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# Relaxed Lagrangian Approach to First-Order Non-Convex Mean Field Type Control Problems

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We address the existence of equilibria for mean field type control problems of first-order with non-convex action functional. Introducing a relaxed Lagrangian approach on the Wasserstein space to handle the lack of convexity, we prove the existence of new relaxed Nash equilibria and we show that our existence result encompasses the classical Mean Field Control problem's existence result under convex data conditions.

*Keywords:* Mean field control, relaxed Lagrangian approach, relaxed controls, residual neural networks.

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## 1. Introduction

The motivation for this work stems from recent advances in deep learning (DL) and the application of techniques from optimal control and mean field theory. Specifically, deep neural networks can be considered as discrete-time nonlinear dynamical systems and their internal parameters can be recast as controls, allowing the formulation of the training process as an optimal control problem. Given a set of training data $(\xi, \zeta) \in \Omega$ distributed according to $\mathcal{M} \in \mathcal{P}(\Omega)$ and an activation function $\varphi$, we construct a neural network with input $z \in \mathbb{R}^d$ and output

$$\frac{1}{n}\sum_{i=1}^{n} \beta_{t,i}\varphi(\alpha_{t,i}z + \rho_{t,i}\zeta_t) = \int_{\mathbb{R}^d} \beta\varphi(\alpha z + \rho\zeta_t)\, \nu_t^n(d\alpha, d\beta, d\rho), \tag{1}$$

where $t$ denotes the index of the layer, $(\alpha, \beta, \rho)$ are the data of the neural network and

$$\nu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\{\alpha_{t,i},\beta_{t,i},\rho_{t,i}\}}.$$

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Thus, setting $a = (\alpha, \beta, \rho)$ and $\psi(z, a, \zeta_t) = \beta\varphi(\alpha z + \rho\zeta_t)$, the architecture of the neural network can be written as

$$X_{t_{k+1}}^{\nu^n,\xi,\zeta} = X_{t_k}^{\nu^n,\xi,\zeta} + \frac{\Delta t}{n}\sum_{i=1}^{n} \psi(X_{t_k}^{\nu^n,\xi,\zeta}, a_{t_k,i}, \zeta_{t_k}),$$

where $\Delta t = t_{k+1} - t_k$. Therefore, as $\Delta t \downarrow 0$ and $n \uparrow \infty$ (heuristically) we obtain

$$X_t^{\nu,\xi,\zeta} = \xi + \int_0^t \int \psi(X_r^{\nu,\xi,\zeta}, a, \zeta_r)\, \nu_r(da)dr$$

and the training procedure of such a model can be recast as an optimization problem on the Wasserstein space

$$\min_{\nu} \int_{\Omega} g(\xi, \zeta, \nu, X_T^{\nu,\xi,\zeta})\, \mathcal{M}(d\xi, d\zeta). \tag{2}$$

So, looking at the minimization problem (2), in this work we aim to extend the possibility of studying neural network architectures by means of the relaxed Lagrangian approach to the case of mean field state equation modeling, for instance, congestion of data or sparsity constraint.

Having these models in mind, Mean Field Control (MFC) is a recent branch of control theory that deals with the optimal control of a large number of interacting agents. It arises when considering systems with a population of individual decision-makers (agents), where each agent's dynamics and objectives depend not only on their own states and actions but also on the aggregate behavior of the entire population. From a variational viewpoint, a general MFC problem of first order (having deterministic dynamics) can be written as a minimization problem of the following type

$$\inf_{\dot\gamma} \int_0^T L(\gamma(t), \dot\gamma(t), m(t))\, dt + u_T(\gamma(T)),$$

where $m \in C([0,T], \mathcal{P}(\Omega))$ is the probability distribution of all agents and $L$ is the Legendre transform of $H$. Here, the Lagrangian $L$ is convex in the control variable $\dot\gamma$.

We consider the MFC problem with general nonlinear controlled dynamics, whose state equations directly depend on the distribution of players, of the form

$$\min_{u:[0,T]\to\mathbb{R}^n} \int_0^T L(\gamma(t), u(t), m(t))\, dt \quad \text{subject to} \quad \dot\gamma(t) = f(\gamma(t), u(t), m(t)), \tag{3}$$

where the Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^n \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is non-convex with respect to the control variable $u(t) \in \mathbb{R}^n$. The convexity assumption of the Lagrangian function, with respect to the control, is needed even if this is not always the case when considering some real life applications such as biological modeling, pricing dynamics, multi-agent reinforcement learning and deep learning ([13, 17, 18, 20, 22, 23, 26]). For the case that $L$ is convex, such a problem is classically solved by using the so-called Lagrangian approach, see for instance [7, 9, 19, 24]. In case of stochastic systems, such an analysis has been developed in [2, 12, 13, 14, 15]. However, to the best of the authors' knowledge, no direct results are available for the first-order case.

We address this challenge by generalizing the Lagrangian approach to non-convex mean field optimization problems taking inspiration from the well-known relaxed control approach [4, 5, 6]. The relaxation method, we propose here, transforms the non-convex optimal control problem (3) into an optimal control problem over probability measures on the control space (6). In particular, once the Nash equilibria is found we have, consequently, constructed the optimal neural network architecture for the data we are interested in.

### 1.1. Relaxation method

Here, we proceed to show the relaxation procedure on Wasserstein space.

Fix a time horizon $T > 0$. Let $m_0 \in \mathcal{P}\left(\mathbb{T}^d\right)$ denote the initial distribution of all agents. Denote by $\mathcal{M}([0,T] \times \mathbb{R}^n)$ the space of all Borel measures on $[0,T] \times \mathbb{R}^n$. Define the set $\mathcal{P}_U$ by

$$\mathcal{P}_U := \left\{\mu \in \mathcal{M}([0,T] \times \mathbb{R}^n) \,\middle|\, \exists\, (\mu_t)_{t\in[0,T]} \subset \mathcal{P}\left(\mathbb{R}^n\right), s.t.\ \mu = \mathcal{L}_{[0,T]} \otimes \mu_t \right\},$$

where $\mu = \mathcal{L}_{[0,T]} \otimes \mu_t$ means that for any Borel map $g : [0,T] \times \mathbb{R}^n \to [0,+\infty]$, the following holds

$$\int_{[0,T]\times\mathbb{R}^n} g(t,u)\mu(dt,du) = \int_0^T \int_{\mathbb{R}^n} g(t,u)\mu_t(du)dt.$$

Similar to (3), we consider the following relaxed state equation:

$$\begin{cases} \dot{\gamma}(t) = \int_{\mathbb{R}^n} f\left(\gamma(t),u,m(t)\right)\mu_t(du), & t \in [0,T] \\ \gamma(0) = x, & x \in \mathbb{T}^d \end{cases} \tag{4}$$

where $\mu \in \mathcal{P}_U$ represents the strategy and $m \in \mathcal{C}([0,T], \mathcal{P}(\mathbb{T}^d))$ is the distribution of all agents with $m(0) = m_0$. Therefore, the aim of each agent is to choose a strategy such that the cost function attains the minimum, which is defined by

$$J^m(\gamma,\mu) = \int_0^T \int_{\mathbb{R}^n} L\left(\gamma(t),u,m(t)\right)\mu(dt,du).$$

Define

$$\Gamma_T := \left\{ \gamma \in AC\left([0,T],\mathbb{T}^d\right) \,\middle|\, \begin{array}{l} \exists\,(x,\mu)\in\mathbb{T}^d\times\mathcal{P}_U, \exists\, m\in\mathcal{C}\left([0,T],\mathcal{P}\left(\mathbb{T}^d\right)\right) \\ \text{with } m(0) = m_0,\ s.t.\ \gamma \text{ is the} \\ \text{solution of the state equation (4)} \end{array} \right\}.$$

Let $\Gamma_T$ be endowed with the uniform norm $\|\cdot\|_\infty$. Let $m \in \mathcal{C}\left([0,T],\mathcal{P}\left(\mathbb{T}^d\right)\right)$ be the map such that $m(0) = m_0$, and let $P \in \mathcal{P}\left(\Gamma_T \times \mathcal{P}_U\right)$ be the probability measure such that every state-measure pair $(\gamma,\mu) \in \mathrm{spt}(P)$ satisfies

$$\dot{\gamma}(t) = \int_{\mathbb{R}^n} f\left(\gamma(t),u,m(t)\right)\mu_t(du). \tag{5}$$

The total cost of all agents is defined by

$$J\left(m,P\right) := \int_{\Gamma_T\times\mathcal{P}_U} J^m\left(\gamma,\mu\right)P\left(d\gamma,d\mu\right).$$

Since the map $m$ represents the state distribution of all agents while the support of $P$ is the state processes and the strategies of all agents, the MFC problem is

$$\inf_P J\left(\left(e_t\sharp\pi_1\sharp P\right)_{t\in[0,T]}, P\right), \tag{6}$$

where the infimum is taken over all the measures $P \in \mathcal{P}(\Gamma_T \times \mathcal{P}_U)$ such that any state-measure pair $(\gamma,\mu) \in \operatorname{spt}(P)$ satisfies (5). See Appendix A for details of the evaluation map $e_t$ and the push-forward measure $\pi_1\sharp P \in \mathcal{P}(\Gamma_T)$.

### 1.2. Assumptions

From now on, fix two constants $p, q \in \mathbb{R}$ with $q > p \geq 1$.

Let $L(x,u,\nu) : \mathbb{T}^d \times \mathbb{R}^n \times \mathcal{P}\left(\mathbb{T}^d\right) \to \mathbb{R}$ be a Lagrangian satisfying the following assumptions.

(L1) For any $\nu \in \mathcal{P}\left(\mathbb{T}^d\right)$, the function $(x,u) \mapsto L(x,u,\nu)$ is of class $\mathcal{C}^2$.

(L2) There is a modulus function $\omega : [0,+\infty) \to [0,+\infty)$ such that

$$|L(x,u,\nu_1) - L(x,u,\nu_2)| \leq \omega\left(d_1(\nu_1,\nu_2)\right), \ \forall (x,u) \in \mathbb{T}^d \times \mathbb{R}^n, \ \forall \nu_1,\nu_2 \in \mathcal{P}(\mathbb{T}^d).$$

See Appendix A and B for details of the modulus function $\omega$ and the Kantorovich-Rubinstein distance $d_1$.

(L3) There is a constant $C_1 \in \mathbb{R}$ such that

$$|L(x,u,\nu)| \leq C_1\left(1+|u|^q\right), \quad \forall (x,u,\nu) \in \mathbb{T}^d \times \mathbb{R}^n \times \mathcal{P}(\mathbb{T}^d).$$

(L4) There is a constant $C_2 \in \mathbb{R}$ such that

$$\left|\frac{\partial L}{\partial x}(x,u,\nu)\right| \leq C_2\left(1+|u|^q\right), \quad \forall (x,u,\nu) \in \mathbb{T}^d \times \mathbb{R}^n \times \mathcal{P}(\mathbb{T}^d).$$

Let $f(x,u,\nu) : \mathbb{T}^d \times \mathbb{R}^n \times \mathcal{P}\left(\mathbb{T}^d\right) \to \mathbb{R}^d$ be a continuous map satisfying the following assumptions.

