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A Cautionary Tale – Every Set is an Interval Set and an Equiset

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Given a metric space (X, d) with distinct points $x, y \in X$, we consider the interval set $I_d(x, y) = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ and the equiset $E_d(x, y) = \{z \in X : d(x, z) = d(y, z)\}$. The topology and geometry of these regions have been studied, in particular when the metric arises from a norm. In this note, we consider the question of which sets arise as interval sets and equisets in general metrizable spaces. We show that, subject to some obvious restrictions, every closed set in every metrizable space occurs as an interval set or equiset, with respect to some compatible metric. We also consider the case of geodesic spaces, and show that any arc or closed union of geodesics between two points of a complete locally compact geodesic space arises as the interval set with respect to some compatible geodesic metric.

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1. Introduction

In [2, 3, 4] the authors investigate the following regions of a metric space (X, d):

The *interval set* $I_d(x, y) = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$
The *equiset* $E_d(x, y) = \{z \in X : d(x, z) = d(y, z)\}$
The *midset* $M_d(x, y) = E_d(x, y) \cap I_d(x, y)$
The *comparative nearness region* $R_d(x, y) = \{z \in X : d(x, z) < d(y, z)\}$

The interval sets generalize the notion of the linear segment between two points of a vector space. This allows us to talk about convexity when we have only a metric.

It is natural to ask what the above regions can look like from a topological point of view. The existing literature contains only partial results. The paper [3] considers the convexity of the four types of regions above. It also deals with the special case of normed spaces. The paper [4] studies continuity of the pair-to-interval map, but only for metric spaces that are proper, normed, or geodesic. The paper [2] deals with the even more rigid context of hyperspaces of a normed space.

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All existing works either study only the geometry of the above regions, or restrict to a specific class of metrics. No work to date considers the case of general metric spaces and purely topological properties. In this work, we show there is a good reason for this. Some sort of restriction is necessary to develop a general theory as, from a purely topological point of view, the above regions are uninteresting. Indeed, the range of topological types for the interval set and equiset covers all metric spaces. Moreover, given an ambient metrizable space X and distinct points $a, b \in X$ we can, subject to some obvious conditions, choose a compatible metric ρ to make $I_\rho(a, b)$ or $E_\rho(a, b)$ whatever we want.

Theorem 4.2 *Let X be a metrizable space and $a, b \in X$ be distinct points. Then the following are equivalent:*

(1) *B is a closed subset of X with $a, b \in B$.*

(2) *$B = I_\rho(a, b)$ with respect to some compatible metric ρ on X.*

To state our main theorem about midsets, we recall the notion of a quasicomponent. The quasicomponent of a point is the intersection of all the clopen sets containing that point. The quasicomponent of a point contains the connected component of that point. Two points are in different quasicomponents exactly if some clopen set contains one point but not the other.

Theorem 5.1 *Let X be a metrizable space and $a, b \in X$ be distinct points. Then the following are equivalent:*

(1) *B is a closed subset of X with a, b in different quasicomponents of $X - B$.*

(2) *$B = E_\rho(a, b) = M_\rho(a, b)$ with respect to some compatible metric ρ on X.*

Moreover, in the case of Theorem 5.1, we also have the freedom to determine the regions $R_\rho(a, b)$ and $R_\rho(b, a)$ according to any disconnection of X by B. This is detailed in the more technical Theorem 5.2, whose statement is deferred until Section 5.

Our main result for geodesic spaces is an analogue of Theorem 4.2.

Theorem 6.4 *Suppose (X, d) is a complete and locally compact geodesic space with distinct points $a, b \in X$. Let $B \subset X$ be closed with $a, b \in B$, and B is the union of a family of geodesics from a to b. Then there exists a compatible metric ρ on X such that (X, ρ) is a complete and locally compact geodesic space, and $B = I_\rho(a, b)$.*

The above is a partial converse to Proposition 3.6 of [4], which says that the interval set between any two points in a geodesic space equals the union of the geodesics between those two points. Hence, it is easy to come up with sets that do not accur as interval sets under any geodesic metric. At the same time, our Lemma 6.7 implies that any loop in $\mathbb{R}^n$ can be made into the interval set between any of its two points, under some geodesic metric. This is surprising due to the existence of so-called wild arcs in $\mathbb{R}^3$, meaning arcs that are not ambiently homeomorphic to a straight line segment. See, for example, Chapter 2 of [19].

