_\theta(u)\right\};$$

from which $\overset{\langle n+1\rangle}{\varphi}_{R_\theta}(x,y) = +\infty$, whenever $\theta \in [\frac{\pi}{n+1}, \pi]$.

For $\theta \in (0, \frac{\pi}{n+1})$, $\sin((n+1)\theta) > 0$. According to (75), (79) and the identity

$$\tfrac{\sin(n\theta)}{2\sin((n+1)\theta)} - \cos\theta = -\tfrac{\sin((n+2)\theta)}{2\sin((n+1)\theta)}$$

we have

$$\begin{aligned}
\overset{\langle n+1\rangle}{\varphi}_{R_\theta}(x,y) &= \sup_{u\in\mathbb{R}^2}\left\{\tfrac{\sin(n\theta)}{2\sin((n+1)\theta)}\|R_\theta(u)-R_\theta(x)\|^2 + x\cdot R_\theta(u) + u\cdot(y-R_\theta(u))\right\}\\
&= \sup_{u\in\mathbb{R}^2}\left\{\tfrac{\sin(n\theta)}{2\sin((n+1)\theta)}\|u-x\|^2 + x\cdot R_\theta(u) + u\cdot(y-R_\theta(u))\right\}\\
&= \sup_{v\in\mathbb{R}^2}\left\{\tfrac{\sin(n\theta)}{2\sin((n+1)\theta)}\|v\|^2 - v\cdot R_\theta(v) + v\cdot(y-R_\theta(x)) + x\cdot y\right\}\\
&= \sup_{v\in\mathbb{R}^2}\left\{(\tfrac{\sin(n\theta)}{2\sin((n+1)\theta)} - \cos\theta)\|v\|^2 + v\cdot(y-R_\theta(x))\right\} + x\cdot y\\
&= \sup_{v\in\mathbb{R}^2}\left\{-\tfrac{\sin((n+2)\theta)}{2\sin((n+1)\theta)}\|v\|^2 + v\cdot(y-R_\theta(x))\right\} + x\cdot y\\
&= \begin{cases} \tfrac{\sin((n+1)\theta)}{2\sin((n+2)\theta)}\|y-R_\theta(x)\|^2 + x\cdot y, & \sin((n+2)\theta) > 0\\ \iota_{\{0\}}(y-R_\theta(x)) + x\cdot y, & \sin((n+2)\theta) = 0\\ +\infty & \sin((n+2)\theta) < 0\end{cases}\\
&= \begin{cases} \tfrac{\sin((n+1)\theta)}{2\sin((n+2)\theta)}\|y-R_\theta(x)\|^2 + x\cdot y, & \theta\in(0,\frac{\pi}{n+2})\\ c_{R_\theta}(x,y), & \theta = \frac{\pi}{n+2}\\ +\infty & \theta\in(\frac{\pi}{n+2},\frac{\pi}{n+1})\end{cases}
\end{aligned}$$

and the inductive step is complete. Therefore (84) is true.

Note that, from (87), for $n \in \mathbb{N}$, $n \geq 3$, R_θ is n-monotone iff $\overset{\langle n\rangle}{\rtimes}_{R_\theta} \geq c$ which comes to $\theta \in (-\frac{\pi}{n-1}, \frac{\pi}{n-1})$ and either $\theta = 0$ or $\sin(n\theta) > 0$, that is, $\theta \in [-\frac{\pi}{n}, \frac{\pi}{n}]$. □

A subset A of $Z = X \times Y$ is a *double-cone* if $\mathbb{R}A = A$, i.e., for every $\lambda \in \mathbb{R}$, $z \in A$ we have $\lambda z \in A$. A subset A of Z is called *non-negative* if, for every $z \in A$, $c(z) \geq 0$, and *skew* if, for every $z \in A$, $c(z) = 0$.

For $A \subset Z$ we set

$$\neg A := \{z = (x,y) \in Z \mid \neg z := (x,-y) \in A\},$$

$$A^\perp := \{z' \in Z \mid \forall z \in A,\ z\cdot z' = 0\},\ \text{and}$$

Note that $\neg A \neq -A := (-1)A$, $A^\perp = (\mathrm{cl}_{\sigma(Z,Z)}\operatorname{lin}A)^\perp$, $A^{\perp\perp} = \mathrm{cl}_{\sigma(Z,Z)}\operatorname{lin}A$, $A^\perp$ is a $\sigma(Z,Z)$-closed linear subspace of Z, and $\neg(A^\perp) = (\neg A)^\perp$. Here $\operatorname{lin}A$ is the linear hull of A.