(F1) There is a constant $C > 0$ such that

$$|f(x,u,\nu)| \leq C\left(1+|u|^p\right), \quad \forall (x,u,\nu) \in \mathbb{T}^d \times \mathbb{R}^n \times \mathcal{P}(\mathbb{T}^d).$$

(F2) For any $(u,\nu) \in \mathbb{R}^n \times \mathcal{P}\left(\mathbb{T}^d\right)$, the map $x \mapsto f(x,u,\nu)$ is Lipschitz continuous and

$$\operatorname{Lip}_x(f) := \sup_{\substack{u\in\mathbb{R}^n,\ \nu\in\mathcal{P}\left(\mathbb{T}^d\right) \\ x_1\neq x_2\in\mathbb{T}^d}} \frac{|f(x_1,u,\nu) - f(x_2,u,\nu)|}{|x_1-x_2|} < +\infty.$$

(F3) For any $(x,u) \in \mathbb{T}^d \times \mathbb{R}^n$, the map $\nu \mapsto f(x,u,\nu)$ is Lipschitz continuous and

$$\operatorname{Lip}_\nu(f) := \sup_{\substack{x\in\mathbb{T}^d,\ u\in\mathbb{R}^n \\ \nu_1\neq\nu_2\in\mathcal{P}\left(\mathbb{T}^d\right)}} \frac{|f(x,u,\nu_1) - f(x,u,\nu_2)|}{d_1(\nu_1,\nu_2)} < +\infty.$$

### 1.3. Main results

Let $R$ be a positive constant. Define the compact subset $\mathcal{P}_U^R \subset \mathcal{P}_U$ by

$$\mathcal{P}_U^R := \left\{ \mu \in \mathcal{P}_U \,\middle|\, \int_0^T \int_{\mathbb{R}^n} |u|^q \mu(dt, du) \le R \right\},$$

where $q$ is the positive constant defined as in Section 1.2. Define the metric $\tilde{d}_1$ on $\mathcal{P}_U$ by

$$\tilde{d}_1(\mu_1, \mu_2) := d_1\left(\frac{\mu_1}{T}, \frac{\mu_2}{T}\right), \quad \forall \mu_1, \mu_2 \in \mathcal{P}_U.$$

For any $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}_U^R$ and any $m \in \mathcal{C}\left([0,T], \mathcal{P}\left(\mathbb{T}^d\right)\right)$ such that $m(0) = m_0$, we denote by $\gamma$ the solution of the state equation (4). By Lemma 2.1, there exists a positive constant $K$, independent of $x$, $\mu$ and $m$, such that

$$\|\dot{\gamma}\|_{L_{q/p}(\mathcal{L}_{[0,T]})} := \left( \int_0^T |\dot{\gamma}(t)|^{q/p}\, dt \right)^{p/q} \le K.$$

Since $q > p \ge 1$, there exists a positive constant $r > 1$ such that $p/q + 1/r = 1$. Define the subset $\mathbb{M}_r \subset \mathcal{C}\left([0,T], \mathcal{P}\left(\mathbb{T}^d\right)\right)$ by the space of $1/r$-Hölder-continuous maps with Hölder seminorm $K$, i.e.,

$$\mathbb{M}_r := \left\{ m \in \mathcal{C}\left([0,T], \mathcal{P}(\mathbb{T}^d)\right) \,\middle|\, \sup_{t_1 \ne t_2 \in [0,T]} \frac{d_1(m(t_1), m(t_2))}{|t_1 - t_2|^{1/r}} \le K,\ m(0) = m_0 \right\}.$$

Define the metric $\hat{d}_1$ on $\mathcal{C}\left([0,T], \mathcal{P}\left(\mathbb{T}^d\right)\right)$ by

$$\hat{d}_1\left(m^1, m^2\right) := \sup_{t \in [0,T]} d_1\left(m^1(t), m^2(t)\right), \quad \forall m^1, m^2 \in \mathcal{C}\left([0,T], \mathcal{P}(\mathbb{T}^d)\right).$$

By Lemma 2.3, $\mathbb{M}_r$ is a compact subset of $\mathcal{C}\left([0,T], \mathcal{P}\left(\mathbb{T}^d\right)\right)$ with respect to the $\hat{d}_1$-topology.

Define the subset $\Gamma_T^R \subset \Gamma_T$ by

$$\Gamma_T^R := \left\{ \gamma \in AC\left([0,T], \mathbb{T}^d\right) \,\middle|\, \begin{array}{l} \exists\,(x, \mu, m) \in \mathbb{T}^d \times \mathcal{P}_U^R \times \mathbb{M}_r, \text{ s.t. } \gamma \text{ is} \\ \text{the solution of the state equation (4)} \end{array} \right\}.$$

Let $\Gamma_T^R$ also be endowed with the uniform norm $\|\cdot\|_\infty$. By Lemma 2.4, $\Gamma_T^R$ is a compact subset of $\Gamma_T$. We denote by $\gamma = \gamma(\cdot; x, \mu, m) \in \Gamma_T^R$ the solution of the state equation (4) with the initial value $x \in \mathbb{T}^d$, the measure $\mu \in \mathcal{P}_U^R$ and the function $m \in \mathbb{M}_r$.

**Definition 1.1.** Let $m \in \mathbb{M}_r$. Define $\mathcal{P}_R(m)$ by the set of probability measures $P \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ satisfying the following conditions.

(i) $e_0 \sharp (\pi_1 \sharp P) = m_0$.

(ii) For any $(t, v) \in [0,T] \times \mathbb{R}^d$ and any open set $N \subset \Gamma_T^R \times \mathcal{P}_U^R$,

$$\int_N \left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f(\gamma(s), u, m(s))\, \mu(ds, du) \right\rangle P(d\gamma, d\mu) = 0. \quad (7)$$

For any $m \in \mathbb{M}_r$ and any $P \in \mathcal{P}_R(m)$, define the MFC functional by

$$J(m,P) := \int_{\Gamma_T^R \times \mathcal{P}_U^R} J^m(\gamma,\mu)\, P(d\gamma, d\mu),$$

where
$$J^m(\gamma,\mu) := \int_0^T \int_{\mathbb{R}^n} L(\gamma(t), u, m(t))\, \mu(dt, du).$$

Define the associated minimization problem by

$$R^*(m) := \operatorname*{argmin}_{P \in \mathcal{P}_R(m)} \{J(m,P)\}, \quad \forall m \in \mathbb{M}_r.$$

**Definition 1.2.** Given $m_0 \in \mathcal{P}(\mathbb{T}^d)$, we say that $P$ is a *relaxed MFC equilibrium* for $m_0$ if
$$P \in R^*\left( (e_t \sharp (\pi_1 \sharp P))_{t\in[0,T]} \right) \text{ and } e_0 \sharp (\pi_1 \sharp P) = m_0.$$

The relaxed MFC equilibrium is well defined since $(e_t \sharp (\pi_1 \sharp P))_{t\in[0,T]} \in \mathbb{M}_r$ for any $P \in \mathcal{P}(\Gamma_T^R \times \mathcal{P}_U^R)$. See Remark 2.6 for details.

The first main result can be stated as follows.

**Theorem 1.3.** *Assume* (L1)–(L4) *and* (F1)–(F3). *There exists at least one relaxed MFC equilibrium.*

For any $(x,m) \in \mathbb{T}^d \times \mathbb{M}_r$ and any $u_x : [0,T] \to \mathbb{R}^n$ such that $\int_0^T |u_x(t)|^q\, dt \le R$, denote by $\gamma_x^{u_x} := \gamma_x^{u_x}\left(\cdot; x, \mathcal{L}_{[0,T]} \otimes \delta_{u_x(t)}, m\right)$ the solution of the state equation (4). Define a probability measure $\eta^u \in \mathcal{P}(\Gamma_T^R)$ by

$$\eta^u(A) := \int_{\mathbb{T}^d} \delta_{\{\gamma_x^{u_x}\}}(A) m_0(dx), \quad \forall A \in \mathcal{B}(\Gamma_T^R). \tag{8}$$

Following Lemma 3.1, $\eta^u$ is the unique measure such that the probability measure $P^u := \eta^u \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{u_{\gamma(0)}(t)}\right\}}$ belongs to the set $\mathcal{P}_R(m)$.

**Definition 1.4.** Given $m_0 \in \mathcal{P}(\mathbb{T}^d)$, we say that $P$ is a *strict relaxed MFC equilibrium* for $m_0$ if it satisfies:

(i) $P$ is a relaxed MFC equilibrium for $m_0$.

(ii) There exists a map $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}^n$ such that $\int_0^T |u(t,x)|^q\, dt \le R$ and
$$P = \eta^u \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{u(t,x)}\right\}},$$
where $\eta^u$ is defined as in (8).

The second main result is stated as follows and, as a consequence, we immediately obtain that Theorem 1.3 encompasses the classical existence result for MFC equilibria.

**Theorem 1.5.** *Let $m \in \mathbb{M}_r$ and $P \in R^*(m)$. For any $(t,x) \in [0,T] \times \mathbb{T}^d$, the set*

$$\mathcal{L}(t,x) := \left\{ (\lambda, w, v) \in \mathbb{R}^{d+2} \;\middle|\; \begin{array}{l} \exists\, u \in \mathbb{R}^n,\ s.t.\ \lambda \geq L(x,u,m(t)), \\ w \geq |u|^q,\ v = f(x,u,m(t)) \end{array} \right\} \tag{9}$$

*is convex. Then the following statements hold.*

(1) *For any $x \in \mathbb{T}^d$ there exist maps $u_x^* : [0,T] \to \mathbb{R}^n$ such that the probability measure $P_0 := \eta^{u^*} \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{u^*_{\gamma(0)}(t)}\right\}} \in R^*(m)$, where $\{u_x^*\}_{x \in \mathbb{T}^d}$ is the optimal control for*

$$\inf_u \int_{\mathbb{T}^d} \int_0^T L(\gamma_x^u(t), u(t), m(t))\, dt\, m_0(dx),$$

*subject to the state equation*

$$\begin{cases} \dot\gamma^u(t) = f(\gamma^u(t), u(t), m(t)), & t \in [0,T] \\ \gamma^u(0) = x, & x \in \mathbb{T}^d \end{cases} \tag{10}$$

*where the infimum is taken over all the maps $u_x : [0,T] \to \mathbb{R}^n$ such that $\int_0^T |u_x(t)|^q\, dt \leq R$ for any $x \in \mathbb{T}^d$.*

(2) *There exists at least one strict relaxed MFC equilibrium.*

**Remark 1.6.** In synthesis, according to Theorem 1.3 and Theorem 1.5, we have that all the information of the neural network architecture described in (1) and (2) is captured by the relaxed MFC equilibrium. In particular, it is on the second marginal of a relaxed MFC equilibrium that we can find the parameters of the neural network that minimize the suitable loss function with respect to the analyzed data. □

**Structure of this paper.** In Section 2, we establish the existence of relaxed MFC equilibria. In Section 3, we demonstrate that these relaxed equilibria possess some suitable structures and that under the classical convexity assumption, the new equilibria coincide with the classical MFC equilibria. Moreover, we introduce some notations in Appendix A, recall well-known results from measure theory which will be used throughout this paper in Appendix B and complete the proof of Proposition 3.2 in Appendix C.