2. Other related work

The metric intervals were first studied by Menger [17] where a general theory of convexity in metric spaces is proposed. The interval sets are often called Menger intervals. More recent results on Menger betweenness and intervals include [1] and the works that cite it, which are about counting the number of distinct Menger intervals (therein called 'lines') generated by finite subsets of a metric space. Contrary to our terminology, the term 'midset' is often used in the literature for what we call the equiset. These sets have been studied for example in [13] where the topological types of equisets (therein called 'bisectors') in normed planes are characterized. In [11] higher-dimensional normed spaces are considered. In [5] it is proved that any connected metric space where each pair has a unique midpoint is homeomorphic to an interval of the real line. In [15] sufficient conditions are established for an equiset (between two disjoint closed sets rather than points) to be a manifold. In [16] it is proved that if every equiset is nonempty and connected then the ambient space is also connected. See [10] Section 2 for a survey of open problems related to equisets. In particular, there are many open problems related to the unique-midpoint property, i.e spaces where the equisets are all singletons. See, for example, [12, 18].

3. Terminology and preliminaries

Throughout, X is a metrizable space. The letters d and ρ denote metrics on X. A metric on X is said to be compatible if it generates the topology. Two metrics on the same set are said to be equivalent if they generate the same topology. A stronger condition is bi-Lipschitz equivalence of metrics. Two metrics d, ρ on X are said to be bi-Lipschitz equivalent if there exist scalars $\alpha, \beta > 0$ such that $\alpha d(x, y) \leq \rho(x, y) \leq \beta d(x, y)$ for all $x, y \in X$. Given sets S and K, we denote the set difference by $S - K = \{s \in S : s \notin K\}$. When we write $K \subset S$, we do not assume the containment is proper, meaning we might have $K = S$.

The following two well-known lemmas will be useful to combine a given metric with a continuous function, while keeping it compatible.

Lemma 3.1. *Suppose X is a metrizable space with a compatible metric d. Suppose also that $f : X \to \mathbb{R}$ is continuous. Then $\rho(x, y) = \max\{d(x, y), |f(x) - f(y)|\}$ is a compatible metric on X.*

Proof. Since ρ is the max of a metric and a pseudo-metric, it is a metric. To prove the two metrics are equivalent, we will show they have the same convergent sequences. Since d and f are both continuous with respect to d, and the max of any two continuous functions is continuous, it follows for each $x_0 \in X$ that the function $x \mapsto \rho(x, x_0)$ is continuous with respect to d. In particular, d-convergence of a sequence to x_0 implies ρ-convergence to the same limit. In the other direction, since $\rho(x, y) \geq d(x, y)$ we see ρ-convergence implies d-convergence. □

The proof of the next lemma is almost identical to the previous.

Lemma 3.2. *Suppose X is a metrizable space with a compatible metric d. Suppose also that $f : X \to \mathbb{R}$ is continuous. Then $\rho(x, y) = d(x, y) + |f(x) - f(y)|$ is a compatible metric on X.*

The final lemma we need appears as Proposition 495 of [20]. The given proof for normed spaces readily generalizes.

Lemma 3.3. *Suppose* (X, d) *is a metric space and* $C \subset X$ *is nonempty. Then the function* $x \mapsto d(x, C) := \inf\{d(x, c) : c \in C\}$ *is* 1*-Lipschitz. In particular, it is continuous.*

4. The interval set

The first step to proving Theorem 4.2 is to find a metric where the entire space is the interval set.

Lemma 4.1. *Let* X *be a metrizable space with* $a, b \in X$ *distinct. There exists a compatible metric* ρ *on* X *with the property that* $\rho(a, x) + \rho(x, b) = \rho(a, b)$ *for all* $x \in X$.

Proof. Let d be a compatible metric on X. Since rescaling always produces an equivalent metric, we can assume $d(a, b) = 1$. It is easy to show that $d(x, y)$ is equivalent to the truncated metric $\min\{d(x, y), 1\}$. Hence, we can assume the diameter of X with respect to d is exactly 1.

Define the piecewise function on X:

$$f(x) = \begin{cases} 2d(x, a) & \text{for } d(x, a) \leq 1/2 \\ 2(1 - d(x, b)) & \text{for } d(x, b) \leq 1/2 \\ 1 & \text{for } d(x, a) \geq 1/2 \text{ and } d(x, b) \geq 1/2 \end{cases}$$

Continuity of the metric implies that each of the three parts of the function is continuous on its domain. Since the domains are closed, to show the function is continuous, it is enough to check that the three parts coincide over the intersections.

