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On Abstract Convexity with Respect to a Certain Quadratic Coupling Function: Subsets

Jakub Maksymiuk
Institute of Applied Mathematics, Faculty of Technical Physics and Applied Mathematics
Gdańsk University of Technology, Poland
jakub.maksymiuk@pg.edu.pl

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We investigate abstract convexity generated by the coupling function

$$\varphi(x, A) = |A[x]^2|, \quad x \in \mathbb{R}^n,\ A \in \mathcal{L}^2_{sym}(\mathbb{R}^n, \mathbb{R}).$$

We study the φ-convexity of four classes of sets: subsets of $\mathbb{R}^n$, subsets of $\mathcal{L}$, epigraphic subsets of $\mathbb{R}^n \times \mathbb{R}$ and epigraphic subsets of $\mathcal{L} \times \mathbb{R}$. We show that all families consist of closed, star-shaped sets and subsets of $\mathcal{L}$ and $\mathcal{L} \times \mathbb{R}$ are convex. For epigraphic subsets of $X \times \mathbb{R}$, we derive a condition partially characterizing them in analogy with classical convexity, where line segments are replaced by parabolas. In the one-dimensional case we obtain a complete characterization.

Keywords: Abstract convexity, coupling function, bilinear maps, quadratic functions.

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1. Introduction

Abstract convexity theory, also called convexity without linearity, is a far-reaching generalization of classical convexity. It replaces affine half-spaces and affine functions in the definition of convex objects, such as sets or functions, with more general, not necessarily geometric, objects. This approach allows one to recognize a variety of situations as a special case of abstract convexity. Although it is quite general, it still allows for the definition of basic concepts from classical convexity, including (abstract) convexity of sets, convexity of functions, conjugation, and subdifferential.

The purpose of this paper is to study the properties of abstract convexity of subsets with respect to the coupling function (φ-convexity for short) defined by

$$\varphi\colon X \times \mathcal{L} \to \mathbb{R}, \quad \varphi(x, A) = |A[x]^2|,$$

where $X = \mathbb{R}^n$ and $\mathcal{L} = \mathcal{L}^2_{sym}(\mathbb{R}^n, \mathbb{R})$. We will work within the framework presented in the book by I. Singer [11], applying it to our specific case.

We study the φ-convexity of four classes of sets: subsets of X, subsets of $\mathcal{L}$, epigraphic subsets of $X \times \mathbb{R}$, and epigraphic subsets of $\mathcal{L} \times \mathbb{R}$. These classes of sets are defined as the intersections of sublevel sets (resp. of epigraphs) of certain families of functions induced by the coupling function φ.

The main result of the present paper is to identify the fundamental properties of each of the four classes. These include the following observations: all four families

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consist of closed and star-shaped sets; φ-convex subsets of $\mathcal{L}$ and $\mathcal{L}\times\mathbb{R}$ are convex. Moreover, for epigraphic subsets of $X\times\mathbb{R}$, we provide a condition that gives a partial characterization in the form analogous to classical convexity, replacing segments by parabolas. We also analyze the relations between subsets of X and $X\times\mathbb{R}$, and between subsets of $\mathcal{L}$ and $\mathcal{L}\times\mathbb{R}$. Additionally, in the special case $X=\mathbb{R}^1$, a complete characterization is given.

For each class of sets considered, we provide examples that illustrate what φ-convex sets look like. We also present examples of sets that satisfy necessary conditions but fail to be φ-convex. In such cases, we discuss the obstacles and provide some remarks on possible forms of sufficient conditions.

This work is the first in a planned series of papers; subsequent papers will focus on φ-convexity of functions, conjugate functions, and the subdifferential.

At least two other approaches to abstract convexity are presented in the monograph [10] by A. Rubinov and in [9] by D. Pallaschke and S. Rolewicz. It seems that all three approaches lead to the same results, at least within the scope of this paper. The choice of [11] is due to its convenient and comprehensive treatment of abstract convexity for the purposes of this work. For the reader's convenience, all relevant definitions and theorems used in this paper are collected in the Appendix.

The main motivation for this study arises from several questions posed in the recently published work [8]. That paper examined the second-order differentiability of the modular

$$R_G(u)=\int_\Omega G(u)\,dx$$

in the Orlicz space $L^G(\Omega,\mathbb{R})$ and the Orlicz-Sobolev space $W^1L^G(\Omega,\mathbb{R})$. Here, $G\colon\mathbb{R}^n\to\mathbb{R}$ is a G-function, i.e., it is convex, even and nonnegative. Let $H\colon\mathcal{L}\to\mathbb{R}$ be the Fenchel-Moreau conjugate of G with respect to φ

$$H(A)=\sup_{x\in\mathbb{R}^n}\{\varphi(x,A)-G(x)\}$$

It was shown in [8] that if G satisfies certain growth conditions and its second derivative belongs to the Orlicz space generated by the Fenchel-Moreau conjugate H of G, then the modular R_G is twice differentiable. To obtain this result, the reasoning was set within the framework of abstract convexity generated by the coupling function φ.

Despite the positive result, several questions arise. One concerns the form of the necessary condition for second-order differentiability, denoted as (A_3) in [8] which reads:

$$\text{there exists } M>0 \text{ such that } H(MD^2G(x))\le G(\sqrt{2}x)\quad\text{for all } x\in\mathbb{R}^n.$$

This condition is implied by

(K) there exists a function $K\colon\mathbb{R}^n\to(0,\infty)$ bounded away from 0 such that

$$K(x_0)\left(D^2G(x_0)[x]^2-D^2G(x_0)[x_0]^2\right)\le G(x)-G(x_0)\quad\text{for all } x,x_0\in\mathbb{R}^n.$$

Condition (K) means that $K(x_0)D^2G(x_0)$ is an element of the abstract subdifferential of G at x_0, that is, the set defined by

$$\partial^{\varphi}G(x_0) = \{A \in \mathcal{L}\colon \varphi(x, A) - \varphi(x_0, A) \leq G(x) - G(x_0)\}.$$

The geometric interpretation of (K) is that G can be approximated by the family of quadratic functions of the form

$$x \mapsto \varphi\left(x, K(x_0)D^2G(x_0)[x_0]^2\right) + r = \left|K(x_0)D^2G(x_0)[x]^2\right| + r.$$

In fact, G is the pointwise supremum, or the upper envelope, of this family. In the framework of φ-convexity, this means that G is a φ-convex function.

There is a direct analogy with the well-known results from convex analysis, see for example [6]. Recall that for a convex and differentiable function $f\colon X \to \mathbb{R}$, its gradient $\nabla f(x_0)$ is an element of the subdifferential $\partial f(x_0)$ and the following inequality holds for all $x, x_0 \in \mathbb{R}^n$:

$$\langle \nabla f(x_0), x\rangle - \langle \nabla f(x_0), x_0\rangle \leq f(x) - f(x_0)$$

This is a direct consequence of convexity and differentiability of f. Geometrically, f can be approximated by some family of affine functions. It can be shown (e.g. [2, 3]) that the gradient of a G-function G belongs to the Orlicz space generated by the convex conjugate of G, which is the dual space of L^G, and that the modular is differentiable on Orlicz space L^G. Moreover, $G^*(\nabla G(x)) \leq G(2x)$ holds.

All these results can be obtained from the basic properties of the convex conjugate and the definitions of G-function and of Orlicz space. Therefore, the connection between the gradient of G, the subdifferential and the differentiability of modular in Orlicz spaces can be viewed as a fundamental property of G-functions. Note that the condition similar to (A_3), i.e., $G^*(F_v(x,u,v)) \leq a(u)(b(x) + c\,G(v))$, is one of the differentiability conditions for the more general functional $\int_\Omega F(x,u,v)\,dx$ on the Orlicz space. We refer the reader to [2, 3, 4, 7] for more details.

Comparing these two cases, one may conjecture that just as convexity is essential for first-order differentiability, φ-convexity is essential for second-order differentiability of the modular on Orlicz spaces. In particular, it can be supposed that condition (K) (or (A_3)) is an inherent property of these G-functions that are simultaneously φ-convex. Moreover, it also likely that theorems characterizing twice differentiable functionals in Orlicz spaces can be formulated in a manner analogous to existing characterizations in Lebesgue spaces. See, for example, the books by Dudley and Norvaiša [5] or Appell and Zabrejko [1]. The latter book provides partial affirmative answer for Orlicz spaces.

To resolve conjectures formulated above, a more detailed investigation of φ-convexity is necessary. This involves the study of abstract convex functions, their abstract conjugates and abstract subdifferentials. It turns out that this also requires studying of abstract convex sets and the epigraphs of abstract convex functions.

The reason is that, similarly to the classical theory, a function is φ-convex if and only if its epigraph is φ-convex. Also, the sublevel sets of abstract convex function are abstract convex. The abstract subdifferential of a function also has its counterpart: every element of the subdifferential defines a supporting "hyperplane" of the epigraph, hence some knowledge of the geometry of the epigraph is needed.

One of the notable differences from the classical theory is that the notions of abstract convexity for an epigraph and for a subset of the domain of a function are different. This difference is explained in the Appendix and in the final remarks in Section 4.

Moreover, the study of the abstract convexity of a function and its conjugate leads to the abstract convexity for $X \times \mathbb{R}$ and $\mathcal{L} \times \mathbb{R}$, which is different in these cases. Therefore, it is necessary to analyze each of these cases separately.

Let us announce that, at least in the one-dimensional case, the conjecture stated earlier appears to be true. A twice differentiable φ-convex G-function has its second derivative in the abstract subdifferential and a condition similar to (A_3) holds. The proof of this result will be presented in a forthcoming publication.

Regardless of the main motivation, the geometry and properties of this instance of abstract convexity (i.e., generated by the coupling function φ) are interesting on their own right. One can view these sets and functions as being approximated by quadratic functions, in direct analogy to classical convexity, where convex sets and functions are approximated by affine functions. This suggests the possibility that applications in optimization theory can be found in similar manner as for classical convexity.

The organization of the paper is as follows. In the Preliminaries we establish a notation and collect some results used throughout the paper. In Section 3 we examine properties of abstract linear and abstract affine functions. Section 4 investigates properties of abstract convex subsets of X, $X \times \mathbb{R}$, $\mathcal{L}$ and $\mathcal{L} \times \mathbb{R}$. In Section 5, we provide a characterization of abstract convex sets in the special case $X = \mathbb{R}$. In the Appendix, a short outline of abstract convexity is provided, based on [11, Chapters 1 and 2].

2. Preliminaries

Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the standard norm and inner product on $\mathbb{R}^n$. We denote by $\mathcal{L} = \mathcal{L}^2_{sym}(\mathbb{R}^n, \mathbb{R})$, the space of symmetric bilinear maps equipped with the norm

$$\|A\|_{\mathcal{L}} = \sup\{|A[x]^2| \colon \|x\| \leq 1\}$$

where we abbreviate $A[x,x]$ to $A[x]^2$. We will identify a bilinear map A with its matrix on the standard basis. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $A \in \mathcal{L}$ we will write

$$A[x]^2 = \sum_{i=1}^{n} A_{ij} x_i x_j.$$

In some places, for better readability, we will also write $x = (\alpha_1, \dots, \alpha_n)$, $X_{ij} = A_{ij}$, and

$$A[x]^2 = \sum_{i,j=1}^{n} \alpha_i \alpha_j X_{ij}.$$

We introduce an Euclidean structure in $\mathcal{L}$ by equipping it with the Frobenius inner product.

Let $A, B \in \mathcal{L}$, the Frobenius inner product and the induced norm are given by

$$\langle A, B\rangle_F = \mathrm{tr}(A^T B) = \sum_{i,j=1}^{n} A_{ij}B_{ij}, \text{ and } \|A\|_F = \sqrt{\langle A, A\rangle_F} = \sqrt{\sum_{i,j=1}^{n} A_{ij}^2}\ .$$

For $x \in \mathbb{R}^n$ define a bilinear map $T_x \in \mathcal{L}$ by $T_x[a, b] = \langle x, a\rangle \langle x, b\rangle$. We have

$$\langle T_x, A\rangle_F = \sum_{i,j=1}^{n} A_{ij}x_i x_j = A[x]^2$$

$$\langle T_x, T_y\rangle_F = \sum_{i,j=1}^{n} x_i y_i\, x_j y_j = T_x[y]^2 = \langle x, y\rangle^2, \quad \|T_x\|_F^2 = T_x[x]^2 = \|x\|^4 .$$

Let V be a normed space. Recall that a subset G of V is called

- *balanced*, if $\lambda g \in G$, for all $g \in G, \lambda \in [-1, 1]$,
- *symmetric*, if $G = -G$
- *star-shaped at* $g \in G$, if $(1 - \lambda)v + \lambda g \in G$, for all $v \in G, \lambda \in [0, 1]$.

It follows that G is balanced if and only if G is symmetric and star-shaped at 0. For a subset E of $V \times \mathbb{R}$ we will say that it is

- *V-balanced* if $(\lambda v, d) \in E$ for all $(v, d) \in E$, $\lambda \in [-1, 1]$,
- *bounded below* if there exists $D \in \mathbb{R}$ such that $E \subset V \times [D, \infty)$,
- *V-symmetric* if $(v, d) \in E \implies (-v, d) \in E$.