## 2. Existence of relaxed MFC equilibria

### 2.1. Proof of the properties of $\mathcal{P}_R(m)$

**Lemma 2.1.** *There exists a constant $K > 0$ such that for any $(x,\mu) \in \mathbb{T}^d \times \mathcal{P}_U^R$ and any $m \in \mathcal{C}\left([0,T], \mathcal{P}\left(\mathbb{T}^d\right)\right)$, the solution $\gamma$ of the state equation (4) satisfies*

$$\|\dot\gamma\|_{L_{q/p}(\mathcal{L}_{[0,T]})} := \left( \int_0^T |\dot\gamma(t)|^{q/p}\, dt \right)^{p/q} \leq K.$$

**Proof.** Since $\gamma$ is the solution of the state equation (4), we have

$$\begin{aligned}\int_0^T |\dot\gamma(t)|^{\frac{q}{p}}\,dt &= \int_0^T \left|\int_{\mathbb{R}^n} f\left(\gamma(t),u,m(t)\right)\mu_t(du)\right|^{\frac{q}{p}} dt\\ &\le \int_0^T\int_{\mathbb{R}^n} \left|f\left(\gamma(t),u,m(t)\right)\right|^{\frac{q}{p}}\mu_t(du)dt\\ &\le \int_0^T\int_{\mathbb{R}^n} C^{\frac{q}{p}}\left(1+|u|^p\right)^{\frac{q}{p}}\mu_t(du)dt\\ &\le 2^{\frac{q}{p}-1}TC^{\frac{q}{p}}+2^{\frac{q}{p}-1}C^{\frac{q}{p}}R.\end{aligned}$$

Here, the second line holds by Jensen's inequality, the third one holds by (F1) and the last one holds by the convexity of $x^{q/p}$. Let the constant $K$ equal to $\left(2^{\frac{q}{p}-1}TC^{\frac{q}{p}}+2^{\frac{q}{p}-1}C^{\frac{q}{p}}R\right)^{\frac{p}{q}}$. The proof is complete. □

**Remark 2.2.** For any $(t,x,\mu)\in[0,T]\times\mathbb{T}^d\times\mathcal{P}_U^R$ and any $m\in\mathcal{C}\left([0,T],\mathcal{P}(\mathbb{T}^d)\right)$, denote by $\gamma$ the related solution of the state equation (4). We have

$$\int_0^t\int_{\mathbb{R}^n}|u|^p\,\mu(ds,du)\le\left(\int_0^t\int_{\mathbb{R}^n}|u|^q\,\mu(ds,du)\right)^{\frac{p}{q}}T^{\frac{1}{r}}\le R^{\frac{p}{q}}T^{\frac{1}{r}}$$

and

$$\int_0^t\int_{\mathbb{R}^n}\left|f\left(\gamma(s),u,m(s)\right)\right|\mu(ds,du)\le CT+CR^{\frac{p}{q}}T^{\frac{1}{r}}.$$

**Lemma 2.3.** *$\mathbb{M}_r$ is a compact subset of $\mathcal{C}\left([0,T],\mathcal{P}\left(\mathbb{T}^d\right)\right)$ with respect to the $\hat d_1$-topology, where $\hat d_1$ is defined as in Section 1.3.*

**Proof.** Since $\mathbb{T}^d$ is compact, $\mathcal{P}\left(\mathbb{T}^d\right)$ is also compact with respect to the $d_1$-topology. Thus, $\mathbb{M}_r$ is uniformly bounded. By the construction of $\mathbb{M}_r$, it is uniformly equi-Hölder-continuous. By Ascoli-Arzelà theorem, $\mathbb{M}_r$ is relatively compact. Obviously, $\mathbb{M}_r$ is closed under $\hat d_1$-topology. We conclude that $\mathbb{M}_r$ is compact. □

**Lemma 2.4.** *$\Gamma_T^R$ is a compact subset of $\Gamma_T$ with respect to the uniform norm $\|\cdot\|_\infty$.*

**Proof.** Since $\mathbb{T}^d$ is compact, $\Gamma_T^R$ is uniformly bounded. By Lemma 2.1, for any $s\le t\in[0,T]$, we have

$$\begin{aligned}|\gamma(t)-\gamma(s)| &= \left|\int_s^t\dot\gamma(\tau)d\tau\right|\le\int_s^t|\dot\gamma(\tau)|\,d\tau\\ &\le\left(\int_s^t|\dot\gamma(\tau)|^{\frac{q}{p}}\,d\tau\right)^{\frac{p}{q}}\cdot(t-s)^{\frac{1}{r}}\le K\cdot(t-s)^{\frac{1}{r}},\end{aligned}$$

implying that $\Gamma_T^R$ is uniformly equi-Hölder-continuous. Thus, $\Gamma_T^R$ is relatively compact by Ascoli-Arzelà theorem.

Consider a sequence $\{\gamma_i\}_{i=1}^{\infty} \subset \Gamma_T^R$ converging to some $\tilde{\gamma}$ in the uniform norm $\|\cdot\|_{\infty}$. There exist sequences $\{x_i\}_{i=1}^{\infty} \subset \mathbb{T}^d$, $\{m^i\}_{i=1}^{\infty} \subset \mathbb{M}_r$ and $\{\mu^i\}_{i=1}^{\infty} \subset \mathcal{P}_U^R$ such that

$$\gamma_i(t) = x_i + \int_0^t \int_{\mathbb{R}^n} f\left(\gamma_i(s), u, m^i(s)\right) \mu_s^i(du)ds, \quad \forall t \in [0,T].$$

Since the sets $\mathbb{T}^d$, $\mathbb{M}_r$ and $\mathcal{P}_U^R$ are compact, there exist $x \in \mathbb{T}^d$, $m \in \mathbb{M}_r$ and $\mu \in \mathcal{P}_U^R$ such that the sequences $\{x_i\}_{i=1}^{\infty}$, $\{m^i\}_{i=1}^{\infty}$ and $\{\mu^i\}_{i=1}^{\infty}$ converge to $x$, $m$ and $\mu$ respectively. Define $\gamma(t) := x + \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu_s(du)ds$. We claim that $\gamma = \tilde{\gamma}$. Indeed,

$$\begin{aligned} |\gamma_i(t) - \gamma(t)| \le |x_i - x| &+ \int_0^t \int_{\mathbb{R}^n} \left| f\left(\gamma_i(s), u, m^i(s)\right) - f\left(\gamma(s), u, m(s)\right)\right| \mu_s^i(du)ds \\ &+ \int_0^t \int_{\mathbb{R}^n} \left| f\left(\gamma(s), u, m(s)\right)\right| (\mu_s^i - \mu_s)(du)ds \\ \triangleq I_1 + I_2 + I_3. & \end{aligned}$$

First of all, $\lim_{i\to\infty} I_1 = 0$ obviously. By Proposition B.2, we have $\lim_{i\to\infty} I_3 = 0$. By (F2) and (F3),

$$\begin{aligned} I_2 &\le \int_0^t \int_{\mathbb{R}^n} \left| f\left(\gamma_i(s), u, m^i(s)\right) - f\left(\gamma(s), u, m^i(s)\right)\right| \mu_s^i(du)ds \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \left| f\left(\gamma(s), u, m^i(s)\right) - f\left(\gamma(s), u, m(s)\right)\right| \mu_s^i(du)ds \\ &\le \mathrm{Lip}_x(f) \int_0^t |\gamma_i(s) - \gamma(s)|\, ds + \mathrm{Lip}_\nu(f) T \hat{d}_1\left(m^i, m\right). \end{aligned}$$

Since $\hat{d}_1\left(m^i, m\right) \to 0$ as $i \to \infty$, there exists a sequence of constants $\{c_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} c_i = 0$ and

$$I_1 + I_3 + \mathrm{Lip}_\nu(f) T \hat{d}_1\left(m^i, m\right) \le c_i.$$

By Gronwall's inequality, $|\gamma_i(t) - \gamma(t)| \le c_i e^{\mathrm{Lip}_x(f)t} \le c_i e^{\mathrm{Lip}_x(f)T}$.

In conclusion, we have $\lim_{i\to\infty} \|\gamma_i - \gamma\|_{\infty} = 0$. Furthermore, for any $i \in \mathbb{N}$, we obtain that

$$\|\gamma - \tilde{\gamma}\|_{\infty} \le \|\gamma - \gamma_i\|_{\infty} + \|\gamma_i - \tilde{\gamma}\|_{\infty}.$$

Let $i$ tend to infinity. The limit $\tilde{\gamma}$ of the sequence $\{\gamma_i\}_{i=1}^{\infty}$ is actually the curve $\gamma$ we defined above. Therefore, $\Gamma_T^R$ is compact. □

**Proposition 2.5.** *For each $m \in \mathbb{M}_r$, the set $\mathcal{P}_R(m)$ has the following properties.*

(1) *$\mathcal{P}_R(m)$ is compact with respect to the narrowly convergence.*

(2) *$\mathcal{P}_R(m)$ is non-empty.*

(3) *For any $P \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$, the probability measure $P$ satisfies the condition (7) if and only if for each state-measure pair $(\gamma, \mu) \in \mathrm{spt}(P)$, the pair satisfies*

$$\dot{\gamma}(t) = \int_{\mathbb{R}^n} f\left(\gamma(t), u, m(t)\right) \mu_t(du). \tag{11}$$

**Proof.** (1) Since $\Gamma_T^R$ and $\mathcal{P}_U^R$ are compact, the set $\mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ is also compact with respect to the narrowly convergence. Let $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}_R(m)$ such that $P_i$ converges to some $P \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ narrowly. Then it is clear that $e_0\sharp\left(\pi_1\sharp P\right) = m_0$.

For each $(t, v) \in [0, T] \times \mathbb{R}^d$, define the function $\kappa : \Gamma_T^R \times \mathcal{P}_U^R \to \mathbb{R}$ by

$$(\gamma, \mu) \mapsto \left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu\left(ds, du\right)\right\rangle. \tag{12}$$

We claim that the function $\kappa$ is bounded and continuous. It is obviously that the function

$$(\gamma, \mu) \mapsto \langle v, \gamma(t) - \gamma(0)\rangle$$

is bounded and continuous. Additionally, by Remark 2.2, the function

$$(\gamma, \mu) \mapsto \left\langle v, \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu(ds, du)\right\rangle \tag{13}$$

is bounded as well. On the other hand, let $\gamma_i \to \gamma$ in $\Gamma_T^R$ and $\mu^i \to \mu$ in $\mathcal{P}_U^R$. Consider

$$\begin{aligned} A_1 :=& \left|\int_0^t \int_{\mathbb{R}^n} f\left(\gamma_i(s), u, m(s)\right) \mu^i(ds, du) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu(ds, du)\right| \\ \leq& \int_0^t \int_{\mathbb{R}^n} \left|f\left(\gamma_i(s), u, m(s)\right) - f\left(\gamma(s), u, m(s)\right)\right| \mu^i(du, ds) \\ &+ \left|\int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu^i(ds, du) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu(ds, du)\right| \\ \triangleq& A_2 + A_3. \end{aligned}$$

By (F2),

$$\lim_{i\to\infty} A_2 \leq \lim_{i\to\infty} \int_0^t \int_{\mathbb{R}^n} \mathrm{Lip}_x(f) \left\|\gamma_i - \gamma\right\|_\infty \mu^i(du, ds) \leq \lim_{i\to\infty} \mathrm{Lip}_x(f) T \left\|\gamma_i - \gamma\right\|_\infty = 0.$$

By (F1) and Proposition B.2, $\lim_{i\to\infty} A_3 = 0$. We conclude that the function $\kappa$ is both bounded and continuous.