First, suppose the first and second conditions hold, i.e. $d(x, a), d(b, x) \leq 1/2$. Then we have $d(x, a) + d(b, x) \leq 1 = d(a, b)$ and by the triangle inequality we must have $d(x, a) + d(b, x) = 1$. Since each summand is at most $1/2$, it follows that each equals $1/2$. Hence, we have $d(x, a) = d(x, b) = 1/2$. Hence, the first two parts of the function equal 1, and all three parts coincide over this intersection.

Now, suppose the first and third conditions both hold. By the previous case, we can assume the second condition does not hold. Since $d(x, a) \leq 1/2$ and $d(x, a) \geq 1/2$ we must have $d(x, a) = 1/2$. Hence, the first part of the function equals 1, which agrees with the third part.

The case where the second and third conditions both hold is identical.

We use the function f to define a metric $\rho(x, y) = \max\{d(x, y), |f(x) - f(y)|\}$ on X. Lemma 3.1 says that d and ρ are equivalent.

To see ρ has the betweenness property we want, first observe that $f(a) = 0$ and $f(b) = 2$. Hence, $\rho(a, b) = \max\{d(a, b), 2\} = \max\{1, 2\} = 2$. Consider the three cases from the definition of f. For $d(x, a) \leq 1/2$ we have:

$$\rho(a, x) = \max\{d(a, x), |f(a) - f(x)|\} = \max\{d(a, x), |2d(a, x))|\} = 2d(a, x)$$

$$\rho(b, x) = \max\{d(b, x), |f(b) - f(x)|\} = \max\{d(b, x), 2|1 - d(a, x)|\}.$$

Since $d(a,x) \leq 1/2$, we have $1 - d(a,x) \geq 1/2$ and so

$$2|1 - d(a,x)| = 2(1 - d(a,x)) \geq 1 \geq d(b,x).$$

Hence, we get $\rho(b,x) = 2(1 - d(a,x))$. Adding the two terms, gives

$$\rho(a,x) + \rho(x,b) = 2d(a,x) + 2(1 - d(a,x)) = 2 = \rho(a,b).$$

The case when $d(x,b) \leq 1/2$ is identical to the first case. For the third case, we have:

$$\rho(a,x) = \max\{d(a,x), |0-1|\} = \max\{d(a,x), 1\} = 1$$
$$\rho(b,x) = \max\{d(b,x), |2-1|\} = \max\{d(b,x), 1\} = 1$$

again, since the diameter is 1. Hence, we get $\rho(a,x) + \rho(b,x) = 1 + 1 = 2 = \rho(a,b)$. □

We are ready to prove our first main theorem.

Theorem 4.2. *Let X be a metrizable space and $a, b \in X$ be distinct points. Then the following are equivalent:*

(1) *B is a closed subset of X with $a, b \in B$.*

(2) *$B = I_\rho(a,b)$ with respect to some compatible metric ρ on X.*

Proof. $1 \Rightarrow 2$: First use Lemma 4.1 to see that X admits a compatible metric d such that $d(a,x)+d(x,b) = d(a,b)$ for all $x \in X$. Lemma 3.3 says the function $x \mapsto d(x,B)$ is continuous. Hence, Lemma 3.2 says $\rho(x,y) = d(x,y) + |d(x,B) - d(y,B)|$ is a compatible metric on X. We claim $I_\rho(a,b) = B$. For each $x \in X$ we have:

$$\begin{aligned}\rho(a,x) + \rho(x,b) &= d(a,x) + |d(a,B) - d(x,B)| + d(x,b) + |d(x,B) - d(b,B)| \\ &= d(a,x) + d(x,b) + 2|d(x,B)| = d(a,b) + 2d(x,B) \\ &= \rho(a,b) + 2d(x,B)\end{aligned}$$

Hence, we have $\rho(a,x) + \rho(x,b) = \rho(a,b)$ if and only if $d(x,B) = 0$. Since B is closed, this is equivalent to demanding $x \in B$. □

5. Equisets and comparative nearness regions

Theorem 5.1. *Let X be a metrizable space and $a, b \in X$ be distinct points. Then the following are equivalent:*

(1) *B is a closed subset of X with a, b in different quasicomponents of $X - B$.*

(2) *$B = E_\rho(a,b) = M_\rho(a,b)$ with respect to some compatible metric ρ on X.*

Proof. The direction $2 \Longrightarrow 1$ is easy. The set $E_\rho(x,y)$ is always closed by continuity of the metric. Moreover, we have the separation $X - B = R_\rho(a,b) \cup R_\rho(b,a)$ into comparative nearness regions, which are both open by continuity of the metric. Clearly $a \in R_\rho(a,b)$ and $b \in R_\rho(b,a)$.