Set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For functions in $\overline{\mathbb{R}}^V$ we define

- *sublevel sets:* $S_d(f) = \{v \in V : f(v) \le d\}$, $d \in \mathbb{R}$,
- *the epigraph:* $\operatorname{epi} f = \{(v, r) \in V \times \mathbb{R} : f(v) \le r\}$.

It is well known that the following properties are equivalent:

- f is lower semi-continuous on V;
- $\operatorname{epi} f$ is a closed subset of $V \times \mathbb{R}$.

For any index set I and any $\{f_i\}_{i\in I} \subset \overline{\mathbb{R}}^V$ we define

$$\sup\{f_i : i \in I\} \in \overline{\mathbb{R}}^V, \quad (\sup\{f_i : i \in I\})(v) = \sup\{f_i(v) : i \in I\}, \quad v \in V$$

We adapt the convention that for $I = \emptyset$ we have $\sup\{f_i : i \in I\} = -\infty$. We have

$$\operatorname{epi}\left(\sup_{i\in I} f_i\right) = \bigcap_{i\in I} \operatorname{epi} f_i. \tag{1}$$

A subset $E \subset V \times \mathbb{R}$ is called *epigraphic* if there exists a function $f : V \to \overline{\mathbb{R}}$ such that $E = \operatorname{epi} f$.

Proposition 2.1. ([11, Proposition 3.4]) *A subset $E \subset V \times \mathbb{R}$ is epigraphic if and only if it satisfies the following conditions:*

$$(v, d) \in E \implies \{v\} \times [d, \infty) \subset E,$$
$$\{(v, d_n)\} \subset E, d_n \to d \in \mathbb{R} \implies (v, d) \in E.$$

In what follows, we will need an equation of the parabola passing through two points in space. Let $V = \mathbb{R}^n$, $v \in V \setminus \{0\}$ and let $0 \leq t_0 < t_1$ be fixed. Set $x_0 = t_0 v$ and $x_1 = t_1 v$. It is a matter of routine calculations to obtain the equation of the parabola through $P_0 = (x_0, d_0)$ and $P_1 = (x_1, d_1)$ in $V \times \mathbb{R}$ with axis $\{0\} \times \mathbb{R}$. It is given by

$$\gamma_{P_0P_1} \colon \mathbb{R} \to X \times \mathbb{R}, \quad \gamma_{P_0P_1}(t) = \left(tv, \frac{d_1 - d_0}{t_1^2 - t_0^2}(t^2 - t_0^2) + d_0 \right). \tag{2}$$

Observe that $\gamma_{P_0P_1}(t_0) = (x_0, d_0)$, $\gamma_{P_0P_1}(t_1) = (x_1, d_2)$ and

$$\frac{d_1 - d_0}{t_1^2 - t_0^2}(t^2 - t_0^2) + d_0 = \frac{d_1 - d_0}{t_1^2 - t_0^2}(t^2 - t_1^2) + d_1.$$

The following lemma is an extension of [6, Proposition 1.1.3].

Lemma 2.2. *Let $P_A = (t_A, d_A)$, $P_B = (t_B, d_B)$ and $P_C = (t_C, d_C)$ be three points in $\mathbb{R}^2$ with $0 \leq t_A < t_B < t_C$. Then the following conditions are equivalent:*

(i) *P_B is below parabola $\gamma_{P_AP_C}$;*

(ii) $\dfrac{d_B - d_A}{t_B^2 - t_A^2} \leq \dfrac{d_C - d_A}{t_C^2 - t_A^2}$;

(iii) $\dfrac{d_C - d_A}{t_C^2 - t_A^2} \leq \dfrac{d_C - d_B}{t_C^2 - t_B^2}$.

Proof. From $t_B^2 - t_A^2 > 0$ and (2), we obtain

$$d_B \leq \frac{d_C - d_A}{t_C^2 - t_A^2}(t_B^2 - t_A^2) + d_A \iff \frac{d_B - d_A}{t_B^2 - t_A^2} \leq \frac{d_C - d_A}{t_C^2 - t_A^2}.$$

This proves (i) $\iff$ (ii). From $t_B^2 - t_C^2 < 0$ and (2), we obtain

$$d_B \leq \frac{d_C - d_A}{t_C^2 - t_A^2}(t_B^2 - t_C^2) + d_C \iff \frac{d_B - d_C}{(t_B^2 - t_C^2)} \geq \frac{d_C - d_A}{t_C^2 - t_A^2}.$$

This proves (i) $\iff$ (iii). □

We will also need the following simple observation.

Lemma 2.3. *Let $\alpha, \beta, d_A, d_C \in \mathbb{R}$ and $0 \leq t_A < t_B < t_C$. Assume that*

$$\alpha t_A^2 + \beta \leq d_A \quad \text{and } \alpha t_C^2 + \beta \leq d_C.$$

Then for all $t_B \in [t_A, t_C]$ the point $(t_B v, \alpha t_B^2 + \beta)$ is below $\gamma_{P_AP_C}$.

Proof. If $\dfrac{d_C - d_A}{t_C^2 - t_A^2} \leq \alpha$ then

$$\frac{d_C - d_A}{t_C^2 - t_A^2} \leq \frac{(\alpha t_C^2 + \beta) - (\alpha t_B^2 + \beta)}{t_C^2 - t_B^2} \leq \frac{d_C - (\alpha t_B^2 + \beta)}{t_C^2 - t_B^2}.$$

If $\dfrac{d_C - d_A}{t_C^2 - t_A^2} \geq \alpha$ then

$$\frac{d_C - d_A}{t_C^2 - t_A^2} \geq \frac{(\alpha t_B^2 + \beta) - (\alpha t_A^2 + \beta)}{t_B^2 - t_A^2} \geq \frac{(\alpha t_B^2 + \beta) - d_A}{t_B^2 - t_A^2}.$$

Hence, for all $t_B \in [t_A, t_C]$ the point $(t_B v, \alpha t_B^2 + \beta)$ is below $\gamma_{P_AP_C}$ by Lemma 2.2. □

3. Coupling function and related concepts

Let $X = \mathbb{R}^n$, $n \geq 1$, and let $\mathcal{L} = \mathcal{L}^2_{sym}(X, \mathbb{R})$. We will work with the coupling function:

$$(\varphi) \qquad \varphi\colon X \times \mathcal{L} \to [0, \infty), \quad \varphi(x, A) = |A[x]^2|.$$

We will also consider the dual coupling function:

$$(\psi) \qquad \psi\colon \mathcal{L} \times X \to [0, \infty), \quad \psi(A, x) = \varphi(x, A).$$

In the rest of the paper, as a rule, objects related to the set X (such as subsets, functions, etc.) will be distinguished by incorporating the symbols X or φ into their descriptions. Similarly, objects associated with the set $\mathcal{L}$ will be identified using the symbols $\mathcal{L}$ or ψ.

In what follows, we refer to the concepts of abstract convexity introduced in the Appendix. The reader unfamiliar with abstract convexity should consult the Appendix.

3.1. φ-derived functions

The coupling function φ induces two families of functions:

Definition 3.1.

$$\begin{aligned} \operatorname{Lin}(X) &= \left\{\varphi_A \in \mathbb{R}^X : \varphi_A = \varphi(\cdot, A), \text{ for some } A \in \mathcal{L}\right\}, \\ \operatorname{Aff}(X) &= \left\{\varphi_A + r \in \mathbb{R}^X : \varphi_A \in \operatorname{Lin}(X),\ r \in \mathbb{R}\right\} = \operatorname{Lin}(X) + \mathbb{R}. \end{aligned}$$

The sets $\operatorname{Lin}(X)$ and $\operatorname{Aff}(X)$ are called the set of X*-linear functions* and the set of X*-affine functions*, respectively. $\square$

For further use, below we list some properties of X-affine functions, their sublevel sets and epigraphs. Let $A \in \mathcal{L}$ and $r \in \mathbb{R}$ then:

- $\varphi_A + r$ is even and continuous,
- φ_A is absolute homogeneous of degree 2: $\varphi_A(\lambda x) = \lambda^2 \varphi_A(x)$, $x \in X$, $\lambda \in \mathbb{R}$.

Due to the homogeneity of X-affine functions, if $d > 0$ then $S_d(\varphi_A) = S_1(\varphi_{\frac{1}{d}A})$. Thus, when considering sublevel sets, it is sufficient to consider only the values $d = -1, 0, +1$. However, often it is more natural not to limit the values of d. Note that sublevel sets and epigraphs of φ_A and φ_{-A} are identical.

Observe that for all $A \in \mathcal{L}$ we have

- $S_d(\varphi_A) = \emptyset$, $d < 0$;
- $S_d(\varphi_0) = X$, $d \geq 0$;
- $\{0\} \in S_d(\varphi_A)$, $d \geq 0$;
- $\operatorname{epi}(\varphi_0 + r) = X \times [r, \infty)$, $r \in \mathbb{R}$;
- $\{0\} \times [r, \infty) \subset \operatorname{epi}(\varphi_A + r) \subset X \times [r, \infty)$, $r \in \mathbb{R}$ (cf. Theorem 4.9).

If A is semi-definite then $\varphi_A(x) = \pm A[x]^2$ and $D^2\varphi_A(x)[h]^2 = \pm A[h]^2 \geq 0$. Hence, if A is semi-definite, φ_A is convex. If A is not definite then φ_A is not convex. Therefore, sublevel sets and epigraphs of X-affine functions can be both convex and non-convex.

If $X = \mathbb{R}^1$ then the graph of $\varphi_A + r$ is an upward directed parabola. In higer dimensions, roughly speaking, the graph of $\varphi_A + r$ is a quadratic hypersurface in $X \times \mathbb{R}$, with its negative part reflected along X. It is easy to check that X-affine function is convex on any line through the origin.

To get some further geometric intuition on sublevel sets and epigraphs of X-affine functions, below we examine low-dimensional examples.

Example 3.2. Let $X = \mathbb{R}^1$, then $S_d(\varphi_A)$ is one of the sets: $\emptyset$, X, $\{0\}$ or $[-\alpha, \alpha]$, $\alpha > 0$. The possible epigraphs of $\varphi_A + r$ are

$$X \times [r, \infty) \ \text{ and } \ \{(x, d) \in X \times \mathbb{R} \colon \alpha x^2 + r \leq d,\ \alpha > 0\},$$

the latter being a region bounded by an upward opening parabola. □

Example 3.3. Let $X = \mathbb{R}^2$ and let $A \in \mathcal{L}$. We may assume that $A[x]^2 = x_1^2 + \varepsilon x_2^2$, where $\varepsilon = -1, 0, +1$. All possible sublevel sets are: $\emptyset$, X, and the sets listed in the tables below:

	if $d = 0$ then $S_d(\varphi_A)$ is ...
$\varepsilon = -1$	a union of two nonparallel lines through the origin
$\varepsilon = 0$	a line through the origin
$\varepsilon = +1$	the set $\{0\}$

	if $d > 0$ then $S_d(\varphi_A)$ is a region ...
$\varepsilon = -1$	bounded by two conjugate hyperbolas
$\varepsilon = 0$	bounded by two parallel lines
$\varepsilon = +1$	bounded by an ellipse

The epigraphs corresponding to values $\varepsilon = -1, 0, 1$ are regions bounded from below by, respectively:

- a hyperbolic paraboloid with its negative part reflected along X,
- a parabolic cylinder,
- an elliptic paraboloid. □

Example 3.4. Let $X = \mathbb{R}^3$. Qualitatively different sublevel sets are the empty set, X and the following:

- for $d > 0$, a region bounded by: ellipsoid, a one-sheeted and a two-sheeted hyperboloid, an elliptic cylinder, a cylinder over conjugate hyperbolas, two parallel planes;
- for $d = 0$, the set $\{0\}$, a cone, two nonparallel planes, a plane through the origin, a line through the origin. □

For higher dimensions analogous description can be given using eigenvalues of A. Notably, the variety of possible shapes increases rapidly, making the description of abstract convex subsets of X and $X \times \mathbb{R}$ quite involved. We want to emphasize that the families of sublevel sets in $X_1 = \mathbb{R}^{n+1}$ and epigraphs in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ are different, with the only common elements being the empty set and $\mathbb{R}^{n+1}$.