For any open set $N \subset \Gamma_T^R \times \mathcal{P}_U^R$, we have

$$\begin{aligned} &\left|\int_N \left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu\left(ds, du\right)\right\rangle P\left(d\gamma, d\mu\right)\right| \\ \leq& \liminf_{i\to\infty} \int_{\Gamma_T^R \times \mathcal{P}_U^R} \mathbf{1}_N\left(\gamma, \mu\right) \left|\left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu(ds, du)\right\rangle\right| P_i(d\gamma, d\mu) \\ =& \liminf_{i\to\infty} \int_{N^+} \left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu\left(ds, du\right)\right\rangle P_i\left(d\gamma, d\mu\right) \\ &- \int_{N^-} \left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) \mu\left(ds, du\right)\right\rangle P_i\left(d\gamma, d\mu\right) = 0, \end{aligned}$$

where the sets $N^+$ and $N^-$ are defined by

$$N^+ := \{(\gamma,\mu) \in N \mid \kappa(\gamma,\mu) > 0\},$$

$$N^- := \{(\gamma,\mu) \in N \mid \kappa(\gamma,\mu) < 0\}.$$

Since the function $\kappa$ is continuous, both $N^+$ and $N^-$ are open sets. In conclusion, $P \in \mathcal{P}_R(m)$, and $\mathcal{P}_R(m)$ is closed. Furthermore, since $\mathcal{P}_R(m) \subset \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ and $\mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ is compact, the set $\mathcal{P}_R(m)$ is also compact with respect to the narrowly convergence.

(2) Let $\mu_0 = \mathcal{L}_{[0,T]} \otimes \delta_{\{0\}} \in \mathcal{P}_U^R$. The state equation (4) is transformed into an autonomous ordinary differential equation

$$\begin{cases} \dot{\gamma}(t) = f(\gamma(t), 0, m(t)), \\ \gamma(0) = x. \end{cases} \tag{14}$$

For any $x \in \mathbb{T}^d$, there exists a unique solution $\gamma_x(t) = \gamma_x(t; x, \mu_0, m)$ of (14) by (F2). Define the map $p\colon \mathbb{T}^d \to \Gamma_T^R$ by $x \mapsto p(x) = \gamma_x$ and $P := p\sharp m_0 \otimes \delta_{\{\mu_0\}} \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$. It is clear that $P \in \mathcal{P}_R(m)$.

(3) Let $P$ be the probability measure that satisfies the condition (7). Suppose there exist $(\gamma_0, \mu^0) \in \operatorname{spt}(P)$ and $t_0 \in [0,T]$ such that

$$\gamma_0(t_0) - \gamma_0(0) \neq \int_0^{t_0} \int_{\mathbb{R}^n} f(\gamma_0(s), u, m(s))\, \mu^0(ds, du).$$

Without loss of generality, assume there exists $v_0 \in \mathbb{R}^d$ such that

$$\left\langle v_0, \gamma_0(t_0) - \gamma_0(0) - \int_0^{t_0} \int_{\mathbb{R}^n} f(\gamma_0(s), u, m(s))\, \mu^0(ds, du) \right\rangle > 0.$$

Similar to the proof of the continuity of $\kappa$, the map

$$(\gamma,\mu) \mapsto \left\langle v_0, \gamma(t_0) - \gamma(0) - \int_0^{t_0} \int_{\mathbb{R}^n} f(\gamma(s), u, m(s))\, \mu(ds, du) \right\rangle$$

is also continuous. Hence, there exists an open neighborhood $N$ of the point $(\gamma_0, \mu^0)$ such that $P(N) > 0$ and

$$\int_N \left\langle v_0, \gamma(t_0) - \gamma(0) - \int_0^{t_0} \int_{\mathbb{R}^n} f(\gamma(s), u, m(s))\, \mu(ds, du) \right\rangle P(d\gamma, d\mu) > 0,$$

which contradicts (7). Therefore, we have

$$\gamma(t) - \gamma(0) = \int_0^t \int_{\mathbb{R}^n} f(\gamma(s), u, m(s))\, \mu(ds, du), \quad \forall (\gamma,\mu) \in \operatorname{spt}(P), \forall t \in [0,T].$$

Taking the derivative with respect to $t$ on both sides gives us (11).

The other point can be easily proved by definition. □

### 2.2. Proof of the existence of relaxed MFC equilibria

In this section, we prove the existence of relaxed MFC equilibria by Kakutani's theorem.

Define a subset $\mathcal{P}_0\left(\Gamma_T^R\right) \subset \mathcal{P}\left(\Gamma_T^R\right)$ by

$$\mathcal{P}_0\left(\Gamma_T^R\right) := \left\{\eta \in \mathcal{P}\left(\Gamma_T^R\right) \mid e_0\sharp\eta = m_0\right\}.$$

Obviously, $\mathcal{P}_0\left(\Gamma_T^R\right)$ is compact with respect to the $d_1$-topology. Define the set-valued map by

$$E : \left(\mathcal{P}_0\left(\Gamma_T^R\right), d_1\right) \rightrightarrows \left(\mathcal{P}_0\left(\Gamma_T^R\right), d_1\right), \quad \eta \mapsto E(\eta),$$

where

$$E(\eta) := \left\{\pi_1\sharp P \mid P \in R^*\Big( (e_t\sharp\eta)_{t\in[0,T]} \Big)\right\}.$$

**Remark 2.6.** Let $\eta \in \mathcal{P}_0\left(\Gamma_T^R\right)$. Define the map $m : [0,T] \to \mathcal{P}\left(\mathbb{T}^d\right)$ by $m(t) = e_t\sharp\eta$ for any $t \in [0,T]$. It is clear that $m \in \mathbb{M}_r$. More precisely, it is clear that $m(0) = e_0\sharp\eta = m_0$. For any $t_1 \neq t_2 \in [0,T]$ and any 1-Lipschitz function $g : \mathbb{T}^d \to \mathbb{R}$, we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d} g(x)e_{t_1}\sharp\eta(dx) - \int_{\mathbb{T}^d} g(x)e_{t_2}\sharp\eta(dx)\right| \\
&\leq \int_{\Gamma_T^R} \left|g\left(\gamma(t_1)\right) - g\left(\gamma(t_2)\right)\right| \eta\left(d\gamma\right) \\
&\leq \int_{\Gamma_T^R} \left|\gamma(t_1) - \gamma(t_2)\right| \eta\left(d\gamma\right) \leq K\left|t_1 - t_2\right|^{\frac{1}{r}},
\end{aligned}$$

where the last inequality comes from Lemma 2.1. By the arbitrariness of the function $g$, we have

$$d_1\left(m(t_1), m(t_2)\right) \leq K\left|t_1 - t_2\right|^{\frac{1}{r}}.$$

Thus, $m \in \mathbb{M}_r$. Therefore, the set-valued map $E$ is well-defined.

**Lemma 2.7.** *For each $m \in \mathbb{M}_r$,*

(1) *the family of functions $\left\{J^m\left(\cdot, \mu\right) : \Gamma_T^R \to \mathbb{R} \mid \mu \in \mathcal{P}_U^R\right\}$ is uniformly equi-continuous.*

(2) *Let $\gamma \in \Gamma_T^R$. If $\mu^i$ converges to $\mu$ narrowly, then*

$$\liminf_{i\to\infty} J^m\left(\gamma, \mu^i\right) \geq J^m\left(\gamma, \mu\right). \tag{15}$$

**Proof.** (1) Fix a sequence $\{\gamma_i\}_{i=1}^\infty$ that converges to some $\gamma$ in $\Gamma_T^R$. For any $\mu \in \mathcal{P}_U^R$, we have

$$\begin{aligned}
&\left|J^m\left(\gamma_i, \mu\right) - J^m\left(\gamma, \mu\right)\right| \leq \int_0^T \int_{\mathbb{R}^n} \left|L\left(\gamma_i(t), u, m(t)\right) - L\left(\gamma(t), u, m(t)\right)\right| \mu(dt, du) \\
&\leq \|\gamma_i - \gamma\|_\infty \int_0^T \int_{\mathbb{R}^n} \int_0^1 \left|\frac{\partial L}{\partial x}\left(\gamma(t) + \lambda\left(\gamma_i(t) - \gamma(t)\right), u, m(t)\right)\right| d\lambda\, \mu(dt, du).
\end{aligned}$$

By (L4), the proof is completed since we obtain that

$$\int_0^T \int_{\mathbb{R}^n} \int_0^1 \left|\frac{\partial L}{\partial x}\left(\gamma(t) + \lambda\left(\gamma_i(t) - \gamma(t)\right), u, m(t)\right)\right| d\lambda\, \mu(dt, du) \leq C_2\left(T + R\right).$$

(2) For any $\varepsilon > 0$, we obtain that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} L(\gamma(t), u, m(t))\, \mu^i(dt, du) &= \int_0^T \int_{\mathbb{R}^n} \frac{L(\gamma(t), u, m(t))}{1+\varepsilon|u|^q} \left(1+\varepsilon|u|^q\right) \mu^i(dt, du) \\ &\geq \int_0^T \int_{\mathbb{R}^n} \frac{L(\gamma(t), u, m(t))}{1+\varepsilon|u|^q} \mu^i(dt, du).\end{aligned}$$

By (L1) and (L3), the map

$$(t, u) \mapsto \frac{L(\gamma(t), u, m(t))}{1+\varepsilon|u|^q}$$

is bounded and continuous. Thus,

$$\liminf_{n\to\infty} \int_0^T \int_{\mathbb{R}^n} L(\gamma(t), u, m(t))\, \mu^i(dt, du) \geq \int_0^T \int_{\mathbb{R}^n} \frac{L(\gamma(t), u, m(t))}{1+\varepsilon|u|^q} \mu(dt, du).$$

Let $\varepsilon$ tend to 0. The proof is complete. □

For any $x \in \mathbb{T}^d$, denote by $\Gamma_m^*(x)$ the set of curves associated with an optimal control $\mu^*$, i.e.,

$$\Gamma_m^*(x) := \left\{\gamma^* \in \Gamma_T^R \mid \gamma^* = \gamma^*(\cdot; x, \mu^*, m),\ J^m(\gamma^*, \mu^*) = \inf_{\substack{\mu\in\mathcal{P}_U^R \\ \gamma=\gamma(\cdot;x,\mu,m)}} J^m(\gamma, \mu)\right\}.$$

Define the map $F_m : \mathbb{T}^d \to \mathcal{B}\left(\Gamma_T^R\right)$ by $F_m(x) := \Gamma_m^*(x)$ for all $x \in \mathbb{T}^d$.