To prove $1 \Longrightarrow 2$ we first rescale the metric to assume $d(a,b) = 1$. We then replace the metric with the truncation $\min\{d(x,y), 1\}$. These operations always produce equivalent metrics.

By assumption, we can write $X - B = O_a \cup O_b$ as a disjoint union of clopen sets with $a \in O_a$ and $b \in O_b$. Define $B_a = O_a \cup B$ and $B_b = O_b \cup B$ and observe that the two closed sets B_a, B_b have union X and intersection B.

First we deal with the case that B is nonempty. Define the piecewise function:

$$f(x) = \begin{cases} 1 - \dfrac{d(x,B)}{d(a,B)} & \text{for } x \in B_a \\ 1 + \dfrac{d(x,B)}{d(b,B)} & \text{for } x \in B_b \end{cases}$$

The function is continuous over each subdomain and each part equals 1 over the intersection B of the two subdomains. Hence, we have a continuous function.

Lemma 3.1 says the metric $\rho(x,y) = \max\{d(x,y), |f(x) - f(y)|\}$ is compatible with X. Observe that $f(a) = 0$ and $f(b) = 2$. Hence, for each $x \in B$, we have

$$\rho(x,a) = \max\{d(x,a), |f(x) - f(a)|\} = \max\{d(x,a), |1 - 0|\} = \max\{d(x,a), 1\} = 1$$

where we have used how the diameter of X is equal to 1. Likewise, we have $\rho(x,b) = |2-1| = 1$. We also have $\rho(a,b) = |2-0| = 2$. Hence, we have $B \subset M_\rho(a,b)$.

To see $E_\rho(a,b) \subset B$, suppose we have $\rho(a,x) = \rho(x,b)$. By the triangle inequality $\rho(a,x) + \rho(x,b) \geq \rho(a,b) = 2$. So we must have $\rho(a,x), \rho(b,x) \geq 1$. Consider two cases.

In the first case, assume $\rho(a,x), \rho(b,x) > 1$. Since the diameter with respect to d equals 1, we must have

$$\rho(a,x) = |f(a) - f(x)| = |f(x)| \quad \text{and} \quad \rho(x,b) = |f(x) - f(b)| = |2 - f(x)|$$

and so $|f(x)| = |2 - f(x)|$. Thus either $f(x) = 2 - f(x)$ which implies $f(x) = 1$, or $f(x) = f(x) - 2$. Since the latter has no solutions for $f(x)$, we must have $f(x) = 1$.

In the second case, assume $\rho(a,x), \rho(b,x) = 1$. We must have both

$$|f(a) - f(x)| = |f(x)| \leq 1 \quad \text{and} \quad |f(b) - f(x)| = |2 - f(x)| \leq 1.$$

In other words, $f(x) \in [-1,1]$ and $f(x) \in [1,3]$. The only such value is $f(x) = 1$.

The two cases show that $x \in E_\rho(a,b)$ implies $f(x) = 1$. Since B is closed, this occurs only for $x \in B$. It follows $E_\rho(a,b) \subset B$. Thus, we have $E_\rho(a,b) \subset B \subset M_\rho(a,b)$. Since we always have $M_\rho(a,b) \subset E_\rho(a,b)$, we get $M_\rho(a,b) \subset E_\rho(a,b) \subset B \subset M_\rho(a,b)$, and so $M_\rho(a,b) = E_\rho(a,b) = B$ as required.

Finally we consider the case that B is empty. Consider the metrizable space $\widetilde{X} = X \cup \{c\}$ that is the union with a singleton, and the closed set $\widetilde{B} = \{c\}$. The result we just proved for nonempty sets gives a metric on $\widetilde{X}$ with $\widetilde{B}$ equal to the midset. Restrict the metric to X to get the desired result. □

The next theorem captures some further details of the construction above. We remark that, given a set B that disconnects the space, there might very well exist more than one different disconnection. The theorem says we can choose the comparative nearness regions according to any disconnection we like.