3.2. ψ-derived functions

The coupling function ψ induces two families of functions:

Definition 3.5.

$$\operatorname{Lin}(\mathcal{L}) = \left\{\psi_x \in \mathbb{R}^{\mathcal{L}} \colon \psi_x = \psi(\cdot, x),\ \text{for some } x \in X\right\},$$
$$\operatorname{Aff}(\mathcal{L}) = \left\{\psi_x + r \in \mathbb{R}^{\mathcal{L}} \colon \psi_x \in \operatorname{Lin}(\mathcal{L}),\ r \in \mathbb{R}\right\} = \operatorname{Lin}(\mathcal{L}) + \mathbb{R}.$$

The sets $\operatorname{Lin}(\mathcal{L})$ and $\operatorname{Aff}(\mathcal{L})$ are called the set of $\mathcal{L}$-*linear* functions and the set of $\mathcal{L}$-*affine* functions, respectively. □

For further use, below we list some properties of $\mathcal{L}$-affine functions, their sublevel sets and epigraphs. Let $x \in X$ and $r \in \mathbb{R}$ then:

- $\psi_x + r$ is even and continuous,
- ψ_x is absolute homogeneous of degree 1: $\psi_x(\lambda A) = |\lambda|\psi_x(A),\ A \in \mathcal{L}, \lambda \in \mathbb{R}$
- ψ_x is convex, since for any A_0, A_1 and $\lambda \in [0,1]$ we have

$$\psi_x(\lambda A_1 + (1-\lambda)A_0) = |(\lambda A_1 + (1-\lambda)A_0)[x]^2| \leq$$
$$\leq \lambda|A_1[x]^2| + (1-\lambda)|A_0[x]^2| = \lambda\psi_x(A_1) + (1-\lambda)\psi_x(A_0).$$

Due to the homogeneity of $\mathcal{L}$-affine functions, if $d > 0$ then $S_d(\psi_x) = S_1(\psi_{\frac{1}{d}x})$. Therefore, when considering sublevel sets, it is sufficient to consider only values $d = -1, 0, +1$. However, it is often more natural not to limit the values of d. Note also that sublevel sets and epigraphs of ψ_x and ψ_{-x} are identical.

Observe that for all $x \in X$ we have

- $S_d(\psi_x) = \emptyset$ for all $x \in X$ and $d < 0$;
- $S_d(\psi_0) = \mathcal{L}$ for all $d \geq 0$;
- $\{0\} \in S_d(\psi_x), d \geq 0$;
- $\operatorname{epi}(\psi_0 + r) = \mathcal{L} \times [r, \infty)$, $r \in \mathbb{R}$;
- $\{0\} \times [r, \infty) \subset \operatorname{epi}(\varphi_A + r) \subset \mathcal{L} \times [r, \infty)$ (cf. Theorem 4.29).

Recall from Section 2 that we endowed $\mathcal{L}$ with the Frobenius inner product. For $x = (\alpha_1, \ldots, \alpha_n) \in X$, and A represented by a matrix $[X_{ij}]$ we have

$$\langle T_x, A\rangle_F = A[x]^2 = \sum_{i,j=1}^{n} \alpha_i \alpha_j X_{ij}.$$

Having this representation, we can provide a geometric interpretation of $\mathcal{L}$-affine functions and their sublevel sets.

An $\mathcal{L}$-affine function $\psi_x(A) = |A[x]^2| + r = |\langle T_x, A\rangle_F| + r$ can be considered as an affine function of the variables X_{ij}, with its negative part reflected along $\mathcal{L}$.

Let $d > 0$ and $x \neq 0$. Then

$$S_0(\psi_x) = \{A \in \mathcal{L} \colon \langle T_x, A\rangle_F = 0\}$$

is a hyperplane through 0 with a normal vector T_x, and

$$S_d(\psi_x) = \{A \in \mathcal{L} \colon |\langle T_x, A\rangle_F| \leq d\}$$

is a region bounded by two parallel hyperplanes in $\mathcal{L}$, orthogonal to the vector T_x.

Note that the direction of T_x is not arbitrary, since only n out of the $\frac{n(n+1)}{2}$ coordinates are independent. Hence, not every hyperplane can occur in the description of sublevel sets of $\mathcal{L}$-linear functions.

If $e_i = (0, \ldots, 1, \ldots, 0)$ then the hyperplane $S_0(\psi_{e_i})$ is parallel to the coordinates hyperplane $X_{ii} = 0$, since $\langle T_{e_i}, A\rangle_F = X_{ii}$. In other cases the direction of the normal vector is limited to the cone $\{A \in \mathcal{L} \colon X_{ii} \geq 0, i = 1, \ldots, n\}$.

If $X = \mathbb{R}^1$, then the possible forms of $S_d(\psi_x)$ are $\emptyset, \mathcal{L} \simeq \mathbb{R}, \{0\}$, and intervals of the form $[-a, a]$, $a > 0$. In any dimension, the epigraph of $\psi_x + r$ is equal to $\mathcal{L} \times [r, \infty)$ if $x = 0$ and it is the analogue of the epigraph of $\xi \mapsto a|\xi| + b$ if $x \neq 0$.

Although the above description of $\mathcal{L}$-affine functions, sublevel sets and epigraphs is relatively simple, it is not entirely satisfactory, and a more refined one could be considering the specific structure of T_x. The rapid growth of the dimension of $\mathcal{L}$ suggests that only an algebraic description is feasible.

4. Abstract convexity of subsets

Let φ and ψ be a pair of coupling functions defined by (φ) and (ψ), respectively, with corresponding families of abstract linear and abstract affine functions. As described in the Appendix, these families of functions define four classes of sets:

- $\mathrm{Lin}(X)$-convex subsets of X,
- $\mathrm{Aff}(X)$-convex subsets of $X \times \mathbb{R}$,
- $\mathrm{Lin}(\mathcal{L})$-convex subsets of $\mathcal{L}$,
- $\mathrm{Aff}(\mathcal{L})$-convex subsets of $\mathcal{L} \times \mathbb{R}$.

In this section, we investigate the properties of these four families. For the reader's convenience, we will rewrite the definition of abstract convexity and the separation property using the notation specific to each case.

The general approach presented in the Appendix does not require any assumptions on either the sets X and $\mathcal{L}$ or the function φ. Therefore, the obtained results provide limited geometric or analytical intuitions. In our case, we have equipped the sets X and $\mathcal{L}$ with a structure of the Euclidean space and the coupling function is given explicitly. These additional structures allow us to give a geometric meaning to the abstract concepts and to derive further properties specific to this particular choice of X, $\mathcal{L}$, and φ.

4.1. Subsets of X

Definition 4.1. We say that the subset $G \subset X$ is $\mathrm{Lin}(X)$-convex if

$$G = \bigcap \left\{S_d(\varphi_A) \in \mathcal{S}(\mathrm{Lin}(X) \times \mathbb{R}) \colon G \subset S_d(\varphi_A)\right\}.$$

We will denote by $\mathcal{K}(X)$ the family of all $\mathrm{Lin}(X)$-convex subsets of X. □

By Propositions A.6 and A.8, every set $G \in \mathcal{K}(X)$ has the following separation

property: $\quad \forall_{\overline{x} \notin G}\ \exists_{A \in \mathcal{L}}\ \exists_{d \in \mathbb{R}}\ G \subset S_d(\varphi_A) \wedge \overline{x} \notin S_d(\varphi_A)$

or, equivalently, $\quad \forall_{\overline{x} \notin G}\ \exists_{A \in \mathcal{L}}\ \exists_{d \in \mathbb{R}}\ \forall_{x \in G}\ \varphi_A(x) \leq d \wedge \varphi_A(\overline{x}) > d.$

The separation property for $\mathrm{Lin}(X)$-convex sets has clear geometric interpretation: the condition $G \subset S_d(\varphi_A)$ means that G is contained in the set $S_d(\varphi_A)$, the region in $X = \mathbb{R}^n$ bounded by two (possibly degenerate) quadrics, and the point $\overline{x}$ lies outside this region. The eigensystem of the matrix of A determines its shape and the parameter d determines its size.

By Proposition A.5, we have that

$$S_d(\varphi_A) \in \mathcal{K}(X), \quad \text{for all } A \in \mathcal{L} \text{ and } d \in \mathbb{R}.$$

This case also includes $\emptyset, X$ and $\{0\} \in \mathcal{K}(X)$.

Basic properties of $\mathrm{Lin}(X)$-convex sets follow directly from the definition of $\mathcal{K}(X)$ and the properties of φ_A.

Theorem 4.2. *Let G be a* $\mathrm{Lin}(X)$*-convex subset of X. Then*

- *G is closed in X,*
- *G is balanced.*

Proof. The theorem holds if $G = \emptyset$ or $G = X$. Let G be a proper subset of X and let

$$G = \bigcap \mathcal{M}_G, \quad \mathcal{M}_G = \{S_d(\varphi_A) \colon G \subset S_d(\varphi_A)\}.$$

Since φ_A is continuous, $S_d(\varphi_A) = \varphi_A^{-1}((\infty, d])$ is closed and, consequently, G is closed. Let $g \in G$ and $\lambda \in [-1, 1]$. For any X-linear function we have $\varphi_A(\lambda g) = \lambda^2 \varphi_A(g) \leq d$. Hence, $\lambda g \in G$. □

It follows that a nonempty $\mathrm{Lin}(X)$-convex subset always contains 0, is symmetric, and star-shaped at 0. Moreover, if $x \in G$ then the set G contains the interval $[-1, 1]\{x\}$. In particular, the only $\mathrm{Lin}(X)$-convex singleton is $\{0\}$.

Intersection of a convex set with arbitrary affine subspace is convex. For a $\mathrm{Lin}(X)$-convex subset G, this need not be the case. It can be easily seen by considering the set $S_1(\varphi_A)$ with $A[x]^2 = x_1^2 \pm x_2^2$. However, the intersection of $\mathrm{Lin}(X)$-convex subsets of X with an arbitrary subspace remains abstract convex. Moreover, it can be viewed as a lower dimensional abstract convex sets.

Proposition 4.3. *Let V be a linear subspace of X. Then $V \in \mathcal{K}(X)$.*

Proof. If $V = \{0\}$ or $V = X$ then the theorem holds. Assume that $\dim V = k$, where $0 < k < n = \dim X$. Let $\{g_1, \ldots, g_k\}$ be an orthonormal basis of V and let $\{f_1, \ldots, f_{n-k}\}$ be an orthonormal basis of $V^\perp$. Then for $j = 1, \ldots, n-k$ and $x = \sum_{i=1}^k x_i g_i \in V$ we have

$$T_{f_j}[x]^2 = \left\langle f_j \sum_{i=1}^k x_i g_i \right\rangle^2 = \sum_{i=1}^k x_i \left\langle f_j, g_i \right\rangle^2 = 0.$$

Hence, $V \subset S_0(\varphi_{A_j})$, where $A_j = T_{f_j}$. Define

$$A[x]^2 = \sum_{j=1}^{n-k} A_j[x]^2 = \sum_{j=1}^{n-k} \left\langle f_j, x \right\rangle^2.$$

Clearly, $A \in \mathcal{L}$. It follows that $V \subset \bigcap_{i=1}^{n-k} S_0(\varphi_{A_j}) = S_0(\varphi_A)$.

For the reverse inclusion, observe that if $x \in S_0(\varphi_{A_j})$ then $\langle f_j, x\rangle = 0$ for all j. Hence, $x \in (V^\perp)^\perp = V$. $\square$

It follows that if $G \in \mathcal{K}(X)$ then $V \cap G \in \mathcal{K}(X)$, by Proposition A.5. Abusing the notation slightly, one can say that $V \cap G \in \mathcal{K}(V)$.

Proposition 4.4. *If $x \in X$ then the interval $[-1, 1]\{x\}$ is* Lin(X)*-convex.*

Proof. Let $x \in X$ and $\lambda \in [-1, 1]$. If $x = 0$ then $[-1, 1]\{x\} = \{0\} \in \mathcal{K}(X)$. Assume that $x \neq 0$. Since $T_x[\lambda x]^2 = \lambda^2 \|x\|^4 \leq \|x\|^4$, we have $\lambda x \in S_{\|x\|^4}(\varphi_{T_x})$. It follows that for $V = \mathbb{R}\{x\}$ we have $[-1, 1]\{x\} = S_{\|x\|^4}(\varphi_{T_x}) \cap V$. $\square$

In a similar manner one can show that a rectangle, a cube, and other similar polyhedra are abstract convex. See also Proposition 4.5 below.

A Lin(X)-convex set is not necessarily convex in the classical sense. A notable exception is the case $X = \mathbb{R}^1$, where $\mathcal{K}(\mathbb{R}^1)$ is the same as the class of closed, symmetric and convex subsets of $\mathbb{R}$, i.e., closed intervals centered at 0. In higher dimensions, the class of closed, symmetric and convex sets is a proper subset of $\mathcal{K}(X)$.

Proposition 4.5. *If $G \subset X$ is symmetric, closed and convex then $G \in \mathcal{K}(X)$.*

Proof. We may assume that G is proper. Let $\overline{x} \notin G$. Since G is closed and convex subset of $X = \mathbb{R}^n$, there exists $s \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that for any $x \in G$

$$\langle s, x\rangle \leq r \text{ and } \langle s, \overline{x}\rangle > r.$$

Since G is symmetric, for any $x \in G$ we have $\langle s, x\rangle + r \geq 0$. Since $\overline{x} \neq 0$ and $0 \in G$, we obtain $0 = \langle s, 0\rangle \leq r < \langle s, \overline{x}\rangle$ and $\langle s, -\overline{x}\rangle < 0 \leq r$. Thus $\langle s, \overline{x}\rangle + r > 0$.