**Lemma 2.8.** *Let $m \in \mathbb{M}_r$. Let the sequence $\{x_i\}_{i=1}^\infty$ converge to $x$ in $\mathbb{T}^d$. If there exists $\gamma_{x_i} \in F_m(x_i)$ for each $i \in \mathbb{N}$ such that the sequence $\{\gamma_{x_i}\}_{i=1}^\infty$ converges to some $\gamma$ in $\Gamma_T^R$, then $\gamma \in F_m(x)$.*

**Proof.** Denote by $\mu_{x_i}$ the measure such that $\gamma_{x_i} = \gamma_{x_i}(\cdot; x_i, \mu_{x_i}, m)$ for any $i \in \mathbb{N}$. Due to the compactness of $\mathcal{P}_U^R$, there exists a measure $\bar{\mu}$ such that $\tilde{d}_1(\mu_{x_i}, \bar{\mu})$ converges to 0 as $i \to \infty$. Let $\tilde{\gamma} = \tilde{\gamma}(\cdot; x, \tilde{\mu}, m)$ and $\tilde{\gamma}_{x_i} = \tilde{\gamma}_{x_i}(\cdot; x_i, \tilde{\mu}, m)$ for any $\tilde{\mu} \in \mathcal{P}_U^R$. Since $x_i$ converges to $x$ in $\mathbb{T}^d$, we have $\tilde{\gamma}_{x_i}$ converges to $\tilde{\gamma}$ in $\Gamma_T^R$. Moreover, since $\gamma_{x_i} \in \Gamma_m^*(x_i)$, we obtain that

$$J^m(\gamma_{x_i}, \mu_{x_i}) \leq J^m(\tilde{\gamma}_{x_i}, \tilde{\mu}). \tag{16}$$

Let $i$ tend to infinity. By Lemma 2.7, we have $J^m(\gamma, \bar{\mu}) \leq J^m(\tilde{\gamma}, \tilde{\mu})$.

The only thing left to prove is that $\gamma = \gamma(\cdot; x, \bar{\mu}, m)$, i.e.,

$$\gamma(t) = x + \int_0^t \int_{\mathbb{R}^n} f(\gamma(s), u, m(s))\, \bar{\mu}(ds, du).$$

Since $\gamma_{x_i}(t) = x_i + \int_0^t \int_{\mathbb{R}^n} f(\gamma_{x_i}(s), u, m(s))\, \mu_{x_i}(ds, du)$, the proof is complete by (F1), (F2) and Proposition B.2. □

**Proposition 2.9.** *For each $m \in \mathbb{M}_r$,*

(1) *the set $\Gamma^*_m(x)$ is non-empty for any $x \in \mathbb{T}^d$;*

(2) *the map $F_m$ has a closed graph.*

**Proof.** The non-emptiness of $\Gamma^*_m(x)$ can easily be proved by using the convexity of $J^m(\gamma,\mu)$ with respect to $\mu$. See [10] for examples. Point (2) is a direct consequence of Lemma 2.8. □

**Proposition 2.10.** *For any $\eta \in \mathcal{P}_0\left(\Gamma^R_T\right)$, the set $E(\eta)$ is non-empty and convex.*

**Proof.** Let $m(t) = e_t\sharp\eta$ for any $t \in [0,T]$. By Proposition 2.9 and [8, Proposition 9.5], the map $F_m$ is measurable. Thus, there exists a measurable selection $\tilde{\gamma}_x \in \Gamma^*_m(x)$ by [16, Chapter3, Theorem 5.3]. Define the measure $\tilde{\eta} \in \mathcal{P}_0\left(\Gamma^R_T\right)$ by

$$\tilde{\eta}(A) := \int_{\mathbb{T}^d} \delta_{\{\tilde{\gamma}_x\}}(A) m_0(dx), \quad \forall A \in \mathcal{B}\left(\Gamma^R_T\right).$$

We claim that $\tilde{\eta} \in E(\eta)$. Define the map $h : \left\{\tilde{\gamma}_x \mid x \in \mathbb{T}^d\right\} \to \Gamma^R_T \times \mathcal{P}^R_U$ by

$$\tilde{\gamma}_x \mapsto (\tilde{\gamma}_x, \mu^x), \text{where } \dot{\tilde{\gamma}}_x(t) = \int_{\mathbb{R}^n} f\left(\tilde{\gamma}_x(t), u, m(t)\right)\mu^x_t(du), \quad \forall t \in [0,T].$$

It is clear that $\tilde{\eta} = \pi_1\sharp\left(h\sharp\tilde{\eta}\right)$. Hence, we only need to verify that $h\sharp\tilde{\eta} \in R^*(m)$. Indeed, we have $h\sharp\tilde{\eta} \in \mathcal{P}\left(\Gamma^R_T \times \mathcal{P}^R_U\right)$ and $e_0\sharp\tilde{\eta} = m_0$ by definition. For any pair $(\gamma,\mu) \in \operatorname{spt}\left(h\sharp\tilde{\eta}\right)$, the pair satisfies $\dot{\gamma} = \gamma\left(\cdot;\gamma(0),\mu,m\right)$. By Proposition 2.5(3), we obtain that $h\sharp\tilde{\eta} \in \mathcal{P}_R(m)$.

Regarding the minimization property of $h\sharp\tilde{\eta}$, we first define $\Gamma^R_T(m) \subset \Gamma^R_T$ by

$$\Gamma^R_T(m) := \left\{\gamma \in \Gamma^R_T \,\middle|\, \exists (x,\mu) \in \mathbb{T}^d \times \mathcal{P}^R_U, s.t.\ \gamma = \gamma\left(\cdot;x,\mu,m\right)\right\}.$$

By Proposition 2.5(3), for any $\tilde{P} \in \mathcal{P}_R(m)$, the support of $\tilde{P}$ satisfies the inclusion $\operatorname{spt}(\tilde{P}) \subset \Gamma^R_T(m) \times \mathcal{P}^R_U$. Thus, we obtain that

$$\begin{aligned} J\left(m, h\sharp\tilde{\eta}\right) &= \int_{\Gamma^R_T \times \mathcal{P}^R_U} J^m(\gamma,\mu)\, h\sharp\tilde{\eta}\,(d\gamma,d\mu) = \int_{\mathbb{T}^d} J^m\left(\tilde{\gamma}_x,\mu^x\right) m_0(dx) \\ &= \int_{\mathbb{T}^d} J^m\left(\tilde{\gamma}_x,\mu^x\right) e_0\sharp\left(\pi_1\sharp\tilde{P}\right)(dx) = \int_{\Gamma^R_T(m) \times \mathcal{P}^R_U} J^m\left(\tilde{\gamma}_{\gamma(0)},\mu^{\gamma(0)}\right)\tilde{P}\,(d\gamma,d\mu) \\ &\le \int_{\Gamma^R_T(m) \times \mathcal{P}^R_U} J^m(\gamma,\mu)\,\tilde{P}\,(d\gamma,d\mu) = \int_{\Gamma^R_T \times \mathcal{P}^R_U} J^m(\gamma,\mu)\,\tilde{P}\,(d\gamma,d\mu) = J\left(m,\tilde{P}\right). \end{aligned}$$

Therefore, the measure $\tilde{\eta} \in E(\eta)$, which implies the non-emptiness of $E(\eta)$.

The convexity can be easily proved by definition. □

**Proposition 2.11.** *Let* $\{\eta_i\}_{i=1}^{\infty} \subset \mathcal{P}_0\left(\Gamma_T^R\right)$ *and* $\eta \in \mathcal{P}_0\left(\Gamma_T^R\right)$ *such that* $\eta_i$ *converges to* $\eta$ *narrowly. If there exists* $\hat{\eta}_i \in E(\eta_i)$ *for any* $i \in \mathbb{N}$ *such that* $\hat{\eta}_i$ *converges to some* $\hat{\eta}$ *narrowly, then* $\hat{\eta} \in E(\eta)$.

**Proof.** For any $t \in [0,T]$, define $m^i(t) = e_t\sharp\eta_i$ and $m(t) = e_t\sharp\eta$. By (L2),

$$\left|J^{m^i}(\gamma,\mu) - J^m(\gamma,\mu)\right| \leq \int_0^T \int_{\mathbb{R}^n} \left|L\left(\gamma(t),u,m^i(t)\right) - L\left(\gamma(t),u,m(t)\right)\right| \mu_t(du)dt$$

$$\leq \int_0^T \int_{\mathbb{R}^n} \omega\left(d_1\left(m^i(t),m(t)\right)\right)\mu_t(du)dt = \int_0^T \omega\left(d_1\left(m^i(t),m(t)\right)\right)dt.$$

Indeed, since $\eta_i$ converges to $\eta$ narrowly and $\Gamma_T^R$ is compact, we have $d_1(\eta_i,\eta) \to 0$. Furthermore, we obtain that $d_1(m^i(t),m(t)) \leq d_1(\eta_i,\eta)$ for any $t \in [0,T]$ by definition. We obtain that

$$\left|J^{m^i}(\gamma,\mu) - J^m(\gamma,\mu)\right| \leq T\omega\left(d_1(\eta_i,\eta)\right). \tag{17}$$

Thererfore, the function $J^{m^i}(\gamma,\mu)$ uniformly converges to the function $J^m(\gamma,\mu)$ as $i \to \infty$. By (17), the function $\tilde{P} \mapsto J\big(m^i,\tilde{P}\big)$ also uniformly converges to the function $\tilde{P} \mapsto J\big(m,\tilde{P}\big)$ as $i \to \infty$, i.e.,

$$\lim_{i\to\infty} \sup_{\tilde{P}\in\mathcal{P}\left(\Gamma_T^R\times\mathcal{P}_U^R\right)} \left|J\left(m^i,\tilde{P}\right) - J\left(m,\tilde{P}\right)\right| = 0. \tag{18}$$

For any $i \in \mathbb{N}$, since $\hat{\eta}_i \in E(\eta_i)$, there exists $P_i \in \mathcal{P}_R(m^i)$ such that $\pi_1\sharp P_i = \hat{\eta}_i$. Since $\mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ is compact with respect to the $d_1$-topology, there exists some $P \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ such that $d_1(P_i,P) \to 0$ as $i \to \infty$. It is obviously that $\pi_1\sharp P = \hat{\eta}$ and $e_0\sharp\pi_1\sharp P = e_0\sharp\hat{\eta} = m_0$. For any $(t,v) \in [0,T]\times\mathbb{R}^d$ and any open set $N \subset \Gamma_T^R\times\mathcal{P}_U^R$, consider

$$\begin{aligned}&\left|\int_N \left\langle v, \gamma(t)-\gamma(0) - \int_0^t\int_{\mathbb{R}^n} f\left(\gamma(s),u,m^i(s)\right)\mu(ds,du)\right\rangle P_i(d\gamma,d\mu)\right.\\&\left.- \int_N \left\langle v, \gamma(t)-\gamma(0) - \int_0^t\int_{\mathbb{R}^n} f\left(\gamma(s),u,m(s)\right)\mu(ds,du)\right\rangle P(d\gamma,d\mu)\right| \leq B_1 + B_2,\end{aligned}$$

where

$$\begin{aligned}B_1 :=& \left|\int_N \left\langle v, \gamma(t)-\gamma(0) - \int_0^t\int_{\mathbb{R}^n} f\left(\gamma(s),u,m^i(s)\right)\mu(ds,du)\right\rangle P_i(d\gamma,d\mu)\right.\\&\left.- \int_N \left\langle v, \gamma(t)-\gamma(0) - \int_0^t\int_{\mathbb{R}^n} f\left(\gamma(s),u,m(s)\right)\mu(ds,du)\right\rangle P_i(d\gamma,d\mu)\right|,\end{aligned}$$

$$\begin{aligned}B_2 :=& \left|\int_N \left\langle v, \gamma(t)-\gamma(0) - \int_0^t\int_{\mathbb{R}^n} f\left(\gamma(s),u,m(s)\right)\mu(ds,du)\right\rangle P_i(d\gamma,d\mu)\right.\\&\left.- \int_N \left\langle v, \gamma(t)-\gamma(0) - \int_0^t\int_{\mathbb{R}^n} f\left(\gamma(s),u,m(s)\right)\mu(ds,du)\right\rangle P(d\gamma,d\mu)\right|.\end{aligned}$$