Theorem 5.2. *Let X be a metrizable space with $a, b \in X$ distinct points, and $B \subset X$ a closed set with the following property: We can write $X - B = O_a \cup O_b$ as a disjoint union of two open sets with $a \in O_a$ and $b \in O_b$. Then there exists a compatible metric ρ on X such that $B = E_\rho(a,b) = M_\rho(a,b)$, $O_a = R_\rho(a,b)$, and $O_b = R_\rho(b,a)$.*

Proof. The construction of the metric ρ is the same as in Theorem 5.1. We just need to show $O_a = R_\rho(a,b)$ and $O_b = R_\rho(b,a)$.

For $x \in R_\rho(a,b)$ we have $\rho(a,x) < \rho(x,b)$. Since $\rho(a,b) = 2$ by construction, the triangle inequality gives $\rho(x,b) > 1$. Since the diameter with respect to d is 1, it follows $\rho(x,b) = |f(x) - f(b)| = |2 - f(x)|$. Since $\rho(a,x) \geq |f(a) - f(x)| = |f(x)|$, the inequality $\rho(a,x) < \rho(x,b)$ gives $|f(x)| < |2 - f(x)|$. In other words, $f(x)$ is closer to zero than it is to 2, meaning $f(x) < 1$. By construction of $f(x)$ we get $x \in B_a$. Since $B = E_\rho(a,b)$ is disjoint from $R_\rho(a,b)$, we cannot have $x \in B$. Hence, we get $x \in O_a$.

It follows that $R_\rho(a,b) \subset O_a$. By symmetry, we have $R_\rho(b,a) \subset O_b$. Since $\{O_a, O_b\}$ and $\{R_\rho(a,b), R_\rho(b,a)\}$ are both partitions of $X - B = X - E_\rho(a,b)$, we must have $O_a = R_\rho(a,b)$ and $O_b = R_\rho(b,a)$ as required. $\square$

Instead of choosing the equiset and midset directly, we can choose the comparative nearness regions.

Corollary 5.3. *Suppose X is a metrizable space and $A, C \subset X$ are open, nonempty, and disjoint. For every $a \in A$ and $b \in C$, there exists a compatible metric ρ on X with $R_\rho(a,b) = A$, $R_\rho(b,a) = C$ and $E_\rho(a,b) = M_\rho(a,b) = X - (A \cup C)$.*

Proof. Apply Theorem 5.2 to the closed set $B = X - (A \cup C)$. $\square$

6. Geodesic spaces

Since there are no restrictions on the topological types of interval sets and equisets, it follows that, in order to develop a theory, we must either study their metric geometry rather than topology, or restrict the class of allowed metrics. One classical result due to Menger [17] is our Theorem 6.2.

Definition 6.1. The metric space (X,d) is called a *geodesic space* if, for every pair of distinct points $a, b \in X$, there exists a geodesic from a to b, i.e a distance-preserving map $\gamma : [0, d(a,b)] \to X$ with $\gamma(0) = a$ and $\gamma(d(a,b)) = b$, where the domain carries the absolute value metric.

We will abuse notation and use the term 'geodesic' to refer to both the path γ and the arc in X which is the image of γ, without confusion. See [9] Example 2.7 for a proof of the following theorem in English.

Theorem 6.2. *Suppose (X,d) is a complete metric space with the property that, for every pair $a, b \in X$ of distinct points there exists $c \in I_d(a,b) - \{a,b\}$. Then (X,d) is a geodesic space.*

We note there is another definition of a geodesic in the literature. See for example [4] and [7] where they are called 'shortest paths'. Here, a geodesic is defined as a path $\gamma : [t, s] \to X$ with length equal to $d(\gamma(t), \gamma(s))$, where the length of a path p is defined as

$$\mathrm{Length}^d(p) = \sup_{\overline{x}} \sum_{i=0}^{n-1} d(p(x_i), p(x_{i+1})) \tag{1}$$

where $\overline{x}$ ranges over all increasing finite sequences of indices $t = x_0 < x_1 < \ldots < x_{n-1} < x_n = s$. By Remark 1.22 of [6], the two definitions are equivalent, in the sense that an arc from a to b is the image of a geodesic in the isometric sense iff it is the image of a geodesic in the length sense.

6.1. Interval sets in geodesic spaces

The following lemma, which appears as Proposition 3.6 of [4], shows that, if we restrict to the class of geodesic spaces, we certainly do not get all possible closed sets as interval sets.