Set $A = T_s$, then for every $x \in G$ we have

$$\varphi_A(x) = A[x]^2 = \langle s, x\rangle^2 \leq r^2 \text{ and } \varphi_A(\overline{x}) = A[\overline{x}]^2 = \langle s, \overline{x}\rangle^2 > r^2.$$

Hence G is Lin(X)-convex by the separation property. $\square$

The examples given in Section 3 show that the geometry of $S_d(\varphi_A)$, and thus of Lin(X)-convex sets, may be quite complicated. Consider, for example, subsets of $\mathbb{R}^2$ that are intersections of $S_1(\varphi_{A_i})$, $i = 1, 2, 3$, where $A_1[x]^2 = x_1^2 - x_2^2$, $A_2[x]^2 = x_1x_2$ and $A_3[x]^2 = x_1^2/2 + x_2^2/2$.

Observe that $S = \{x \in \mathbb{R}^2 \colon \max\{|x_1|, |x_2|\} \leq 1\} = S_1(\varphi_{T_{(1,0)}}) \cap S_1(\varphi_{T_{(0,1)}})$ is Lin(X)-convex. Removing a small part of an abstract convex set may preserve abstract convexity. For example, the set

$$S \cap S_1(\varphi_A), \quad A[x] = 2x_1 - x_2^2$$

remains abstract convex. However, if we remove too large a part of abstract convex set, the resulting set will no longer be abstract convex while still satisfying the properties given in Theorem 4.2.

Example 4.6. Let $G = \{x \in \mathbb{R}^2\colon \max\{||x_1|, |x_2|\} \le 1 \ \wedge\ (x_1 - x_2)(x_1 + x_2) \ge 0\}$. The set G is closed, balanced and bounded. Then every point outside the square $S = \{x \in \mathbb{R}^2\colon \max\{||x_1|, |x_2|\} \le 1\}$ can be separated from G by $S_1(\varphi_A)$ where either $A = T_{(0,1)}$ or $A = T_{(1,0)}$.

There also exist points in $S \setminus G$ that can be separated from G. Obviously, in such cases, the corresponding sublevel set must be a region between two conjugate hyperbolas. For example, the point $(\frac{9}{10}, 0)$ can be separated by $S_{16}(\varphi_A)$, where $A[x]^2 = 25x_1^2 - 9x_2^2$. This can be easily seen by drawing a picture.

However, G is not abstract convex. In fact, no point on the hyperbola $2x_1^2 - x_2^2 = 1$ with $x_2 \in (-1, 1)$ can be separated from G.

Let $\overline{x} = (x_1, x_2)$ be a point satisfying $2x_1^2 - x_2^2 = 1$, $x_1 > 0$ and $x_2 \in (-1, 1)$. It is easy to check that $\overline{x} \notin G$. Assume there exists $A \in \mathcal{L}$ and $d > 0$ such that $G \subset S_d(\varphi_A)$ and $\overline{x} \notin G$. Write $A[x]^2 = A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2$. Without loss of generality we may assume that $d = 1$ and that $A_{12} \ge 0$.

Substituting $(1, 1), (1, -1), (0, 1) \in G$ into $\varphi_A(x) \le 1$ we get

$$|A_{11} + 2A_{12} + A_{22}| \le 1 \wedge |A_{11} - 2A_{12} + A_{22}| \le 1 \wedge |A_{22}| \le 1.$$

For $\overline{x} = \left(x_1, \sqrt{2x_1^2 - 1}\right)$ we obtain

$$1 < \left|A_{11}x_1^2 + 2A_{12}x_1\sqrt{2x_1^2 - 1} + A_{22}\left(2x_1^2 - 1\right)\right|.$$

We show that this system of inequalities has no solution. Since $A_{12} \ge 0$, we have

$$2A_{12}x_1\sqrt{2x_1^2 - 1} \le A_{12}\left(x_1^2 + (2x_1^2 - 1)\right).$$

Hence

$$\begin{aligned}
1 &< A_{11}x_1^2 + 2A_{12}x_1\sqrt{2x_1^2 - 1} + A_{22}(2x_1^2 - 1) \le A_{11}x_1^2 + A_{12}(3x_1^2 - 1) + A_{22}(2x_1^2 - 1) \\
&= \frac{1 - x_1^2}{4}(A_{11} - 2A_{12} + A_{22}) + \frac{5x_1^2 - 1}{4}(A_{11} + 2A_{12} + A_{22}) + (x_1^2 - 1)A_{22} \\
&\le \frac{1 - x_1^2}{4} + \frac{5x_1^2 - 1}{4} + (x_1^2 - 1) = 2x_1^2 - 1
\end{aligned}$$

and

$$\begin{aligned}
-1 &> A_{11}x_1^2 + 2A_{12}x_1\sqrt{2x_1^2 - 1} + A_{22}(2x_1^2 - 1) \ge A_{11}x_1^2 - A_{12}(3x_1^2 - 1) + A_{22}(2x_1^2 - 1) \\
&= \frac{5x_1^2 - 1}{4}(A_{11} - 2A_{12} + A_{22}) + \frac{1 - x_1^2}{4}(A_{11} + 2A_{12} + A_{22}) + (x_1^2 - 1)A_{22} \\
&\ge -\frac{5x_1^2 - 1}{4} - \frac{1 - x_1^2}{4} - (x^2 - 1) = -(2x_1^2 - 1).
\end{aligned}$$

A contradiction, since $2x_1^2 - 1 < 1$. □

Hence, properties given in Theorem 4.2 do not fully characterize $\mathrm{Lin}(X)$-convex subsets.

4.2. Epigraphic subsets of $X \times \mathbb{R}$

Definition 4.7. We say that $E \subset X \times \mathbb{R}$ is $\mathrm{Aff}(X)$-*convex* if

$$E = \bigcap \{\mathrm{epi}(\varphi_A + r) \in \mathcal{E}(\mathrm{Aff}(X))\colon E \subset \mathrm{epi}(\varphi_A + r)\}\,.$$

We will denote by $\mathcal{K}(X \times \mathbb{R})$ the family of all $\mathrm{Aff}(X)$-convex subsets of $X \times \mathbb{R}$. □

By Propositions A.6 and A.8, every set $E \in \mathcal{K}(X \times \mathbb{R})$ is characterized by the following separation property:

$$\forall_{(\overline{x},\overline{d})\notin E}\ \exists_{A\in\mathcal{L}}\ \exists_{r\in\mathbb{R}}\ E \subset \mathrm{epi}(\varphi_A + r)\ \wedge (\overline{x},\overline{d}) \notin \mathrm{epi}(\varphi_A + r)$$

or, equivalently,

$$\forall_{(\overline{x},\overline{d})\notin E}\ \exists_{A\in\mathcal{L}}\ \exists_{r\in\mathbb{R}}\ \forall_{(x,d)\in E}\ \varphi_A(x) + r \le d\ \wedge \varphi_A(\overline{x}) + r > \overline{d}.$$

The separation property for $\mathrm{Aff}(X)$-convex sets has a clear geometric interpretation: it means that E lies above the graph of the function $x \mapsto |A[x]^2| + r$ and the point $(\overline{x}, \overline{d})$ lies below it.

It follows from the general theory that $\emptyset$, $X \times \mathbb{R}$ and

$$\mathrm{epi}(\varphi_A + r), \quad A \in \mathcal{L} \text{ and } r \in \mathbb{R}$$

are $\mathrm{Aff}(X)$-convex subsets of $X \times \mathbb{R}$.

Basic properties of $\mathrm{Aff}(X)$-convex sets follow directly from the definition of $\mathcal{K}(X\times\mathbb{R})$ and the properties of $\varphi_A + r$.

Theorem 4.8. *Let $E \subset X \times \mathbb{R}$ be a proper $\mathrm{Aff}(X)$-convex set. Then*

- *E is epigraphic;*
- *E is closed in $X \times \mathbb{R}$;*
- *E is X-balanced.*

Proof. Define $\quad \mathcal{W}_E = \{\varphi_A + r \in \mathrm{Aff}(X)\colon E \subset \mathrm{epi}(\varphi_A + r)\}. \quad (3)$

By definition, $E = \bigcap\{\mathrm{epi}(\varphi_A + r) \in \mathcal{E}(\mathrm{Aff}(X))\colon \varphi_A + r \in \mathcal{W}_E\}$. By (1), the set E is the epigraph of $f\colon X \to \mathbb{R}$ given by $f(x) = \sup\{\varphi_A(x) + r\colon \varphi_A + r \in \mathcal{W}_E\}$. The set E is closed, since every set $\mathrm{epi}(\varphi_A + r)$ is closed. Let $(x, d) \in E$, and let $\lambda \in [-1, 1]$. For every $\varphi_A + r \in \mathcal{W}_E$ we have $\varphi_A(\lambda x) + r = \lambda^2\varphi_A(x) + r \le d$. Hence $(\lambda x, d) \in E$. □

Since every X-affine function has a global minimum at 0, every proper $\mathrm{Aff}(X)$-convex subset of $X \times \mathbb{R}$ is bounded below. Moreover, its lowest point lies on the $\mathbb{R}$-axis.

Theorem 4.9. *Let E be a proper $\mathrm{Aff}(X)$-convex subset of $X \times \mathbb{R}$.*
Define $d_E = \inf\{d \in \mathbb{R}\colon \exists_{x\in X}\ (x, d) \in E\}$, then

- *$d_E = \min\{d \in \mathbb{R}\colon (0, d) \in E\}$;*
- *$\{0\} \times [d_E, \infty) \subset E \subset X \times [d_E, \infty)$.*

Proof. Set $S = \{d \in \mathbb{R} \colon \exists_{x \in X}\ (x, d) \in E\}$ and $S_0 = \{d \in \mathbb{R} \colon (0, d) \in E\}$. Clearly, $S_0 \subset S$. Since E is X-balanced, for every $(x, d) \in E$ we have $(0, d) \in E$. Hence, $S \subset S_0$ and $d_E = \inf S = \inf S_0$. Since E is proper, closed and bounded below $d_E \in \mathbb{R}$. By Proposition 2.1, we get $(0, d_E) \in E$.

The second claim follows immediately. □

The restriction of an X-affine function $\varphi_A + r$ to a line $\mathbb{R}\{v\}$, $v \in X \setminus \{0\}$, is a quadratic function $t \mapsto \varphi_A(v)t^2 + r$, possibly degenerated to a constant function. By examining its epigraph in the radial direction, we can see that it is a region bounded by either an upward opening parabola or a halfplane. This property is inherited by $\mathrm{Aff}(X)$-convex sets, which, in the radial direction, are either halfplanes or at least as convex as parabola. The following theorem clarifies this statement.

Theorem 4.10. *Let $E \in \mathcal{K}(X \times \mathbb{R})$ be a proper subset of $X \times \mathbb{R}$. For every $v \in X \setminus \{0\}$ and every two points $(t_0 v, d_0)$, $(t_1 v, d_1) \in E$, satisfying $0 \le t_0 < t_1$ and $d_0 < d_1$, we have*

$$(\Gamma) \qquad \gamma(t) = \Big(tv, \frac{d_1 - d_0}{t_1^2 - t_0^2}\,(t^2 - t_0^2) + d_0\Big) \in E, \ \textit{for every } t \in [t_0, t_1]$$

Proof. Write $\gamma(t) = (tv, q(t))$. Suppose that there exists $\bar{t} \in (t_0, t_1)$ such that $(\bar{t}v, q(\bar{t})) \notin E$. By the separation property, there exists an X-affine function $\varphi_A + r$ satisfying

$$\varphi_A(v)t_0^2 + r \le d_0 = q(t_0),\ \ \varphi_A(v)t_1^2 + r \le d_1 = q(t_1) \text{ and } \varphi_A(\bar{t}v) + r > q(\bar{t}).$$

By Lemma 2.3, $\varphi_A(v)t^2 + r \le q(t)$ for every $t \in [t_0, t_1]$. A contradiction. □

The formula (Γ) describes the arc of an upward opening parabola passing through the points $P_0 = (t_0 v, d_0)$ and $P_1 = (t_1 v, d_1)$, with axis $\{0\} \times \mathbb{R}$. Since the set E is epigraphic and γ is convex, the segment joining P_0 and P_1 is contained in E. In particular, by choosing $P_0 = (0, d_E)$, it follows that E is starshaped with respect to $(0, d_E)$.

An $\mathrm{Aff}(X)$-convex set need not be a convex subset of $X \times \mathbb{R}$. However, if there exists a family $\mathcal{W}_E \subset \mathrm{Aff}(X)$ consisting only of convex functions (i.e., $\varphi_A + r$ with A semidefinite) such that $E = \bigcap \mathcal{E}(\mathcal{W}_E)$ then E is convex. In the case $X = \mathbb{R}$ all $\mathrm{Aff}(X)$-convex sets are convex. Moreover, it turns out that the properties described above fully characterize $\mathrm{Aff}(\mathbb{R})$-convexity (see Section 5).

The results obtained so far allow us to say that, roughly speaking, an $\mathrm{Aff}(X)$-convex subset of $X \times \mathbb{R}$ resembles the epigraph of a function $f \colon X \to \mathbb{R}$ that is l.s.c., even, bounded below and radially at least as convex as a quadratic function. However, this is not a complete description of $\mathrm{Aff}(X)$-convexity.