Similar to the proof of Proposition 2.5(1), we have $\lim_{i\to\infty} B_2 = 0$. At the same time, since

$$\begin{aligned}B_1 &= \left|\int_N \left\langle v, \int_0^t \int_{\mathbb{R}^n} f\left(\gamma(s), u, m(s)\right) - f\left(\gamma(s), u, m^i(s)\right) \mu(ds, du)\right\rangle P_i\left(d\gamma, d\mu\right)\right| \\ &\leq \int_N |v| \int_0^t \mathrm{Lip}_\nu(f) d_1\left(\eta_i, \eta\right) ds\, P_i\left(d\gamma, d\mu\right) \leq |v|\, \mathrm{Lip}_\nu(f) T d_1\left(\eta_i, \eta\right),\end{aligned}$$

we have $\lim_{i\to\infty} B_1 = 0$. Since $P_i \in \mathcal{P}_R\left(m^i\right)$, we obtain that $P \in \mathcal{P}_R(m)$.

Consider

$$J\left(m^i, P_i\right) - J\left(m, P\right) = \left(J\left(m^i, P_i\right) - J\left(m, P_i\right)\right) + \left(J\left(m, P_i\right) - J\left(m, P\right)\right) \triangleq B_3 + B_4.$$

By (18), $\lim_{i\to\infty} B_3 = 0$. Since the function $(\gamma, \mu) \mapsto J^m\left(\gamma, \mu\right)$ is bounded and lower semi-continuous by Lemma 2.7, it follows from Corollary B.4 that $\liminf_{i\to\infty} B_4 \geq 0$.

In conclusion, for any $\hat{P} \in \mathcal{P}_R(m)$, we have

$$J\left(m, P\right) \leq \liminf_{i\to\infty} J\left(m^i, P_i\right) \leq \liminf_{i\to\infty} J\left(m^i, \hat{P}\right) = J\left(m, \hat{P}\right).$$

Due to the arbitrariness of $\hat{P} \in \mathcal{P}_R(m)$, we complete the proof. □

**Corollary 2.12.** *For any $\eta \in \mathcal{P}_0\left(\Gamma_T^R\right)$, the set $E\left(\eta\right)$ is compact.*

**Proof.** By Proposition 2.11, the set-valued map $E$ has a closed graph, which ensures that $E\left(\eta\right)$ is a closed set. Furthermore, $E(\eta)$ is contained within the compact set $\mathcal{P}_0\left(\Gamma_T^R\right)$. Consequently, $E\left(\eta\right)$ is also compact. □

**Proof of Theorem 1.3.** Combining Proposition 2.10 and Corollary 2.12, the set $E\left(\eta\right)$ is non-empty, convex and compact for any $\eta \in \mathcal{P}_0\left(\Gamma_T^R\right)$. Furthermore, $\mathcal{P}_0\left(\Gamma_T^R\right)$ is non-empty and compact. By Proposition 2.11, the set-valued map $E$ has a closed graph, which indicates the continuity of the map. Consequently, there exists a measure $\bar{\eta} \in \mathcal{P}_0\left(\Gamma_T^R\right)$, such that $\bar{\eta} \in E\left(\bar{\eta}\right)$ by Kakutani's theorem. Thus, there exists $P \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ such that $\bar{\eta} = \pi_1 \sharp P$ and $P \in R^*\left(\left(e_t \sharp \bar{\eta}\right)_{t\in[0,T]}\right)$. Therefore, the probability measure $P$ belongs to the set $R^*\left(\left(e_t \sharp \pi_1 \sharp P\right)_{t\in[0,T]}\right)$, indicating that $P$ is a relaxed MFC equilibrium. □

## 3. Existence of strict relaxed MFC equilibria

**Lemma 3.1.** *Fix $m \in \mathbb{M}_r$. For any $x \in \mathbb{T}^d$ and any $u_x : [0,T] \to \mathbb{R}^n$ such that $\int_0^T \left|u_x(t)\right|^q dt \leq R$, denote by $\gamma_x^{u_x} = \gamma_x^{u_x}\left(\cdot; x, \mathcal{L}_{[0,T]} \otimes \delta_{u_x(t)}, m\right)$ the solution of the state equation (4). Define a probability measure $\eta^u \in \mathcal{P}_0\left(\Gamma_T^R\right)$ by*

$$\eta^u(A) := \int_{\mathbb{T}^d} \delta_{\left\{\gamma_x^{u_x}\right\}}(A) m_0(dx), \quad \forall A \in \mathcal{B}\left(\Gamma_T^R\right).$$

*For any $\eta \in \mathcal{P}_0\left(\Gamma_T^R\right)$, define $P_\eta := \eta \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{u_{\gamma(0)}(t)}\right\}}$. Then, $P_\eta \in \mathcal{P}_R(m)$ if and only if $\eta = \eta^u$. Denote $P^u := \eta^u \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{u_{\gamma(0)}(t)}\right\}} \in \mathcal{P}_R(m)$.*

**Proof.** Obviously, $P^u \in \mathcal{P}_R(m)$ by definition.

Suppose $\eta \in \mathcal{P}_0\left(\Gamma_T^R\right)$ and $P_\eta \in \mathcal{P}_R(m)$. For any $\gamma \in \operatorname{spt}(\eta)$, by Proposition 2.5(4), there exists $x \in \mathbb{T}^d$ such that $\gamma(0) = x$ and $\gamma = \gamma_x^{u_x}$. By $e_0\sharp\eta = e_0\sharp\eta^u = m_0$, for any $g \in \mathcal{C}_b\left(\Gamma_T^R\right)$,

$$\int_{\Gamma_T^R} g(\gamma)\eta(d\gamma) = \int_{\Gamma_T^R} g(\gamma_{\gamma(0)}^{u_{\gamma(0)}})\eta(d\gamma) = \int_{\mathbb{T}^d} g(\gamma_x^{u_x})m_0(dx) = \int_{\Gamma_T^R} g(\gamma)\eta^u(d\gamma).$$

Thus, $\eta = \eta^u$. $\square$

**Proposition 3.2.** *Fix $m \in \mathbb{M}_r$ and $P \in R^*(m)$. There exists a map*

$$\hat{q} : [0,T] \times \mathbb{T}^d \to \mathcal{P}(\mathbb{R}^n)$$

*such that $\hat{P} := \pi_1\sharp P \otimes \delta_{\left\{\mathcal{L}_{[0,T]}\otimes\hat{q}_{t,\gamma(t)}\right\}} \in R^*(m)$, where we denote $\hat{q}_{t,x} = \hat{q}(t,x)$ for convenience.*

**Proof.** Define a measure $\eta$ on $[0,T] \times \mathbb{T}^d \times \mathbb{R}^n$ by

$$\eta(C) := \frac{1}{T}\int_{\Gamma_T^R\times\mathcal{P}_U^R}\int_0^T\int_{\mathbb{R}^n} \mathbf{1}_C(t,\gamma(t),u)\,\mu(dt,du)\,P(d\gamma,d\mu),$$

for all $C \in \mathcal{B}\left([0,T]\times\mathbb{T}^d\times\mathbb{R}^n\right)$. Define $\hat{\eta}_t := e_t\sharp(\pi_1\sharp P)$, which is a probability measure on $\mathbb{T}^d$. Define $\eta_{1,2} := \frac{1}{T}\mathcal{L}_{[0,T]} \otimes \hat{\eta}_t(dx)$, which is a probability measure on $[0,T]\times\mathbb{T}^d$. By Theorem B.5, construct the map $\hat{q}$ by

$$\eta(dt,dx,du) = \eta_{1,2}(dt,dx) \otimes \hat{q}_{t,x}(du).$$

Define the measure $\hat{P}$ by $\hat{P} := \pi_1\sharp P \otimes \delta_{\left\{\mathcal{L}_{[0,T]}\otimes\hat{q}_{t,\gamma(t)}\right\}} \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$. Then, we obtain that $\hat{P} \in \mathcal{P}_R(m)$ by Appendix C. Furthermore,

$$\begin{aligned}
J(m,\hat{P}) &= \int_{\Gamma_T^R}\int_0^T\int_{\mathbb{R}^n} L(\gamma(t),u,m(t))\,\hat{q}_{t,\gamma(t)}(du)\;dt\,\pi_1\sharp P(d\gamma)\\
&= \int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}^n} L(x,u,m(t))\,\hat{q}_{t,x}(du)\,e_t\sharp(\pi_1\sharp P)(dx)\,dt\\
&= T\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}^n} L(x,u,m(t))\,\eta(dt,dx,du)\\
&= \int_{\Gamma_T^R\times\mathcal{P}_U^R}\int_0^T\int_{\mathbb{R}^n} L(\gamma(t),u,m(t))\,\mu(dt,du)P(d\gamma,d\mu) = J(m,P).
\end{aligned}$$

Therefore, we have $\hat{P} \in R^*(m)$. $\square$

**Proof of Theorem 1.5.** (1) Define the map $\hat{q} : [0,T]\times\mathbb{T}^d \to \mathcal{P}(\mathbb{R}^n)$ by Proposition 3.2. By the convexity condition, for any $(t,x) \in [0,T]\times\mathbb{T}^d$, we have

$$\int_{\mathbb{R}^n} \left(L(x,u,m(t)), |u|^q, f(x,u,m(t))\right)\hat{q}_{t,x}(du) \in \mathcal{L}(t,x).$$

By [21, Theorem A.9], there exist negative measurable functions

$$z_1 : [0,T] \times \mathbb{T}^d \to \mathbb{R}^- \quad \text{and} \quad z_2 : [0,T] \times \mathbb{T}^d \to \mathbb{R}^-$$

and a measurable map $\hat{\alpha} : [0,T] \times \mathbb{T}^d \to \mathbb{R}^n$ such that

$$\begin{aligned} L\left(x, \hat{\alpha}(t,x), m(t)\right) &= \int_{\mathbb{R}^n} L\left(x, u, m(t)\right) \hat{q}_{t,x}(du) + z_1(t,x), \\ |\hat{\alpha}(t,x)|^q &= \int_{\mathbb{R}^n} |u|^q \hat{q}_{t,x}(du) + z_2(t,x), \\ f\left(x, \hat{\alpha}(t,x), m(t)\right) &= \int_{\mathbb{R}^n} f\left(x, u, m(t)\right) \hat{q}_{t,x}(du). \end{aligned}$$