Lemma 6.3. *Let (X, d) be a geodesic space with $a, b \in X$. Then $I_d(a, b)$ is the union of all geodesics from a to b.*

Lemma 6.3 implies that interval sets in geodesic spaces are arc-connected. Moreover, it forbids some arc-connected sets such as $C = ([-1, 1] \times \{0\}) \cup (\{0\} \times [-1, 1])$ occuring as the interval set $I_d(a, b)$ for $a = (-1, 0)$ and $b = (1, 0)$ under any metric d which makes the plane into a geodesic space. This is because C contains no arc from a to b through the point $c = (0, -1)$, so it certainly contains no such geodesic.

One can ask for a converse to Lemma 6.3 – is it possible to turn a closed union of geodesics between two points into the interval set between those two points? If we impose some further restrictions on the metric space, the answer is yes. The proof is similar to Theorem 5.1. If we want some union B of geodesics to be the interval set, we can use a metric that gives the space the geometry of the graph of $x \mapsto d(x, B)$. All the hard work is showing that this geometry gives a geodesic space.

Theorem 6.4. *Suppose (X, d) is a complete and locally compact geodesic space with distinct points $a, b \in X$. Let $B \subset X$ be closed with $a, b \in B$, and B is the union of a family of geodesics from a to b. Then there exists a compatible metric ρ on X such that (X, ρ) is a complete and locally compact geodesic space, and $B = I_\rho(a, b)$.*

Proof. Lemma 3.2 says the metric $d'(x, y) = d(x, y) + |d(x, B) - d(y, B)|$ is compatible on X. It is straightforward to show that $I_{d'}(a, b) = B$. However, there is no guarantee that (X, d') is a geodesic space. Instead, consider the length metric $\rho = \overline{d'}$ associated to d', i.e. $\rho(x, y) = \inf\{\mathrm{Length}^{d'}(p) : p \text{ is a path from } x \text{ to } y\}$ where $\mathrm{Length}^{d'}(p)$ is defined according to (1). This metric is discussed in [6] Chapter I.3, Proposition 3.2, where the following properties are proved.

(a) $\overline{d'}$ is indeed a metric.

(b) $d' \le \overline{d'}$.

(c) The length of a path with respect to $\overline{d'}$ is the same as the length with respect to d'.

Now, if we can show that $\rho = \overline{d'}$ is bi-Lipschitz equivalent to d, it will follow that (X, ρ) is complete and locally compact. It will then follow from the Hopf-Rinow theorem (see [6] Chapter I.3, Proposition 3.7) that (X, ρ) is a geodesic space as required.

Lemma 3.3 says the function $x \mapsto d(x, B)$ is 1-Lipschitz with respect to d. Hence, we have $d'(x, y) = d(x, y) + |d(x, B) - d(y, B)| \leq 2d(x, y)$ and so $d' \leq 2d$. It is clear from the definition that the operation of taking the length metric is monotone and commutes with rescaling by a positive constant. Thus, we have $\overline{d'} \leq \overline{2d} = 2\overline{d} = 2d$ where the last equality follows from how (X, d) is assumed to be a geodesic space, and so the length metric coincides with the original metric. From this, we get $\overline{d'} \leq 2d$. Since $d \leq d'$ by construction, we can use (b) to get $d \leq \overline{d'}$. We conclude that $d \leq \rho \leq 2d$ and the metrics are bi-Lipschitz equivalent, as required.

Now we will show $I_\rho(a, b) = B$. By construction we have $d'(x, y) = d(x, y)$ for all $x, y \in B$. Hence, an arc from a to b inside B is a d-geodesic iff it is a d'-geodesic. Since B is the union of a family of d-geodesics from a to b, it follows that B is also the union of a family of d'-geodesics from a to b. We claim every such d'-geodesic is also a ρ-geodesic, and therefore $B \subset I_\rho(a, b)$ by Lemma 6.3. By (c) it is enough to show $\rho(a, b) = d'(a, b)$. By definition $\rho(a, b)$ is the infimal d'-length among paths from a to b. Since there is at least one d'-geodesic from a to b, the infimal length is just $d'(a, b)$ as required.