Example 4.7. (Skew Double-Cone) Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system and let $D \subset Z := X \times Y$ be a skew double-cone. Then

(i) $$\overset{\langle 1\rangle}{\Join}_D = \iota_D,\ \overset{\langle 1\rangle}{\varphi}_D = \iota_{D^\perp},\ \overset{\langle 1\rangle}{\psi}_D = \iota_{D^{\perp\perp}}, \tag{89}$$

D is monotone iff $D \subset D^\perp$; D is representable iff $D = D^{\perp\perp}$ iff D is a $\sigma(Z,Z)$-closed linear subspace of Z; D is maximal monotone iff D is a $\sigma(Z,Z)$-closed linear subspace of Z and $\neg D^\perp$ is monotone.

(ii) $$\overset{\langle 2\rangle}{\varphi}_D(x,y) = \sup_{u \in \mathrm{Pr}_X(D)} \iota_{D^\perp}(u,y) + \iota_{(\mathrm{Pr}_Y(D))^\perp}(x),\ (x,y) \in Z; \tag{90}$$

where $\mathrm{Pr}_X(x,y) := x$ and $\mathrm{Pr}_Y(x,y) := y$, $(x,y) \in X \times Y$ are the projections of $X \times Y$ onto X and Y, respectively.

(iii) D is 3-monotone iff $\mathrm{Pr}_X(D) \subset (\mathrm{Pr}_Y(D))^\perp$ iff, for every $(x,y) \in \mathrm{Pr}_X(D) \times \mathrm{Pr}_Y(D)$, $\langle x,y\rangle = 0$ iff $D \subset S \times S^\perp$, for some S a $\sigma(X,Y)$-closed linear subspace of X iff D is cyclically monotone.

(iv) D is maximal 3-monotone iff $D = S \times S^\perp$, for some S a $\sigma(X,Y)$-closed linear subspace of X iff D is maximal cyclically monotone. In this case $\overset{\langle 2\rangle}{\varphi}_D = \overset{\langle\infty\rangle}{\varphi}_D = \iota_{S\times S^\perp}$.

Proof. (i) Since D is a skew double-cone, we have $\overset{\langle 1\rangle}{\Join}_D = c_D = \iota_D$, and

$$\overset{\langle 1\rangle}{\varphi}_D(z) = \sup\{z\cdot z' - c(z') \mid z' \in D\} = \sup\{z \cdot z' \mid z' \in D\} = \iota_{D^\perp}(z),\ z \in Z.$$

Because $D^\perp$ is a linear subspace of Z we get $\overset{\langle 1\rangle}{\psi}_D = (\overset{\langle 1\rangle}{\varphi}_D)^\square = \iota_{D^\perp}^\square = \sigma_{D^\perp} = \iota_{D^{\perp\perp}}$.

According to Theorem 2.2, D is monotone iff $\iota_{D^\perp} = \overset{\langle 1\rangle}{\varphi}_D \le \overset{\langle 1\rangle}{\Join}_D = \iota_D$ iff $D \subset D^\perp$.

According to Theorem 3.3, D is representable iff $\overset{\langle 1\rangle}{\psi}_D = \iota_{D^{\perp\perp}} \ge c$ and $D = [\iota_{D^{\perp\perp}} = c]$. Clearly, since D is skew, if $D = D^{\perp\perp}$ then D is representable.

Conversely, assume that D is representable, that is, for every

$z \in L := D^{\perp\perp} = \mathrm{cl}_{\sigma(Z,Z)} \operatorname{lin} D$, we have $c(z) \le 0$, and $D = [\iota_L = c]$.

It suffices to prove that L is skew, since, in that case, $[\iota_L = c] = L$. To this end note that, because D is skew monotone (recall (54)), for every $z, w \in D$, we have

$$c(z - w) = c(z) + c(w) + z \cdot w = z \cdot w \ge 0.$$

Since $-z \in D$, we know that $z \cdot w = 0$. Therefore, because $\neg D$ is skew, for every $u, v \in \neg D$

$$c(u - v) = c(u) + c(v) - u \cdot v = \neg u \cdot \neg v = 0,$$

in particular, $\neg D$ is a skew monotone double-cone. According to (89) and Theorem 2.6, $\overset{\langle 1\rangle}{\psi}_{\neg D} = \iota_{(\neg D)^{\perp\perp}} = \iota_{\neg L} \ge c$, which comes to, for every $z \in \neg L$, $c(z) \le 0$. Hence, for every $z \in L$, $c(z) = -c(\neg z) \ge 0$, and so, L is skew.

According to Theorem 3.1, D is maximal monotone iff $\overset{\langle 1\rangle}{\varphi}_D = \iota_{D^\perp} \ge c$ and D is representable. Because $D^\perp$ is linear, $\iota_{D^\perp} \ge c$ is equivalent to $\neg D^\perp$ is monotone.