Define the probability measure $P_0 := \pi_1 \sharp P \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{\hat{\alpha}(t,\gamma(t))}\right\}} \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$. The measure $P_0$ is well-defined since for any $\gamma \in \operatorname{spt}(\pi_1 \sharp P)$, we have

$$\begin{aligned} \int_0^T |\hat{\alpha}(t,\gamma(t))|^q \, dt &= \int_0^T \int_{\mathbb{R}^n} |u|^q \hat{q}_{t,\gamma(t)}(du) dt + \int_0^T z_2(t,\gamma(t)) dt \\ &\leq \int_0^T \int_{\mathbb{R}^n} |u|^q \hat{q}_{t,\gamma(t)}(du) dt \leq R. \end{aligned}$$

It is clear that $e_0 \sharp \left(\pi_1 \sharp P_0\right) = e_0 \sharp \left(\pi_1 \sharp P\right) = m_0$ and $P_0 \in \mathcal{P}_R(m)$. Besides,

$$\begin{aligned} J\left(m, P_0\right) &= \int_{\Gamma_T^R} \int_0^T \int_{\mathbb{R}^n} L\left(\gamma(t), u, m(t)\right) \delta_{\hat{\alpha}(t,\gamma(t))}\left(du\right) \, dt \, \pi_1 \sharp P\left(d\gamma\right) \\ &= \int_{\Gamma_T^R} \int_0^T L\left(\gamma(t), \hat{\alpha}\left(t, \gamma(t)\right), m(t)\right) \, dt \, \pi_1 \sharp P\left(d\gamma\right) \\ &= \int_{\Gamma_T^R} \int_0^T \int_{\mathbb{R}^n} L\left(\gamma(t), u, m(t)\right) \hat{q}_{t,\gamma(t)}(du) \, dt \, \pi_1 \sharp P\left(d\gamma\right) \\ &\quad + \int_{\Gamma_T^R} \int_0^T z_1\left(t, \gamma(t)\right) dt \, \pi_1 \sharp P\left(d\gamma\right) \leq J\left(m, \hat{P}\right) = J\left(m, P\right). \end{aligned}$$

In conclusion, $P_0 \in R^*(m)$. Since $P_0 \in \mathcal{P}_R(m)$, for any $\gamma \in \operatorname{spt}(\pi_1 \sharp P)$, it satisfies

$$\dot{\gamma}(t) = f\left(\gamma(t), \hat{\alpha}\left(t, \gamma(t)\right), m(t)\right). \tag{19}$$

Denote by $\gamma_x^{\hat{\alpha}}$ the solution of (19) with $\gamma_x^{\hat{\alpha}}(0) = x$. For any $x \in \mathbb{T}^d$ we define the maps $u_x^*(t) := \hat{\alpha}\left(t, \gamma_x^{\hat{\alpha}}(t)\right)$. Then, by $e_0 \sharp \pi_1 \sharp P = m_0$, it is clear that

$$J(m, P_0) = \int_{\mathbb{T}^d} \int_0^T L\left(\gamma_x^{u_x^*}(t), u_x^*(t), m(t)\right) dt \, m_0(dx),$$

where $$\dot{\gamma}_x^{u_x^*}(t) = f\left(\gamma_x^{u_x^*}(t), u_x^*(t), m(t)\right), \quad \gamma_x^{u_x^*}(0) = x.$$

Following Lemma 3.1, for any $x \in \mathbb{T}^d$ and any map $u_x : [0,T] \to \mathbb{R}^n$ such that $\int_0^T |u_x(t)|^q \, dt \leq R$, we have $P^u = \eta^u \otimes \delta_{\left\{\mathcal{L}_{[0,T]} \otimes \delta_{u_{\gamma(0)}(t)}\right\}} \in \mathcal{P}_R(m)$. Thus,

$$\int_{\mathbb{T}^d} \int_0^T L\left(\gamma_x^{u_x}(t), u_x(t), m(t)\right) dt \, m_0(dx) \geq \int_{\mathbb{T}^d} \int_0^T L\left(\gamma_x^{u_x^*}(t), u_x^*(t), m(t)\right) dt \, m_0(dx).$$

(2) By Theorem 1.3, there exists a relaxed MFC equilibrium $P \in R^*(m)$, where $m(t) = e_t\sharp(\pi_1\sharp P)$ for any $t \in [0,T]$. As a consequence of the point (1) of this theorem, $P_0 \in R^*(m)$. Define $m^0(t) := e_t\sharp(\pi_1\sharp P_0)$ for any $t \in [0,T]$. By the definition of $P_0$, we have $m(t) = m^0(t)$ for any $t \in [0,T]$. Thus, $P_0 \in R^*(m^0)$, which means $P_0$ is also a relaxed MFC equilibrium, and it is strict obviously. □

## A. Notations

We list some notations which are used in this paper as follows.

**(1)** For any set $A$, denote by $\mathcal{B}(A)$ the family of Borel subsets of $A$, by $\mathcal{P}(A)$ the family of Borel probability measures on $A$, by $\mathcal{M}(A)$ the family of Borel measures on $A$. The support of a measure $P \in \mathcal{M}(A)$, denoted by $\operatorname{spt}(P)$, is a closed set defined by

$$\operatorname{spt}(P) := \{x \in A \mid P(V_x) > 0 \text{ for every open neighborhood } V_x \text{ of } x\}.$$

**(2)** Denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^-$ the set of negative real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers. For any $n \in \mathbb{N}$, denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^n$, by $|\cdot|$ the usual norm in $\mathbb{R}^n$. Denote by $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the standard $n$-dimensional flat torus. Let $\mathcal{P}(\mathbb{T}^d)$ be endowed with the narrowly convergence. It is convenient to put a metric, i.e., the Kantorovich-Rubinstein distance $d_1$ on $\mathcal{P}(\mathbb{T}^d)$.

**(3)** Fix a constant $T > 0$. For any absolutely continuous map $\gamma : [0,T] \to \mathbb{R}^d$, denote by $\|\gamma\|_\infty$ the uniform norm of the map $\gamma$, i.e.,

$$\|\gamma\|_\infty = \sup_{t\in[0,T]} |\gamma(t)|.$$

Denote by $\mathcal{L}_{[0,T]}$ the Lebesgue measure on $[0,T]$. For any constant $1 \leq \alpha < +\infty$, denote by $\|\gamma\|_{L_\alpha(\mathcal{L}_{[0,T]})}$ the $L_\alpha$ norm of the map $\gamma$ with respect to the measure $\mathcal{L}_{[0,T]}$, i.e.,

$$\|\gamma\|_{L_\alpha(\mathcal{L}_{[0,T]})} = \left(\int_0^T |\gamma(t)|^\alpha \, dt\right)^{\frac{1}{\alpha}}.$$

**(4)** Let $\Gamma_T$ be a set of absolutely continuous maps $\gamma : [0,T] \to \mathbb{T}^d$. For any $t \in [0,T]$, denote by $e_t$ the evaluation map, i.e.,

$$e_t(\gamma) = \gamma(t), \quad \forall\, \gamma \in \Gamma_T.$$

**(5)** The function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus function if it is a nondecreasing and upper semi-continuous function such that $\lim_{r\to 0+} \omega(r) = 0$.

**(6)** Let A be a set. For any subset $B \subset A$, denote by $\mathbf{1}_B : A \to \{0,1\}$ the indicator function of $B$, i.e.,

$$\mathbf{1}_B(x) = \begin{cases} 1 & x \in B, \\ 0 & x \notin B. \end{cases}$$

Let $x \in A$. Denote by $\delta_{\{x\}}$ the Dirac mass at point $x$.

**(7)** For any sets $N_1, N_2$ and any $P \in \mathcal{P}(N_1 \times N_2)$, denote by $\pi_1 : N_1 \times N_2 \to N_1$ the canonical projection of the first variable. Define the push-forward measure $\pi_1 \sharp P \in \mathcal{P}(N_1)$ by

$$\pi_1 \sharp P(B) := P\left(\pi_1^{-1}(B)\right), \quad \forall B \in \mathcal{B}(N_1).$$

We consider a probability measure $\eta \in \mathcal{P}(N_1)$ and a family of probability measures $(\nu_x)_{x \in N_1} \subset \mathcal{P}(N_2)$. We write $P = \eta \otimes \nu_x$ if these measures satisfy

$$\int_{N_1 \times N_2} g(x,y) P(dx,dy) = \int_{N_1} \int_{N_2} g(x,y) \nu_x(dy) \eta(dx),$$

for all Borel maps $g\colon N_1 \times N_2 \to [0,+\infty]$. Similarly, consider a probability measure $\nu \in \mathcal{P}(N_2)$ and a family of probability measures $(\eta_y)_{y \in N_2} \subset \mathcal{P}(N_1)$. We write $P = \eta_y \otimes \nu$ if these measures satisfy

$$\int_{N_1 \times N_2} g(x,y) P(dx,dy) = \int_{N_2} \int_{N_1} g(x,y) \eta_y(dx) \nu(dy),$$

for all Borel maps $g\colon N_1 \times N_2 \to [0,+\infty]$.

**(8)** For metric spaces $([0,t], d_A)$ and $(B, d_B)$ with $t > 0$, denote by $AC([0,t], B)$ the space of absolutely continuous maps from $[0,t]$ to $B$. For metric spaces $(A, d_A)$ and $(B, d_B)$, denote by $\mathcal{C}(A,B)$ the space of continuous maps from $A$ to $B$. Denote by $\mathcal{C}(A)$ the space of continuous functions on $A$. Denote by $\mathcal{C}_b(A)$ the space of bounded and continuous functions on $A$. Denote by $\mathcal{C}^k(A)$ the space of $k$-times continuously differentiable functions on $A$.

## B. Measure theory

In this section, we recall some results from measure theory that will be useful in this paper. Throughout this section, the space $(M, \rho)$ is a separable metric space.

For a sequence $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(M)$, $\mu_n$ narrowly converges to some $\mu \in \mathcal{P}(M)$ as $n \to \infty$ if

$$\lim_{n \to \infty} \int_M f(x) \mu_n(dx) = \int_M f(x) \mu(dx), \quad \forall f \in \mathcal{C}_b(M).$$

For each $p \in [1, +\infty)$, the Wasserstein space of order $p$ is defined by

$$\mathcal{P}_p(M) := \left\{ \mu \in \mathcal{P}(M) \,\middle|\, \int_M \rho^p(x_0, x)\, \mu(dx) < +\infty \right\},$$

where $x_0 \in M$ is an arbitrary point. The Monge-Kantorovich distance on $\mathcal{P}_p(M)$ is defined by

$$d_p(\mu_1, \mu_2) := \inf_{\eta \in \Pi(\mu_1, \mu_2)} \left( \int_{M^2} \rho^p(x,y) \eta(dx,dy) \right)^{1/p}, \quad \forall \mu_1, \mu_2 \in \mathcal{P}_p(M),$$

where $\Pi(\mu_1, \mu_2)$ is the set of Borel probability measures on $M^2$ with the properties $\eta(A \times M) = \mu_1(A)$ and $\eta(M \times A) = \mu_2(A)$ for any Borel set $A \in \mathcal{B}(M)$.