To see $I_\rho(a, b) \subset B$, let $z \in X - B$ be arbitrary. Since B is closed, we have $d(z, B) > 0$. Since $a, b \in B$ we have $d'(a, b) = d(a, b)$. Now expand:

$$\begin{aligned} d'(a, z) + d'(z, b) &= d(a, z) + |d(a, B) - d(z, B)| + d(z, b) + |d(z, B) - d(b, B)| \\ &= d(a, z) + d(z, b) + 2d(z, B) \\ &\geq d(a, b) + 2d(z, B) > d(a, b) = d'(a, b). \end{aligned}$$

Thus, we have $d'(a, z) + d'(z, b) > d'(a, b)$. Since we have already shown $\rho(a, b) = d'(a, b)$ and (b) gives $\rho(a, z) + \rho(z, b) \geq d'(a, z) + d'(z, b)$ we get $\rho(a, z) + \rho(z, b) > \rho(a, b)$ and so $z \notin I_\rho(a, b)$, as required. □

While the Hopf-Rinow theorem is essential to our proof of Theorem 6.4, we see no immediate reason why completeness and local compactness should be needed for the result to hold.

Conjecture 6.5. *Suppose (X, d) is a geodesic space with distinct points $a, b \in X$. Let $B \subset X$ be closed with $a, b \in B$, and B is the union of a family of geodesics from a to b. Then there exists a compatible metric ρ on X such that (X, ρ) is a geodesic space and $B = I_\rho(a, b)$.*

Even for locally compact spaces, Theorem 6.4 is not entirely satisfying. We would prefer a purely topological characterisation of the sets that can be made into interval sets with respect to some compatible geodesic metric. The obvious idea is to replace the condition that the set be a union of geodesics, with it being simply a union of arcs. The following example shows that this will not work. The idea is to make the proposed interval set contain points $c_n \to a$ but have a 'bottleneck' which keeps the shortest path distances from a to c_n bounded from below.

Example 6.6. Let $X = [0, 1]^2$ be the unit square and $B \subset X$ the boundary of the square, together with the segments $I_n = \{1 - 1/2^n\} \times [0, 1]$ for $n = 0, 1, 2, \ldots$. Let $a = (1, 2/3), b = (1, 1/3), c = (1, 1)$, and each $c_n = (1 - 1/2^n, 2/3)$. See Figure 1.

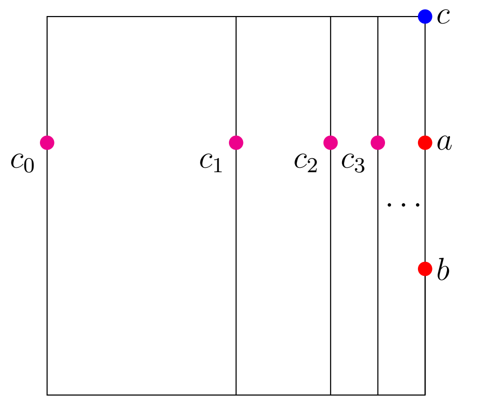

Figure 1: Diagram of Example 6.6

Clearly, the closed set B is a union of arcs from a to b. In particular there is, for each n, a unique arc from a to b through c_n, consisting of a vertical segment upwards from a, then a horizontal segment to the left, then downwards along I_n, then a horizontal segment to the right, and finally a vertical segment up to b. It follows from Lemma 6.3 that if $B = I_d(a, b)$ for some geodesic metric d, each of these arcs must be a geodesic. However, since each such geodesic hits c before it hits c_n, this would imply $d(c_n, a) > d(c, a)$. This contradicts how $c_n \to a$ and so $d(c_n, a) \to 0$. $\square$

Despite Theorem 6.4 not generalizing to unions of arcs, we can use existing results to turn any single arc into the interval set. This is surprising due to the existence of so-called wild arcs in $\mathbb{R}^3$, meaning arcs that are not ambiently homeomorphic to a straight line segment. See for example Chapter 2 of [19]. We thank Moishe Kohan [14] for making us aware of the main result, which allows us to turn a finite family of arcs into geodesics. Then we can use Theorem 6.4 to make sure they are the only geodesics.

Lemma 6.7. *Suppose (X, d) is a complete and locally compact geodesic space with $a, b \in X$. Let Γ be a union of finitely many arcs from a to b that are disjoint except at their endpoints. Then there exists a compatible metric ρ on X such that (X, ρ) is a complete and locally compact geodesic space and $I_\rho(a, b) = \Gamma$.*

Proof. Since X is connected and locally compact it is, in the terminology of [8], a generalized continuum. It follows from the definitions that every open ball in a geodesic space is connected, hence the space is locally connected. The space Γ admits the shortest path metric d_Γ where each of the finitely many arcs has length equal to 1. This metric is convex, meaning that for all $x, y \in \Gamma$ there exists $z \in \Gamma - \{x, y\}$ with $d_\Gamma(x, y) = d_\Gamma(x, z) + d_\Gamma(z, y)$. Under this metric, each of the arcs that make up Γ is a geodesic from a to b. Corollary 4.4 of [8] says d_Γ extends to a complete convex metric ρ on X. By Theorem 6.2 we see (X, ρ) is a geodesic space. Since ρ restricts to d_Γ, we see that each arc that makes up Γ is a geodesic with respect to ρ. Finally, apply Theorem 6.4 for $B = \Gamma$ to complete the proof. $\square$

6.2. Equisets and midsets in geodesic spaces

Just like with interval sets, there are some restrictions a set must satisfy to have any hope of being made into an equiset or midset with respect to a geodesic metric.