Example 4.11. Let $X = \mathbb{R}^2$. Define $f \colon X \to \mathbb{R}$ by

$$f(x_1, x_2) = \sqrt{|x_1 x_2 (x_1^2 - x_2^2)|}$$

and set $E = \operatorname{epi} f$. Clearly, E is epigraphic, closed, X-balanced and bounded below.

For every $(v_1, v_2) \in X \setminus \{0\}$, we have $f(v_1 t, v_2 t) = t^2 \sqrt{|v_1 v_2 (v_1^2 - v_2^2)|}$.

Hence, f is radially quadratic and E satisfies (Γ), by Lemma 2.3.

However, E is not an element of $\mathcal{K}(X \times \mathbb{R})$. Let $r \in \mathbb{R}$ and $A \in \mathcal{L}$ be such that $|A[x]^2| + r \leq f(x)$ for every $x \in X$. In particular, for every $t \in \mathbb{R}$, we have

$$t^2|A_{11}| + r = |A[(t,0)]^2| + r \leq f(t,0) = 0,$$
$$t^2|A_{22}| + r = |A[(0,t)]^2| + r \leq f(0,t) = 0,$$
$$t^2|A_{11} + 2A_{12} + A_{22}| + r = |A[(t,t)]^2| + r \leq f(t,t) = 0.$$

It follows that $A = 0$ and $r \leq 0$. Since $f(1,2) = \sqrt{6} > 2$, the point $(1,2,2) \notin E$ cannot be separated from E. Thus the set E is not an $\mathrm{Aff}(X)$-convex subset of $X \times \mathbb{R}$. $\square$

The argument used in the example above is based on the observation that f is constant on four lines through the origin, whereas a function $x \mapsto |A[x]^2| + r$ can be constant along at most two such directions. However, this is not the only obstacle.

Example 4.12. Define $g\colon X \to \mathbb{R}$ by

$$g(x_1,x_2) = \sqrt{|x_1x_2(x_1^2 + x_2^2)|}$$

and set $E = \operatorname{epi} g$. Clearly, E is epigraphic, closed, X-balanced and bounded below. For every $(v_1,v_2) \in X \setminus \{0\}$, we have

$$g(v_1t, v_2t) = t^2\sqrt{|v_1v_2(v_1^2 + v_2^2)|}.$$

Hence, g is radially quadratic and E satisfies (Γ), by Lemma 2.3.

However, E is not an element of $\mathcal{K}(X \times \mathbb{R})$. Let $r \in \mathbb{R}$ and $A \in \mathcal{L}$ be such that $|A[x]^2| + r \leq g(x)$ for every $x \in X$. In particular, for every $t \in \mathbb{R}$, we have

$$t^2|A_{11}| + r = |A[(t,0)]^2| + r \leq g(t,0) = 0,$$
$$t^2|A_{22}| + r = |A[(0,t)]^2| + r \leq g(0,t) = 0,$$
$$t^2|A_{11} + 2A_{12} + A_{22}| + r = |A[(t,t)]^2| + r \leq g(t,t) = \sqrt{2}t^2.$$

This yields $A_{11} = A_{22} = 0$, $|A_{12}| \leq \frac{1}{\sqrt{2}}$, $r \leq 0$.

Consider $\overline{P} = (1,4,6)$. Since $g(1,4) = 2\sqrt{17} \simeq 8$, we have $\overline{P} \notin E$. If $\varphi_A + r$ separates E from $\overline{P}$ then we must have

$$|A_{12}| \leq \frac{1}{\sqrt{2}}, \quad r \leq 0 \ \text{ and } \ 6 < 8|A_{12}| + r.$$

A contradiction. Hence $E = \operatorname{epi} g$ is not $\mathrm{Aff}(X)$-convex. Note that g is constant only on two lines. $\square$

The argument used here is based on examining the graph of g on circles and lines, where this function behaves differently from X-affine functions.

Example 4.13. Let $f(x) = \frac{1}{p}\|x\|^p$, $p \geq 2$. If $p = 2$ then $\operatorname{epi} f \in \mathcal{K}(X \times \mathbb{R})$ since $f = \varphi_{\frac{1}{2}I} \in \mathrm{Aff}(X)$. Fix $p > 2$, $\overline{x} \in X$ and set

$$A_{\overline{x}}[x]^2 = \frac{1}{2(p-1)}D^2f(\overline{x})[x]^2 \quad \text{and} \quad r = \frac{2-p}{2}f(\overline{x}).$$

Clearly, $\varphi_{A_{\overline{x}}} + r \in \mathrm{Aff}(X)$. It was proved in [8, Section 5.2] that for all $x \in X$,

$$f(x) - f(\overline{x}) \geq \frac{1}{2(p-1)}\left(D^2 f(\overline{x})[x]^2 - D^2 f(\overline{x})[\overline{x}]^2\right).$$

Note that we have $D^2 f(\overline{x})[\overline{x}]^2 = (p-1)\left\|\overline{x}\right\|^p$. It follows that $f \geq \varphi_{A_{\overline{x}}} + r$ and $f(\overline{x}) = \varphi_{A_{\overline{x}}}(\overline{x}) + r$. Now, if $(\overline{x}, \overline{d}) \notin E$ then

$$\mathrm{epi}\, f \subset \mathrm{epi}(\varphi_{A_{\overline{x}}} + r) \text{ and } (\overline{x}, \overline{d}) \notin \mathrm{epi}(\varphi_{A_{\overline{x}}} + r).$$

Hence $\mathrm{epi}\, f \in \mathcal{K}(X \times \mathbb{R})$. $\square$

The above example shows, in addition, that the function $f(x) = \frac{1}{p}\left\|x\right\|^p$, $p \geq 2$, is abstract convex and that $\frac{1}{2(p-1)} D^2 f(\overline{x})$ belongs to the abstract subdifferential $\partial^{\varphi} f(\overline{x})$. However, this topic exceeds the scope of the present paper and will be continued in the forthcoming publication, see also [8, Section 5].

Theorem 4.9 shows that every proper $\mathrm{Aff}(X)$-convex set E is contained in the upper halfspace $X \times [d_E, \infty)$ and contains the halfline $\{0\} \times [d_E, \infty)$. These two sets represent, in a certain sense, the extremal cases for $\mathrm{Aff}(X)$-convexity. Note that

$$X \times [r, \infty) = \mathrm{epi}(\varphi_0 + r), r \in \mathbb{R}$$

and

$$\{0\} \times [r, \infty) = \bigcap \{\varphi_A + r \in \mathrm{Aff}(X) \colon A \in \mathcal{L}\}.$$

Therefore, one can say that among the proper subsets of $X \times \mathbb{R}$, sets of the form $X \times [r, \infty)$ are the largest, while those of the form $\{0\} \times [r, \infty)$ are the smallest possible $\mathrm{Aff}(X)$-convex sets. Note that both $\{0\}$ and X are $\mathrm{Lin}(X)$-convex. In general, any $\mathrm{Lin}(X)$-convex set generates a family of $\mathrm{Aff}(X)$-convex sets.

Proposition 4.14. *Let G be a proper* $\mathrm{Lin}(X)$*-convex set and let $d_E \in \mathbb{R}$. Then $E = G \times [d_E, \infty) \in \mathcal{K}(X \times \mathbb{R})$.*

Proof. Let $(\overline{x}, \overline{d}) \notin E = \{(x, d) \in X \times \mathbb{R} \colon x \in G \wedge d_E \leq d\}$. We show that there exists an X-affine function that separates E and $(\overline{x}, \overline{d})$.

Assume first that $\overline{d} < d_E$ and $\overline{x}$ is arbitrary. Then

$$E \subset \mathrm{epi}(\varphi_0 + d_E) \wedge (\overline{x}, \overline{d}) \notin \mathrm{epi}(\varphi_0 + d_E).$$

Next assume that $d_E \leq \overline{d}$ and $\overline{x} \notin G$. By the separation property for $\mathrm{Lin}(X)$-convex sets, there exist $\varphi_A \in \mathrm{Lin}(X)$ and $r \in \mathbb{R}$ such that $S_r(\varphi_A)$ separates G from $\overline{x}$. Moreover, for every $\lambda > 0$ we have

$$G \subset S_{\lambda r}(\varphi_{\lambda A}) \wedge \overline{x} \notin S_{\lambda r}(\varphi_{\lambda A})$$

since $\lambda \varphi_A = \varphi_{\lambda A}$. We have $\varphi_A(\overline{x}) > r$, hence

$$\lim_{\lambda \to \infty} (\varphi_{\lambda A}(\overline{x}) - \lambda r) = \lim_{\lambda \to \infty} \lambda(\varphi_A(\overline{x}) - r) = \infty.$$

Therefore, there exists $\overline{\lambda} > 0$ such that $\varphi_{\overline{\lambda} A}(\overline{x}) - \overline{\lambda} r > \overline{d} - d_E$.

Hence $(\overline{x}, \overline{d}) \notin \operatorname{epi}(\varphi_{\overline{\lambda}A}(\overline{x}) + d_E - \overline{\lambda}r)$. For any $(x, d) \in E$ we have

$$\varphi_{\overline{\lambda}A}(x) - \overline{\lambda}r + d_E \leq d_E \leq d.$$

Thus $E \subset \operatorname{epi}(\varphi_{\overline{\lambda}A} + d_E - \overline{\lambda}r)$. □

Intersection of a convex set with an arbitrary affine subspace is convex. However, this need not be the case for an $\operatorname{Aff}(X)$-convex subset E. This can be easily observed by considering the epigraph of φ_A with $A[x]^2 = x_1^2 \pm x_2^2$. Nevertheless, the horizontal and radial slices of $\operatorname{Aff}(X)$-convex subsets of $X \times \mathbb{R}$ remain abstract convex. Moreover, these slices can be regarded as lower dimensional abstract convex subsets.

Proposition 4.15. *Let $E \in \mathcal{K}(X \times \mathbb{R})$ be a proper subset.*

(a) *Let $D \in \mathbb{R}$ and set $\Pi_D = X \times \{D\}$.*
Then $\Pi_D \cap E = G \times \{D\}$ for some $G \in \mathcal{K}(X)$.

(b) *Let V be a subspace of X. Then $(V \times \mathbb{R}) \cap E$ belongs to $\mathcal{K}(X \times \mathbb{R})$.*
If $\dim V = 1$ then $(V \times \mathbb{R}) \cap E$ is convex.

Proof. (a) For any $\varphi_A + r \in \operatorname{Aff}(X)$ we have

$$\Pi_D \cap \operatorname{epi}(\varphi_A + r) = \{(x, D) \colon \varphi_A(x) \leq D - r\} = S_{D-r}(\varphi_A) \times \{D\}.$$

Hence

$$\begin{aligned}\Pi_D \cap E &= \bigcap \{\Pi_D \cap \operatorname{epi}(\varphi_A + r) \colon \varphi_A + r \in \mathcal{W}_E\} \\ &= \bigcap \{S_{D-r}(\varphi_A) \times \{D\} \colon \varphi_A + r \in \mathcal{W}_E\} \\ &= \Big(\bigcap \{S_{D-r}(\varphi_A) \colon \varphi_A + r \in \mathcal{W}_E\}\Big) \times \{D\} = G \times \{D\},\end{aligned}$$

where $\mathcal{W}_E$ is defined by (3). Note that $G \in \mathcal{K}(X)$.

(b) For arbitrary $x \in X$ write $x = v + u$, where $v \in V$ and $u \in V^{\perp}$. Set $T[x]^2 = \|u\|^2$. Clearly, T defines an element of $\mathcal{L}$, hence $\varphi_T \in \operatorname{Lin}(X)$. Note that $V = S_0(\varphi_T)$. By Proposition 4.14, we have $V \times [d_E, \infty) \in \mathcal{K}(X \times \mathbb{R})$, hence $(V \times \mathbb{R}) \cap E \in \mathcal{K}(X \times \mathbb{R})$.

Assume that $\dim V = 1$. Every set $(V \times \mathbb{R}) \cap \operatorname{epi}(\varphi_A + r)$ is convex. Hence

$$(V \times \mathbb{R}) \cap E = \bigcap \{(V \times \mathbb{R}) \cap \operatorname{epi}(\varphi_A + r) \colon \varphi_A + r \in \mathcal{W}_E\}$$

is convex. □

Abusing slightly the notation, we can say that the horizontal slice $\Pi_D \cap E$ is an element of $\mathcal{K}(X)$ and that $(V \times \mathbb{R}) \cap E$ belongs to $\mathcal{K}(V \times \mathbb{R})$.