As for the distance $d_1$, which is often called Kantorovich-Rubinstein distance, can be characterized by a useful duality formula (see, for instance, [11]) as

$$d_1(\mu_1, \mu_2) = \sup \left\{ \int_M g(x)\mu_1(dx) - \int_M g(x)\mu_2(dx) \right\}, \quad \forall \mu_1, \mu_2 \in \mathcal{P}_1(M),$$

where the supremum is taken over all 1-Lipschitz functions $g : M \to \mathbb{R}$.

We now recall the relations between the narrowly convergence and the $d_p$-convergence. See [1, Theorem 7.1.5] and [25, Theorem 7.12] for examples.

**Proposition B.1.** *If a sequence of measures $\{\mu_i\}_{i=1}^{\infty} \subset \mathcal{P}_p(M)$ converges to some $\mu \in \mathcal{P}_p(M)$ in $d_p$-topology, then $\mu_n$ converges to $\mu$ narrowly.*
*Conversely, if $\bigcup_{i=1}^{\infty} \operatorname{spt}\{\mu_i\}$ is contained in a compact subset of $M$ and $\mu_i$ converges to $\mu$ narrowly, then $\mu_i$ converges to $\mu$ in $d_p$-topology.*

**Proposition B.2.** *Let $\mu \in \mathcal{P}_p(M)$ and $\{\mu_i\}_{i=1}^{\infty} \subset \mathcal{P}_p(M)$. The following statements are equivalent.*

(1) $d_p(\mu_i, \mu) \to 0$.

(2) *$\mu_i$ converges to $\mu$ narrowly. There exists $x_0 \in M$ such that*

$$\lim_{r \to +\infty} \sup_{i \in \mathbb{N}} \int_{\{x \,|\, \rho^p(x, x_0) \geq r\}} \rho^p(x, x_0)\, \mu_i(dx) = 0.$$

(3) *For each continuous function $f : M \to \mathbb{R}$ satisfies that there exist $x_0 \in M$ and $c > 0$ such that*

$$|f(x)| \leq c\,(1 + \rho^p(x, x_0)), \quad \forall x \in M,$$

*we have*
$$\lim_{i \to \infty} \int_M f(x)\mu_i(dx) = \int_M f(x)\mu(dx).$$

The next theorem reveals the relation between the narrowly convergence and the almost surely convergence. See, for instance, [3, Theorem 6.7].

**Theorem B.3.** (Skorokhod's Representation Theorem) *Let $\{\mu_i\}_{i=1}^{\infty} \subset \mathcal{P}(M)$ be a sequence of probability measures such that $\mu_i$ converges to some $\mu \in \mathcal{P}(M)$ narrowly. There exist $M$-valued random variables $\{X_i\}_{i=1}^{\infty}$ and $X$, whose distributions are $\{\mu_i\}_{i=1}^{\infty}$ and $\mu$ respectively. Then, $X_i$ converges to $X$ $\mu$-almost surely.*

Based on Theorem B.3, we can easily derive Corollary B.4.

**Corollary B.4.** *Let $\{\mu_i\}_{i=1}^{\infty} \subset \mathcal{P}(M)$ be a sequence of probability measures such that $\mu_i$ converges to some $\mu \in \mathcal{P}(M)$ narrowly. For any lower semi-continuous and bounded function $g : M \to \mathbb{R}$, we have*

$$\liminf_{i \to \infty} \int_M g(x)\mu_i(dx) \geq \int_M g(x)\mu(dx).$$

Finally, we recall the disintegration theorem. See, for instance, [11, Theorem 8.5].

**Theorem B.5.** (Disintegration Theorem) *Let $X$ and $Y$ be Radon separable metric spaces. Let $\mu$ be a Borel probability measure on $X$. Let $\pi : X \to Y$ be a Borel map. Define $\nu = \pi\sharp\mu \in \mathcal{P}(Y)$. Then, there exists a $\mu$-almost everywhere uniquely determined family of Borel probability measures $\{\nu_y\}_{y\in Y} \subset \mathcal{P}(X)$ such that*

$$\nu_y\left(X\backslash\pi^{-1}(y)\right) = 0, \qquad \textit{for } \mu - \textit{a.e. } y \in Y,$$
$$\int_X g(x)\mu(dx) = \int_Y \left(\int_{\pi^{-1}(y)} g(x)\nu_y(dx)\right)\nu(dy)$$

*for every Borel map $g : X \to [0, +\infty]$.*

## C. Completion of the Proof of Proposition 3.2

Define the function $g : \Gamma_T^R \to \bar{\mathbb{R}}$ by

$$\gamma \mapsto \int_0^T \int_{\mathbb{R}^n} |u|^q \, \hat{q}_{t,\gamma(t)}\,(du)\,dt,$$

which is a positive measurable function. For any Borel subset $V \subset \Gamma_T^R \times \mathcal{P}_U^R$, there always exists a measure $\mu_V \in \mathcal{P}_U^R$ such that

$$\begin{aligned}&\int_{\Gamma_T^R\times\mathcal{P}_U^R}\int_0^T\int_{\mathbb{R}^n} \mathbf{1}_V(\gamma,\mu)\,|u|^q\,\mu(dt,du)P\,(d\gamma,d\mu)\\ &\le P(V)\int_0^T\int_{\mathbb{R}^n}|u|^q\,\mu_V(dt,du) \le P(V)R.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\int_{\Gamma_T^R} \mathbf{1}_{\pi_1(V)}(\gamma)\,g\,(\gamma)\,\pi_1\sharp P\,(d\gamma)\\
&= \int_{\Gamma_T^R}\int_0^T\int_{\mathbb{R}^n}\mathbf{1}_{\pi_1(V)}(\gamma)\,|u|^q\,\hat{q}_{t,\gamma(t)}\,(du)\;dt\,\pi_1\sharp P\,(d\gamma)\\
&= T\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}^n}\mathbf{1}_{\pi_1(V)}\left(e_t^{-1}(x)\right)|u|^q\,\hat{q}_{t,x}(du)\,\eta_{1,2}(dt,dx)\\
&= T\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}^n}\mathbf{1}_{\pi_1(V)}\left(e_t^{-1}(x)\right)|u|^q\,\eta(dt,dx,du)\\
&= \int_{\Gamma_T^R\times\mathcal{P}_U^R}\int_0^T\int_{\mathbb{R}^n}\mathbf{1}_{\pi_1(V)}(\gamma)\,|u|^q\,\mu(dt,du)\,P\,(d\gamma,d\mu) \le R\,P(V),\end{aligned}$$

indicating that $g$ is integrable. By Vitali-Carathéodory theorem, for any $\varepsilon > 0$, there exists a lower semi-continuous function $\hat{g} : \Gamma_T^R \to \mathbb{R}$ with $g(\gamma) \le \hat{g}(\gamma)$ for $\pi_1\sharp P$-*a.e.* $\gamma \in \Gamma_T^R$ such that

$$\int_{\Gamma_T^R} \hat{g}(\gamma) - g(\gamma)\;\pi_1\sharp P(d\gamma) \le \varepsilon.$$

For $\gamma_0 \in \operatorname{spt}(\pi_1\sharp P)$, there exists a unique measure $\mu^0 \in \mathcal{P}_U^R$ such that the pair $(\gamma_0, \mu^0) \in \operatorname{spt}(P)$ by Proposition 2.5(3). Denote by $V$ the open neighborhood of $(\gamma_0, \mu^0)$, we have $P(V) > 0$ and

$$\begin{aligned} g(\gamma_0) \le \hat{g}(\gamma_0) &\le \liminf_{V \to (\gamma_0, \mu^0)} \frac{1}{P(V)} \int_V \hat{g}(\gamma) P\,(d\gamma, d\mu) \\ &\le \liminf_{V \to (\gamma_0, \mu^0)} \frac{1}{P(V)} \int_V g(\gamma) P\,(d\gamma, d\mu) + \varepsilon \le R + \varepsilon. \end{aligned}$$

Let $\varepsilon$ tend to 0. We conclude that $\mathcal{L}_{[0,T]} \otimes \hat{q}_{t,\gamma(t)} \in \mathcal{P}_U^R$ for $\pi_1\sharp P$-a.e. $\gamma \in \operatorname{spt}(\pi_1\sharp P)$. Let $\mathcal{L}_{[0,T]} \otimes \hat{q}_{t,\gamma(t)}$ equal any measure that belongs to $\mathcal{P}_U^R$ when $\gamma$ belongs to the null set. We can conclude that $\hat{P} \in \mathcal{P}\left(\Gamma_T^R \times \mathcal{P}_U^R\right)$ since the difference in the null set has no influence on the following analysis by (F1) and (L3).

It is clear that $e_0\sharp\left(\pi_1\sharp \hat{P}\right) = e_0\sharp\,(\pi_1\sharp P) = m_0$. Moreover, for any $t \in [0,T]$ and any open set $N \subset \Gamma_T^R \times \mathcal{P}_U^R$, we can deduce that

$$\begin{aligned} &\int_{\Gamma_T^R} \int_0^t \int_{\mathbb{R}^n} \mathbf{1}_{\pi_1(N)}(\gamma) f\,(\gamma(s), u, m(s))\, \hat{q}_{s,\gamma(s)}\,(du)\; ds\, \pi_1\sharp P\,(d\gamma) \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^n} \mathbf{1}_{\pi_1(N)}\left(e_s^{-1}(x)\right) f\,(x, u, m(s))\, \hat{q}_{s,x}(du)\, e_s\sharp\,(\pi_1\sharp P)\,(dx)\, ds \\ &= T \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^n} \mathbf{1}_{\pi_1(N)}\left(e_s^{-1}(x)\right) \mathbf{1}_{[0,t]}(s) f\,(x, u, m(s))\, \eta(ds, dx, du) \\ &= \int_{\pi_1(N) \times \mathcal{P}_U^R} \int_0^t \int_{\mathbb{R}^n} f\,(\gamma(s), u, m(s))\, \mu(ds, du)\, P\,(d\gamma, d\mu)\,. \end{aligned}$$

Based on this equality, for any $v \in \mathbb{R}^d$, we have

$$\begin{aligned} &\int_N \Big\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\,(\gamma(s), u, m(s))\, \mu\,(ds, du) \Big\rangle \hat{P}\,(d\gamma, d\mu) \\ &= \int_{\Gamma_T^R \times \mathcal{P}_U^R} \mathbf{1}_N\,(\gamma, \mu) \left\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\,(\gamma(s), u, m(s))\, \mu\,(ds, du) \right\rangle \hat{P}\,(d\gamma, d\mu) \\ &= \int_{\Gamma_T^R} \mathbf{1}_{\pi_1(N)}\,(\gamma) \Big\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\,(\gamma(s), u, m(s))\, \hat{q}_{s,\gamma(s)}\,(du)\, ds \Big\rangle \pi_1\sharp P\,(d\gamma) \\ &= \int_{\pi_1(N) \times \mathcal{P}_U^R} \Big\langle v, \gamma(t) - \gamma(0) - \int_0^t \int_{\mathbb{R}^n} f\,(\gamma(s), u, m(s))\, \mu\,(ds, du) \Big\rangle P\,(d\gamma, d\mu) = 0. \end{aligned}$$

Thus, we obtain that $\hat{P} \in \mathcal{P}_R(m)$.

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