For example, the definitions of a geodesic and the sets $E_d(a,b)$ and $M_d(a,b)$ quickly imply the following.

Lemma 6.8. *Let (X,d) be a metric space with distinct $a,b \in X$. Then every geodesic from a to b intersects $E_d(a,b)$ at exactly one point. That point is in $M_d(a,b)$.*

Lemma 6.8 forbids, for example, the set $B = \{-1,1\} \times \mathbb{R}$ occurring as either $E_d(a,b)$ or $M_d(a,b)$ for $a = (-2,0)$ and $b = (2,0)$ under any compatible geodesic metric on the plane.

A second necessary condition concerns the manner in which $E_d(a,b)$ disconnects the space.

Lemma 6.9. *Let (X,d) be a geodesic space with distinct $a,b \in X$. Then $R_d(a,b)$ is a union of geodesics with one endpoint a, and $R_d(b,a)$ is a union of geodesics with one endpoint b.*

Proof. By symmetry, it is enough to prove for each $p \in R_d(a,b)$ that every geodesic from a to p is contained in $R_d(a,b)$. Let γ be any such geodesic. For each t we have

$$d(a,\gamma(t)) + d(\gamma(t),p) = d(a,p) < d(b,p) \leq d(b,\gamma(t)) + d(\gamma(t),p)$$

where the first inequality uses $p \in R_d(a,b)$ and the second uses the triangle inequality. Cancel $d(\gamma(t),p)$ from each side to get $d(a,\gamma(t)) < d(b,\gamma(t))$. □

Lemma 6.9 rules out examples like the following. Let $X = \mathbb{R}^2$ with any compatible geodesic metric, with $a = (-1,1)$, $b = (1,-1)$, and $B = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$. Then $X - B$ has four arc components, so B cannot be made into the equiset by choosing an equivalent geodesic metric.

We can strengthen Lemma 6.9 to describe what happens when we attach the equiset back to either arc component.

Lemma 6.10. *Let (X,d) be a geodesic space with distinct $a,b \in X$. Then $R_d(a,b) \cup E_d(a,b)$ is a union of geodesics with one endpoint a, and $R_d(b,a) \cup E_d(a,b)$ is a union of geodesics with one endpoint b.*

Proof. The proof is the same as Lemma 6.9, except we only have $d(a,p) \leq d(b,p)$ rather than $d(a,p) < d(b,p)$. □

Example 6.11. Let X be the triod $([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$ with the path metric. Let $a = (-1,0)$, $b = (1,0)$, and $B = \{(-1/2,0),(0,1)\}$. There is only one way to express $X - B$ as the disjoint union of two sets O_a, O_b, with O_a the union of a family of arcs beginning at a, and O_b the union of a family of arcs beginning at b. Namely $O_a = [-1,-1/2) \times \{0\}$ and $O_b = ((-1/2,1] \times \{0\}) \cup (\{0\} \times [0,1))$. Hence, if we have $B = E_d(a,b)$ then Lemma 6.9 says $R_d(a,b) = O_a$ and $R_d(b,a) = O_b$. However, the set $R_d(a,b) \cup B$ is not connected. This contradicts Lemma 6.10, and we conclude that the set B cannot be made into the equiset by choosing an equivalent geodesic metric.

Example 6.12. Consider the triod from the previous example. Lemmas 6.9 and 6.10 do not rule out sets like $\{0\} \times [0,1]$ from occurring as the equiset with respect to a geodesic metric. Indeed, the set occurs as the equiset under the standard shortest path metric. However, such a set does not occur as the midset under any geodesic metric. It follows from Lemma 6.3 that every point of $M_d(a,b)$ must be contained in a geodesic from a to b. Since there is no arc from a to b through $c = (0, 1)$, we must have $c \notin M_d(a, b)$ for every geodesic metric d.

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