4.3. Subsets of $\mathcal{L}$

Definition 4.16. We say that the subset $G \subset \mathcal{L}$ is $\operatorname{Lin}(\mathcal{L})$-*convex* if

$$G = \bigcap \left\{S_d(\psi_x) \in \mathcal{S}(\operatorname{Lin}(\mathcal{L}) \times \mathbb{R}) \colon G \subset S_d(\psi_x)\right\}.$$

We will denote by $\mathcal{K}(\mathcal{L})$ the family of all $\operatorname{Lin}(\mathcal{L})$-convex subsets of $\mathcal{L}$. □

By Propositions A.6 and A.8, every set $G \in \mathcal{K}(\mathcal{L})$ has the following separation

property: $\forall_{A \notin G}\ \exists_{x \in X}\ \exists_{d \in \mathbb{R}}\ G \subset S_d(\psi_x) \wedge A \notin S_d(\psi_x)$,

or, equivalently, $\forall_{A \notin G}\ \exists_{x \in X}\ \exists_{d \in \mathbb{R}}\ \forall_{B \in G}\ \psi_x(B) \leq d \wedge \psi_x(A) > d.$

The separation property for $\mathrm{Lin}(\mathcal{L})$-convex sets has a clear geometric interpretation: the condition $G \subset S_d(\psi_x)$ means that G is contained in the set $S_d(\psi_x)$, which is the region bounded by two parallel hyperplanes with distance d from the origin, and the point $\overline{A}$ lies outside this region. The direction of the hyperplane is determined by x, its normal vector is given by T_x.

By Proposition A.5 we have $S_d(\psi_x) \in \mathcal{K}(\mathcal{L}),\ x \in X$ and $d \in \mathbb{R}$.

This include also $\emptyset, \mathcal{L}$ and $\{0\} \in \mathcal{K}(\mathcal{L})$.

Basic properties of $\mathrm{Lin}(\mathcal{L})$-convex sets follow directly from the definition of $\mathcal{K}(\mathcal{L})$ and the properties of ψ_x.

Theorem 4.17. *Assume that G is a* $\mathrm{Lin}(\mathcal{L})$*-convex subset. Then*

(a) *G is closed in $\mathcal{L}$,*

(b) *G is convex,*

(c) *G is balanced.*

Proof. The theorem holds if $G = \emptyset$ or $\mathcal{L}$. Let G be a proper subset of $\mathcal{L}$ and let

$$G = \bigcap \mathcal{M}_G,\ \mathcal{M}_G = \{S_d(\psi_x) \colon G \subset S_d(\psi_x)\}.$$

Since ψ_x is convex and continuous, $S_d(\psi_x) = \psi_x^{-1}((\infty, d])$ is convex and closed. Consequently, G inherits these properties. Let $g \in G$ and $\lambda \in [-1, 1]$. For any $\mathcal{L}$-linear function we have $\psi_x(\lambda g) = |\lambda|\psi_x(g) \leq d$. Hence $\lambda g \in G$. $\square$

Thus $\mathcal{K}(\mathcal{L})$ is a subclass of all convex subsets of $\mathcal{L}$. It follows that any nonempty $\mathrm{Lin}(\mathcal{L})$-convex set always contains 0, is symmetric and star-shaped at 0. Moreover, with every point A it contains interval $[-1, 1]\{A\}$. In particular, the only $\mathrm{Lin}(\mathcal{L})$-convex singleton is $\{0\}$. In contrast to $\mathrm{Lin}(X)$-convex sets, not every segment in $\mathcal{L}$ is $\mathrm{Lin}(\mathcal{L})$-convex, as we will demonstrate in the examples below.

A closed ball under the operator norm $\|\cdot\|_{\mathcal{L}}$ is abstract convex.

Proposition 4.18. *A closed ball $\overline{B}_{\mathcal{L}}(0, R) = \{A \in \mathcal{L} \colon \|A\|_{\mathcal{L}} \leq R\}$, $R > 0$, is* $\mathrm{Lin}(\mathcal{L})$*-convex.*

Proof. Let $\overline{A} \notin \overline{B}_{\mathcal{L}}(0, R)$, i.e. $\|\overline{A}\|_{\mathcal{L}} > R$. There exists $x \in X$, $\|x\| \leq 1$, such that $\psi_x(\overline{A}) = \|\overline{A}\|_{\mathcal{L}} > R$. For every $A \in \overline{B}_{\mathcal{L}}(0, R)$ we have

$$\psi_x(A) \leq \|A\|_{\mathcal{L}} \leq R.$$

Therefore ψ_x separates $\overline{B}_{\mathcal{L}}(0, R)$ from $\overline{A}$. Hence $\overline{B}_{\mathcal{L}}(0, R)$ is $\mathrm{Lin}(\mathcal{L})$-convex. $\square$

The above result can be obtained directly from the definition, providing a more geometric point of view, see also Example 4.26.

Example 4.19. Fix $R > 0$ and set

$$G = \bigcap \{S_R(\psi_x) \colon x \in \overline{B}_X(0,1)\} \subset \mathcal{L}.$$

An element A belongs to G if and only if $|A[x]^2| \le R$, $x \in \overline{B}_X(0,1)$. Hence $G = \overline{B}_{\mathcal{L}}(0,R)$.

Another representation is given by

$$\overline{B}_{\mathcal{L}}(0,R) = \bigcap \{S_1(\psi_x) \colon x \in \overline{B}_X(0, 1/\sqrt{R})\}.$$

This follows from the identity $S_R(\psi_x) = S_1(\psi_{x/\sqrt{R}})$. In fact, in both formulas it is sufficient to consider only x's from the boundary. Let us also note that the above formula remains valid for $\overline{B}_{\mathcal{L}}(0,0) = \{0\}$ and $\overline{B}_{\mathcal{L}}(0,\infty) = \mathcal{L}$. □

The properties given in Theorem 4.17 do not characterize $\mathrm{Lin}(\mathcal{L})$-convex subsets. An exception is the one-dimensional case: if $X = \mathbb{R}^1$, then $\mathcal{L} \simeq \mathbb{R}$. In this case, the ψ-convex sets are precisely intervals of the form $[-\alpha, \alpha]$, $\alpha \ge 0$ (see Section 5).

Below, we provide several examples that provide some insight into the geometry of $\mathrm{Lin}(\mathcal{L})$-convex sets. We restrict our attention to the case $X = \mathbb{R}^2$, where $\mathcal{L} \simeq \mathbb{R}^3$, as this allows for a more intuitive geometric interpretation. For brevity, we will write $A = (X_{11}, X_{12}, X_{22})$ and $x = (\alpha_1, \alpha_2)$.

The coordinate planes

$$S_0(\psi_{(1,0)}) = \{A \in \mathcal{L} \colon X_{11} = 0\} \quad \text{and} \quad S_0(\psi_{(0,1)}) = \{A \in \mathcal{L} \colon X_{22} = 0\}$$

are abstract convex, as they are sublevel sets of $\mathcal{L}$-linear functions.

However, the third coordinate plane $\{A \in \mathcal{L} \colon X_{12} = 0\}$ is not of this form. This is due to the fact that its normal vector cannot be expressed as T_x for any $x \in X$. In fact, it is not $\mathrm{Lin}(\mathcal{L})$-convex and, moreover, is so uniquely oriented in space that it cannot be separated from any point by $\mathcal{L}$-linear function.

Example 4.20. Let $G = \{A \in \mathcal{L} \colon X_{12} = 0\}$ and let $\overline{A} = (\overline{X}_{11}, \overline{X}_{12}, \overline{X}_{22}) \notin G$. Assume that there exists $x = (\alpha_1, \alpha_2) \in X$ and $d \in \mathbb{R}$ such that $S_d(\psi_x)$ separates G from $\overline{A}$. Observe that, in this case, necessarily $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Define

$$A_0 = \begin{bmatrix} \overline{X}_{11} + \frac{\alpha_2}{\alpha_1}\overline{X}_{12} & 0 \\ 0 & \overline{X}_{22} + \frac{\alpha_1}{\alpha_2}\overline{X}_{12} \end{bmatrix} \in G.$$

It is easy to check that $\psi_x(\overline{A}) = \psi_x(A_0)$. However, this gives $d < \psi_x(\overline{A}) = \psi_x(A_0) \le d$. A contradiction. □

The fact that the plane $X_{12} = 0$ is not abstract convex is not due to its unboundedness. Even if we replace this plane with its bounded subset, it still fails to be abstract convex.

Example 4.21. Let $A = (X_{11}, 0, X_{22})$, $X_{11}X_{22} > 0$ and $\overline{A} = (0, \sqrt{X_{11}X_{22}}, 0)$. Assume there exists $S_d(\psi_x)$ that separates $\{A\}$ from $\overline{A}$. As before, it must be that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. We have

$$\psi_x(A) = |\alpha_1^2 X_{11} + \alpha_2^2 X_{22}| \le d \quad \psi_x(\overline{A}) = 2|\alpha_1\alpha_2|\sqrt{X_{11}X_{22}} > d.$$

This system has no solution. Hence, the set $\{A\}$ cannot be separated from $\overline{A}$. An analogous result holds for the case $X_{11}X_{22} < 0$.

It follows that any set $G \subset \mathcal{L}$ satisfying $A \in G$ and $\overline{A} \notin G$ is not $\mathrm{Lin}(\mathcal{L})$-convex. In particular, no segment containing a point $(X_{11}, 0, X_{22})$ with $X_{11}X_{22} \neq 0$ is abstract convex. $\square$

The previous example covers almost all sets that lie entirely on the plane $X_{12} = 0$, but also many sets that are "not thick enough".

Example 4.22. Let $G = [-1, 1] \times [-R, R] \times [-1, 1]$ and $\overline{A} = (0, R+1, 0)$, $R \geq 0$. If $S_d(\psi_x)$ separates G from $\overline{A}$ then $|2(R+1)\alpha_1\alpha_2| > d$. Since $(1, R, 1), (1, -R, 1) \in G$, we obtain

$$|\alpha_1^2 + 2R\alpha_1\alpha_2 + \alpha_2^2| \leq d \text{ and } |\alpha_1^2 - 2R\alpha_1\alpha_2 + \alpha_2^2| \leq d.$$

A simple drawing shows that this system of inequalities has no solutions. A contradiction. $\square$

Hence we see that increasing the "thickness" of the set does not necessarily make it abstract convex.

Example 4.23. Fix $R > 0$. By definition, the following sets are $\mathrm{Lin}(\mathcal{L})$-convex:

- $\{0\} \times \mathbb{R} \times \{0\} = S_0(\psi_{(1,0)}) \cap S_0(\psi_{(0,1)})$;
- $\{0\} \times [-R, R] \times \{0\} = S_0(\psi_{(1,0)}) \cap S_0(\psi_{(0,1)}) \cap S_R(\psi_{(1,1/2)})$;
- $[-R, R] \times \{0\} \times \{0\} = S_0(\psi_{(0,1)}) \cap \bigcap_{n\in\mathbb{N}} S_R(\psi_{(1,n)})$;
- $\{0\} \times \{0\} \times [-R, R] = S_0(\psi_{(1,0)}) \cap \bigcap_{n\in\mathbb{N}} S_R(\psi_{(n,1)})$. $\square$

However, the axes $\mathbb{R} \times \{0\} \times \{0\}$ and $\{0\} \times \{0\} \times \mathbb{R}$ are not abstract convex, since they cannot be separated from $\overline{A} = (0, 1, 0)$.

Example 4.24. Let $A_1 = (1, 0, 0)$, $A_2 = (0, 0, 1)$ and $\overline{A} = (1, 0, -1)$. Assume that $S_d(\psi_x)$ separates $\{A_1, A_2\}$ from $\overline{A}$. Then

$$\psi_x(A_1) = |\alpha_1^2| \leq d \quad \psi_x(A_2) = |\alpha_2^2| \leq d \quad \psi_x(\overline{A}) = |\alpha_1^2 - \alpha_2^2| > d.$$

A simple drawing shows that this system of inequalities has no solutions. It follows that the sets

- $\{A \in \mathcal{L} \colon X_{11}^2 + X_{22}^2 \leq 1, |X_{22}| \leq 1\}$,
- $\{A \in \mathcal{L} \colon X_{11}^2 + X_{22}^2 \leq 1, X_{12} \in \mathbb{R}\}$,
- $\{A \in \mathcal{L} \colon X_{11}^2 + X_{22}^2 \leq (|X_{12}| - 1)^2, |X_{22}| \leq 1\}$,
- $\{A \in \mathcal{L} \colon X_{11}^2 + 2X_{12}^2 + X_{22}^2 \leq 1\}$

are not $\mathrm{Lin}(\mathcal{L})$-convex. Note that the last set is a ball under the Frobenius norm. $\square$

In contrast, observe that the set $S_1(\psi_{(1,0)}) \cap S_1(\psi_{(0,1)}) = [-1, 1] \times \mathbb{R} \times [-1, 1]$ is abstract convex. More similar examples can be easily obtained.

Example 4.25. Let $I[x]^2 = \alpha_1^2 + \alpha_2^2$ and $R > 0$. Let $\overline{A} \in \mathcal{L}$ satisfies $\left\|\overline{A}\right\|_{\mathcal{L}} \leq R$. Assume that there exists $x \in X$ and $d \in \mathbb{R}$ such that $S_d(\psi_x)$ separates RI from $\overline{A}$. Then

$$d < \psi_x(\overline{A}) \leq \left\|\overline{A}\right\|_{\mathcal{L}} \|x\|^2 \leq R \|x\|^2 = \psi_x(A) \leq d.$$

Hence RI cannot be separated from points of $\overline{B}_{\mathcal{L}}(0,R)$. Assume $\left\|\overline{A}\right\|_{\mathcal{L}} > R$. There exists $\|x\| \le 1$ such that $\overline{A}[x]^2 = \left\|\overline{A}\right\|_{\mathcal{L}}$. We have

$$\psi_x(RI) = R\,\|x\|^2 \le R \text{ and } \psi_x(\overline{A}) = \left\|\overline{A}\right\|_{\mathcal{L}} > R.$$

Hence $S_R(\psi_x)$ separates. Hence the smallest $\mathrm{Lin}(\mathcal{L})$-convex set containing $A = RI$ is a closed ball of radius R.