(ii) From (76) and the fact that D is a skew double-cone, we find that

$$\overset{\langle 2\rangle}{\varphi}_D(x,y) = \sup_{(u,v)\in D}\{\iota_{D^\perp}(u,y) + \langle x-u, v\rangle\} = \sup_{(u,v)\in D}\{\iota_{D^\perp}(u,y) + \langle x, v\rangle\}$$
$$= \sup_{u\in \mathrm{Pr}_X(D)} \iota_{D^\perp}(u,y) + \sigma_{\mathrm{Pr}_Y(D)}(x)$$
$$= \sup_{u\in \mathrm{Pr}_X(D)} \iota_{D^\perp}(u,y) + \iota_{(\mathrm{Pr}_Y(D))^\perp}(x),\ (x,y)\in Z,$$

because $\mathrm{Pr}_X(D)$ is a double-cone.

(iii) According to Theorem 2.2, D is 3-monotone iff $\overset{\langle 2\rangle}{\varphi}_D \le \overset{\langle 1\rangle}{\curlyvee}_D = \iota_D$ iff, for every $z\in D$, $\overset{\langle 2\rangle}{\varphi}_D \le 0$.

If $\mathrm{Pr}_X(D) \subset (\mathrm{Pr}_Y(D))^\perp$, i.e., for every $(x,y)\in \mathrm{Pr}_X(D)\times \mathrm{Pr}_Y(D)$, $\langle x,y\rangle = 0$, then, for every $(x,y)\in D$, $u\in \mathrm{Pr}_X(D)$, we have

$$(u,y)\in D^\perp \text{ and } x\in \mathrm{Pr}_X(D)\subset (\mathrm{Pr}_Y(D))^\perp.$$

Hence $\overset{\langle 2\rangle}{\varphi}_D(x,y) = 0$, that is, D is 3-monotone.

Conversely, assume that D is 3-monotone, i.e., for every $z=(x,y)\in D$, $\overset{\langle 2\rangle}{\varphi}_D \le 0$. Since $\iota_{D^\perp}\ge 0$, in particular, that yields $\iota_{(\mathrm{Pr}_Y(D))^\perp}(x)\le 0$, namely, $x\in (\mathrm{Pr}_Y(D))^\perp$. We proved $\mathrm{Pr}_X(D)\subset (\mathrm{Pr}_Y(D))^\perp$.

In this case $D\subset S\times S^\perp = \mathrm{Graph}(\partial\iota_S)$ is cyclically monotone, where $S = \mathrm{Pr}_X(D)^{\perp\perp}$ is linear $\sigma(X,Y)$-closed.

(iv) The direct implications are plain due to (i), (iii). Conversely, assume that D is maximal ∞-monotone. In particular, D is 3-monotone. As previously seen in (iii), $D\subset S\times S^\perp$, where $S=\mathrm{Pr}_X(D)^{\perp\perp}$. Since $S\times S^\perp$ is ∞-monotone, we infer that $D = S\times S^\perp = \mathrm{Graph}(\partial\iota_S) = D^\perp$ is maximal ∞-monotone. But, according to (90) and Theorem 3.13,

$$\overset{\langle 2\rangle}{\varphi}_{S\times S^\perp}(x,y) = \iota_S(x) + \iota_{S^\perp}(y) = \overset{\langle\infty\rangle}{\varphi}_{S\times S^\perp}(x,y),\ (x,y)\in Z.$$

We conclude that D is maximal 3-monotone with the aid of Theorem 3.1, after we note that $D = [\overset{\langle 2\rangle}{\varphi}_D = c]$ and $\overset{\langle 2\rangle}{\varphi} \ge c$. □

Corollary 4.8. *Let $(X,Y,\langle\cdot,\cdot\rangle)$ be a dual system and let $A : D(A)\subset X \rightrightarrows Y$ be a multi-valued linear skew operator, that is, for every $x\in D(A)$, $y\in A(x)$, $\langle x,y\rangle = 0$. Then*

(i) *A is maximal monotone iff* Graph A *is $\sigma(Z,Z)$-closed in Z and $A^*: X\rightrightarrows Y$ is monotone, where the adjoint A^* is defined by* $\mathrm{Graph}(A^*) = \neg(\mathrm{Graph}\,A)^\perp$, *or*
$$(x,y)\in \mathrm{Graph}(A^*) \ \Leftrightarrow\ \forall (u,v)\in \mathrm{Graph}\,A,\ \langle x,v\rangle = \langle u,y\rangle. \tag{91}$$

(ii) *A is 3-monotone iff A is cyclically monotone iff $R(A)\subset D(A)^\perp$.*

(iii) *A is maximal 3-monotone iff A is maximal cyclically monotone iff $D(A)$ is closed convex and $R(A) = D(A)^\perp$.*

Proof. We use Example 4.7 for $D = \mathrm{Graph}\,A$, for which $D(A) = \mathrm{Pr}_X(D)$ and $R(A) = \mathrm{Pr}_Y(D)$. □

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