In fact, we have shown that the abstract convex hull of $\{I\}$, i.e. the smallest abstract convex set containing I (see [11]), is the closed unit ball. It can be easily checked that for $A_1 = (1,0,0)$ and $A_2 = (0,0,1)$ the respective hulls are $[-1,1]\{A_1\}$ and $[-1,1]\{A_2\}$.

We have seen that the closed ball under the Frobenius norm is not abstract convex, while the closed ball under the operator norm is. In the next example, we revisit this fact and provide another geometric description of $\overline{B}_{\mathcal{L}}(0,R)$.

Example 4.26. Recall that $\|A\|_{\mathcal{L}}^2 = \lambda_{max}(A^TA)$, where λ_{max} denotes the largest eigenvalue of A^TA. Direct computations shows that

$$\begin{aligned}
\|A\|_{\mathcal{L}}^2 &= \frac{1}{2}\left(X_{11}^2 + 2X_{12}^2 + X_{22}^2 + |X_{11}+X_{22}|\sqrt{4X_{12}^2 + |X_{11}-X_{22}|^2}\right)\\
&= \frac{|X_{11}+X_{22}|^2}{4} + X_{12}^2 + \frac{|X_{11}-X_{22}|^2}{4} + \frac{1}{2}|X_{11}+X_{22}|\sqrt{4X_{12}^2 + |X_{11}-X_{22}|^2}\\
&= \left|\frac{X_{11}+X_{22}}{2}\right|^2 + \left(\sqrt{X_{12}^2 + \left|\frac{X_{11}-X_{22}}{2}\right|^2}\right)^2 + 2\left|\frac{X_{11}+X_{22}}{2}\right|\sqrt{X_{12}^2 + \left|\frac{X_{11}-X_{22}}{2}\right|^2}\\
&= \left(\left|\frac{X_{11}+X_{22}}{2}\right| + \sqrt{X_{12}^2 + \left|\frac{X_{11}-X_{22}}{2}\right|^2}\right)^2.
\end{aligned}$$

It is well known that the Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

form an orthogonal basis of $\mathcal{L}^2_{sym}(\mathbb{R}^2,\mathbb{R})$. In this basis we have

$$A = \frac{X_{11}+X_{22}}{2}I + X_{12}\,\sigma_1 + \frac{X_{11}-X_{22}}{2}\sigma_3 = XI + Y\sigma_1 + Z\sigma_3$$

and

$$\|A\|_{\mathcal{L}} = |X| + \sqrt{Y^2+Z^2}.$$

Hence the ball $\overline{B}_{\mathcal{L}}(0,R) = \{(X,Y,Z)\colon |X| + \sqrt{Y^2+Z^2} \le R\}$ is the union of two cones.

Let $0 < X_0 < R$, $t_0 \in [0,2\pi]$ and consider the point

$$A_0 = X_0 I + (R-X_0)\cos t_0\sigma_1 + (R-X_0)\sin t_0\sigma_3 \in \partial\overline{B}_{\mathcal{L}}(0,R).$$

The vector given by

$$N = I + \cos t_0\,\sigma_1 + \sin t_0\,\sigma_3 = \begin{bmatrix} 1+\sin t_0 & \cos t_0 \\ \cos t_0 & 1-\sin t_0 \end{bmatrix}$$

is the outward normal to the ball $\overline{B}_{\mathcal{L}}(0, R)$ at A_0 and the plane

$$\langle N, A\rangle_F = (1 + \sin t_0)X_{11} + 2\cos t_0 X_{12} + (1 - \sin t_0)X_{22} = 2R$$

is tangent at A_0. It is easy to check that

$$x_0 = x_0(t_0) = \begin{cases} \left(\sqrt{1+\sin t_0}, \sqrt{1-\sin t_0}\,\right) & t \in [0, \pi/2] \cup [3\pi/2, 2\pi] \\ \left(\sqrt{1+\sin t_0}, -\sqrt{1-\sin t_0}\,\right) & t \in [\pi/2, 3\pi/2] \end{cases}$$

is a solution to the equation $T_x = N$. Moreover, $\|x_0\|^2 = \|T_{x_0}\|_F = \|N\|_F = 2$.

If $A \in \overline{B}_{\mathcal{L}}(0, R)$ then $\psi_{x_0}(A) = |A[x_0]^2| \le \|A\|_{\mathcal{L}} \|x_0\|^2 \le 2R$. Since $\psi_{x_0}(A) = |\langle T_{x_0}, A\rangle_F|$, we obtain $\overline{B}_{\mathcal{L}}(0, R) \subset S_{2R}(\psi_{x_0})$. It follows that

$$\overline{B}_{\mathcal{L}}(0, R) = \bigcap\{S_{2R}(\psi_{x_0}) \colon t_0 \in [0, 2\pi]\} = \bigcap\{S_{2R}(\psi_x) \colon x \in \overline{B}_X(0, \sqrt{2})\}$$

Details are left to the reader. □

As the above examples show, abstract convexity of subsets of $\mathcal{L}$ depends on both the shape and orientation of the set in space. Neither of them can be considered separately, as illustrated by the example of the ball in the operator norm. While in its original orientation it is abstract convex, after a 45-degree rotation it loses this property. Another interesting fact is that the ball in the Frobenius norm is not abstract convex, despite its rounded shape, whereas the ball in the operator norm, which has corners, is abstract convex.

4.4. Epigraphic subsets of $\mathcal{L} \times \mathbb{R}$

Definition 4.27. We say that $E \subset \mathcal{L} \times \mathbb{R}$ is $\mathrm{Aff}(\mathcal{L})$-*convex* if

$$E = \bigcap\{\mathrm{epi}(\psi_x + r) \in \mathcal{E}(\mathrm{Aff}(\mathcal{L})) \colon E \subset \mathrm{epi}(\psi_x + r)\}$$

We will denote by $\mathcal{K}(\mathcal{L} \times \mathbb{R})$ the family of all $\mathrm{Aff}(\mathcal{L})$-convex subsets of $\mathcal{L} \times \mathbb{R}$. □

By Propositions A.6 and A.8, every set $E \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$ is characterized by the following separation property:

$$\forall_{(\overline{A},\overline{d})\notin E}\ \exists_{(x,r)\in X\times\mathbb{R}}\ E \subset \mathrm{epi}(\psi_x + r)\ \wedge (\overline{A}, \overline{d}) \notin \mathrm{epi}(\psi_x + r),$$

or, equivalently,

$$\forall_{(\overline{A},\overline{d})\notin E}\ \exists_{x\in X}\ \exists_{r\in\mathbb{R}}\ \forall_{(A,d)\in E}\ \psi_x(A) + r \le d\ \wedge \psi_x(\overline{A}) + r > \overline{d}.$$

The separation property for $\mathrm{Aff}(\mathcal{L})$-convex sets has a clear geometric interpretation: the condition $E \subset \mathrm{epi}(\psi_x + r)$ means that E lies above the graph of the function $A \mapsto |A[x]^2| + r$ and the point $(\overline{A}, \overline{d})$ lies below it.

It follows from the general theory we know that $\emptyset$, $\mathcal{L} \times \mathbb{R}$ and

$$\mathrm{epi}(\psi_x + R), \quad x \in X \text{ and } r \in \mathbb{R}$$

are $\mathrm{Aff}(\mathcal{L})$-convex subsets of $\mathcal{L} \times \mathbb{R}$.

Basic properties of $\mathrm{Aff}(\mathcal{L})$-convex sets follow directly from the definition of $\mathcal{K}(\mathcal{L}\times\mathbb{R})$ and the properties of $\psi_x + r$.

Theorem 4.28. *Let $E \subset \mathcal{L} \times \mathbb{R}$ be a proper* $\mathrm{Aff}(\mathcal{L})$*-convex set. Then*

- *E is epigraphic;*
- *E is closed in $\mathcal{L} \times \mathbb{R}$;*
- *E is $\mathcal{L}$-balanced;*
- *E is convex.*

Proof. Set
$$\mathcal{W}_E = \{\psi_x + r \in \mathrm{Aff}(\mathcal{L})\colon E \subset \mathrm{epi}(\psi_x + r)\}. \tag{4}$$
By definition, $E = \bigcap\{\mathrm{epi}(\psi_x + r) \in \mathcal{E}(\mathrm{Aff}(\mathcal{L}))\colon \psi_x + r \in \mathcal{W}_E\}$. By (1), the set E is the epigraph of $f\colon \mathcal{L} \to \mathbb{R}$ given by $f(A) = \sup\{\psi_x(A)+r\colon \psi_x+r \in \mathcal{W}_E\}$. The set E is closed and convex, since every set $\mathrm{epi}(\psi_x+r)$ is closed and convex. Let $(A, d) \in E$, and let $\lambda \in [-1, 1]$. For every $\psi_x + r \in \mathcal{W}_E$ we have $\psi_x(\lambda A) + r = |\lambda|\psi_x(A) + r \leq d$. Hence, $(\lambda A, d) \in E$. $\square$

Since every $\mathcal{L}$-affine function attains its global minimum at 0, every proper $\mathrm{Aff}(\mathcal{L})$-convex subset of $\mathcal{L} \times \mathbb{R}$ is bounded from below. Moreover, its lowest point lies on the $\mathbb{R}$-axis.

Theorem 4.29. *Let E be a proper* $\mathrm{Aff}(\mathcal{L})$*-convex subset of $\mathcal{L} \times \mathbb{R}$.*
Set $d_E = \inf\{d \in \mathbb{R}\colon \exists_{A\in\mathcal{L}}\ (A, d) \in E\}$, then

- *$d_E = \min\{d \in \mathbb{R}\colon (0, d) \in E\}$;*
- *$\{0\} \times [d_E, \infty) \subset E \subset \mathcal{L} \times [d_E, \infty)$.*

Proof. Let $S = \{d \in \mathbb{R}\colon \exists_{A\in\mathcal{L}}\ (A, d) \in E\}$ and $S_0 = \{d \in \mathbb{R}\colon (0, d) \in E\}$. Clearly, $S_0 \subset S$. Since E is $\mathcal{L}$-balanced, for every $(A, d) \in E$ we have $(0, d) \in E$. Hence $S \subset S_0$ and $d_E = \inf S = \inf S_0$. Since E is proper, closed and bounded from below $d_E \in \mathbb{R}$. From Proposition 2.1, we get $(0, d_E) \in E$.
The second claim follows immediately. $\square$

The results obtained above allow us to say that, roughly speaking, an $\mathrm{Aff}(\mathcal{L})$-convex subset of $\mathcal{L}\times\mathbb{R}$ resembles the epigraph of a function $f\colon \mathcal{L} \to \mathbb{R}$ which is: l.s.c, even, bounded from below and convex. However, these properties do not provide a full characterization of $\mathrm{Aff}(X)$-convexity. Again, the exception is the case $X = \mathbb{R}$, where $\mathcal{L} \simeq \mathbb{R}$. In this case, the properties given in Theorems 4.28 and 4.28 fully characterize $\mathrm{Aff}(\mathcal{L})$-convexity of subsets of $\mathcal{L} \times \mathbb{R}$ (see Section 5).

Example 4.30. Define $f\colon \mathcal{L} \to \mathbb{R}$ by $f(A) = \frac{1}{p}\|A\|^p_{\mathcal{L}}$, where $1 \leq p < \infty$.

(a) If $p = 1$ then we have
$$f(A) = \|A\|_{\mathcal{L}} = \sup\{|A[x]^2|\colon \|x\| \leq 1\} = \sup\{\psi_x(A)\colon \|x\| \leq 1\}$$
and
$$\mathrm{epi}\, f = \mathrm{epi}\left(\sup\{\psi_x\colon \|x\| \leq 1\}\right) = \bigcap\{\mathrm{epi}\,\psi_x\colon \|x\| \leq 1\}.$$

Hence $\mathrm{epi}\, f \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$.

Let us prove this again by using the separation property. For any $(\overline{A}, \overline{d}) \notin \mathrm{epi}\, f$ we have $\overline{d} < \left\|\overline{A}\right\|_{\mathcal{L}}$, since $\mathrm{epi}\, f = \{(A, d)\colon \|A\|_{\mathcal{L}} \leq d\}$. There exists $\overline{x} \in X$ satisfying $\|\overline{x}\| \leq 1$ and $\left\|\overline{A}\right\|_{\mathcal{L}} = |\overline{A}[\overline{x}]^2|$. Observing that $\psi_{\overline{x}}(A) \leq \|A\|_{\mathcal{L}}$ and $\overline{d} < \psi_{\overline{x}}(\overline{A})$, we obtain $\mathrm{epi}\, f \subset \mathrm{epi}\,\psi_{\overline{x}}$ and $(\overline{A}, \overline{d}) \notin \mathrm{epi}\,\psi_{\overline{x}}$.

(b) Assume that $p > 1$. For any A_0, denote by $y_0 \in X$ a vector such that $\|y_0\| \leq 1$ and $|A_0[y_0]^2| = \|A_0\|_{\mathcal{L}}$. Set

$$x_0 = \|A_0\|_{\mathcal{L}}^{\frac{p-1}{2}} y_0 \quad \text{and} \quad r_0 = \frac{1-p}{p} \|A_0\|_{\mathcal{L}}^p .$$

For all $A \in \mathcal{L}$, we have

$$\psi_{x_0}(A) + r_0 = \left| A \left[\|A_0\|_{\mathcal{L}}^{\frac{p-1}{2}} y_0 \right]^2 \right| + \frac{1-p}{p} \|A_0\|_{\mathcal{L}}^p = \|A_0\|_{\mathcal{L}}^{p-1} \left| A\left[y_0\right]^2 \right| + \frac{1-p}{p} \|A_0\|_{\mathcal{L}}^p$$

$$\leq \|A_0\|_{\mathcal{L}}^{p-1} \|A\|_{\mathcal{L}} + \frac{1-p}{p} \|A_0\|_{\mathcal{L}}^p = \|A_0\|_{\mathcal{L}}^{p-1} (\|A\|_{\mathcal{L}} - \|A_0\|_{\mathcal{L}}) + \frac{1}{p} \|A_0\|_{\mathcal{L}}^p \leq \frac{1}{p} \|A_0\|_{\mathcal{L}}^p = f(A)$$

where the last inequality follows from the convexity of function $t \mapsto \frac{1}{p} t^p$. Hence $\operatorname{epi} f \subset \operatorname{epi}(\psi_{x_0} + r_0)$. If $(A_0, d_0) \notin \operatorname{epi} f$ then $(A_0, d_0) \notin \operatorname{epi}(\psi_{x_0} + r_0)$, since we have $\psi_{x_0}(A_0) + r_0 = f(A_0) > d_0$. Hence $\operatorname{epi} f \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$.

(c) Note that for $p = \infty$, if we set $f = \chi_{\{\|A\|_{\mathcal{L}} \leq 1\}}$, the indicator function of the unit ball, then $\operatorname{epi} f = \overline{B}_{\mathcal{L}}(0,1) \times [0, \infty) \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$. □

The above example shows, in addition, that the function $A \mapsto \frac{1}{p} \|A\|_{\mathcal{L}}^p$ is abstract convex and that $\|A_0\|_{\mathcal{L}}^{\frac{p-1}{2}} y_0$ belongs to the subdifferential $\partial^{\psi} f(A_0)$. However, this topic exceeds the scope of this paper and will be continued in the forthcoming publication. See also [8, Section 5].

In contrast, the Frobenius norm does not define an abstract convex function.

Example 4.31. Let $X = \mathbb{R}^2$. Define $f \colon \mathcal{L} \to \mathbb{R}$ by $f(A) = \|A\|_F$. Set $A_1 = (1, 0, 0)$, $A_2 = (0, 0, 1)$ and $\overline{A} = (1, 0, -1)$. Then $(A_1, 1), (A_2, 1) \in \operatorname{epi} f$ and $(\overline{A}, 1) \notin \operatorname{epi} f$. If $\psi_x + r$ separates $\operatorname{epi} f$ from $(\overline{A}, 1)$ then

$$\psi_x(A_1) + r \leq 1, \ \psi_x(A_2) + r \leq 1, \ \psi_x(\overline{A}) + r > 1.$$

This system of inequalities has no solution. Therefore $\operatorname{epi} f$ is not $\operatorname{Aff}(\mathcal{L})$-convex. □

Theorem 4.29 shows that every $\operatorname{Aff}(\mathcal{L})$-convex set E is contained in the upper half-space $\mathcal{L} \times [d_E, \infty)$ and contains the halfline $\{0\} \times [d_E, \infty)$. These two sets represent, in a certain sense, the extremal cases for $\operatorname{Aff}(\mathcal{L})$-convexity. Note that

$$\mathcal{L} \times [r, \infty) = \operatorname{epi}(\psi_0 + r), r \in \mathbb{R}$$

and
$$\{0\} \times [r, \infty) = \bigcap \{\psi_x + r \in \operatorname{Aff}(\mathcal{L}) \colon x \in X\}.$$

Therefore, one can say that among the proper subsets of $\mathcal{L} \times \mathbb{R}$, sets of the form $\mathcal{L} \times [r, \infty)$ are the largest, while sets of the form $\{0\} \times [r, \infty)$ are the smallest possible $\operatorname{Aff}(\mathcal{L})$-convex sets. Note that both $\{0\}$ and $\mathcal{L}$ are $\operatorname{Lin}(\mathcal{L})$-convex. In general, any $\operatorname{Aff}(\mathcal{L})$-convex set generates an $\mathcal{L} \times \mathbb{R}$ set.

Proposition 4.32. *Let G be a proper $\operatorname{Lin}(\mathcal{L})$-convex set and let $d_E \in \mathbb{R}$. Then $E = G \times [d_E, \infty) \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$.*

Proof. Let $(\overline{A}, \overline{d}) \notin E = \{(A, d) \in \mathcal{L} \times \mathbb{R} \colon A \in G \wedge d_E \leq d\}$. We will show that there exists an $\mathcal{L}$-affine function that separates E from $(\overline{A}, \overline{d})$.

Assume first that $d_E > \overline{d}$ and $\overline{A}$ is arbitrary. Then

$$E \subset \operatorname{epi}(\psi_0 + d_E) \wedge (\overline{A}, \overline{d}) \notin \operatorname{epi}(\psi_0 + d_E).$$

Next, assume that $d_E \leq \overline{d}$ and $\overline{A} \notin G$. By the separation property for $\operatorname{Lin}(\mathcal{L})$-convex sets, there exist $\psi_x \in \operatorname{Lin}(\mathcal{L})$ and $r \in \mathbb{R}$ such that $S_r(\psi_x)$ separates G from $\overline{A}$.

Moreover, for every $\lambda > 0$ we have

$$G \subset S_{\lambda^2 r}(\psi_{\lambda x}) \wedge \overline{A} \notin S_{\lambda^2 r}(\varphi_{\lambda x})$$

since $\lambda^2 \psi_x = \varphi_{\lambda x}$. We have $\psi_x(\overline{A}) > r$, hence

$$\lim_{\lambda \to \infty} (\psi_{\lambda x}(\overline{A}) - \lambda^2 r) = \lim_{\lambda \to \infty} \lambda^2 (\psi_x(\overline{A}) - r) = \infty.$$

Therefore, there exists $\overline{\lambda} > 0$ such that

$$\psi_{\overline{\lambda} x}(\overline{A}) - \overline{\lambda}^2 r > \overline{d} - d_E.$$

Consequently, $(\overline{A}, \overline{d}) \notin \operatorname{epi}(\psi_{\overline{\lambda} x}(\overline{A}) + d_E - \overline{\lambda}^2 r)$. For any $(A, d) \in E$ we have

$$\psi_{\overline{\lambda} x}(A) - \overline{\lambda}^2 r + d_E \leq d_E \leq d.$$

Thus $E \subset \operatorname{epi}(\varphi_{\overline{\lambda} x} + d_E - \overline{\lambda}^2 r)$. □

Since an $\operatorname{Aff}(\mathcal{L})$-convex set is convex, its intersection with arbitrary affine subspace is convex. Similarly to $\operatorname{Aff}(X)$-convexity, the horizontal slices of $\operatorname{Aff}(\mathcal{L})$-convex subsets of $\mathcal{L} \times \mathbb{R}$ are $\operatorname{Lin}(X)$-convex sets.

Proposition 4.33. *Let $E \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$ be a proper subset. Let $D \in \mathbb{R}$ and set $\Pi_D = \mathcal{L} \times \{D\}$. Then $\Pi_D \cap E = G \times \{D\}$ for some $G \in \mathcal{K}(\mathcal{L})$.*

Proof. For any $\psi_x + r \in \operatorname{Aff}(\mathcal{L})$ we have

$$\Pi_D \cap \operatorname{epi}(\psi_x + r) = \{(A, D) \colon \psi_x(A) \leq D - r\} = S_{D-r}(\psi_x) \times \{D\}.$$

We have

$$\begin{aligned} \Pi_D \cap E &= \bigcap \{\Pi_D \cap \operatorname{epi}(\psi_x + r) \colon \psi_x + r \in \mathcal{W}_E\} = \bigcap \{S_{D-r}(\psi_x) \times \{D\} \colon \psi_x + r \in \mathcal{W}_E\} \\ &= \Big(\bigcap \{S_{D-r}(\psi_x) \colon \psi_x + r \in \mathcal{W}_E\}\Big) \times \{D\} = G \times \{D\}. \end{aligned}$$

Note that $G \in \mathcal{K}(\mathcal{L})$. □

Abusing the notation, we can say that the horizontal slice $\Pi_D \cap E$ is an element of $\mathcal{K}(\mathcal{L})$. Intersection of an $\operatorname{Aff}(\mathcal{L})$-convex set with a subspace $V \times \mathbb{R}$ need not be $\operatorname{Aff}(\mathcal{L})$-convex.

Example 4.34. Let $E=\{(A,d)\in\mathcal{L}\times\mathbb{R}\colon \|A\|_{\mathcal{L}}\le d\}\in\mathcal{K}(\mathcal{L}\times\mathbb{R})$ and $V=\{\lambda I:\lambda\in\mathbb{R}\}$. Then $$E_V = (V\times\mathbb{R})\cap E = \{(\lambda I, d)\in\mathcal{L}\times\mathbb{R}\colon \lambda\in\mathbb{R}\wedge|\lambda|\le d\}.$$

Define $\overline{A}\in\mathcal{L}$ by $\overline{A}[x]^2 = 2\alpha_1\alpha_2$. Note that $(\overline{A},1)\notin E_V$. Suppose that there exist $\psi_x + r$ that separates E_V from $(\overline{A},1)$. Then

$$\psi_x(I) + r = |\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2| + r \le 1 \quad\text{and}\quad \psi_x(\overline{A}) + r = |2\alpha_1\alpha_2| + r > 1.$$

This system has no solutions, hence E_V is not $\mathrm{Aff}(\mathcal{L})$-convex.

It is left to the reader to check that if $X=\mathbb{R}^2$ and $W=\mathbb{R}\times\{0\}\times\mathbb{R}$, then $(W\times\mathbb{R})\cap E$ is not $\mathrm{Aff}(X)$-convex. □

Observe that if $G\in\mathcal{K}(\mathcal{L})$ then $(G\times\mathbb{R})\cap E\in\mathcal{K}(\mathcal{L}\times\mathbb{R})$ for any $E\in\mathcal{K}(\mathcal{L}\times\mathbb{R})$. In particular, if a subspace $V\subset\mathcal{L}$ is abstract convex, then the horizontal slice of $\mathrm{Aff}(\mathcal{L})$-convex set remains $\mathrm{Aff}(\mathcal{L})$-convex. Interestingly, if V is one dimensional then $(V\times\mathbb{R})\cap E$, considered as a subset of $V\times\mathbb{R}$ is $\mathrm{Aff}(\mathcal{L}^2_{sym}(\mathbb{R},\mathbb{R}))$-convex (see Section 5).

4.5. Some remarks

For each of the four classes of sets considered in the present section, we have shown that the properties we identified does not provide a full characterization. In each case, the additional conditions required for a complete characterization remain unknown. At present, no criterion except separation property is available for determining whether a set is abstract convex. An exception is the case of $X=\mathbb{R}^1$ considered in the next section.

Below, we present some observations that may provide insight into the missing conditions needed for a full characterization of each class. However, these comments are merely ideas based on examples and require further investigation.

Recall that if a set S is convex, then for any two points $P_0, P_1\in S$, the segment joining them is entirely contained in S. This condition is known as the inner definition of convexity, in contrast to the outer definition: a set is convex if it is the intersection of a family of halfspaces; equivalently, if it can be separated from any external point by a hyperplane.

It is natural to ask whether there exists an internal definition of abstract convexity. That is a statement of the form

$$\begin{aligned}&S \text{ is abstract convex}\\ &\iff \text{every two points can be joined by some curve contained in } S\end{aligned}$$

in addition to the conditions obtained so far. We discuss this idea separately for each of the classes considered in this section, restricting our considerations to the case $X=\mathbb{R}^2$ only.

For $\mathrm{Lin}(X)$-convex subsets of X the necessary conditions are given in Theorem 4.2. For convex $\mathrm{Lin}(X)$-subsets they are also sufficient as shown in Proposition 4.5.

If a $\mathrm{Lin}(X)$-convex subset is not convex then it is the intersection of some family $\mathcal{M}$ that contains non convex sublevel set $S_d(\varphi_A)$. This sublevel set is either a region bounded by a pair of conjugate hyperbolas or the sum of two non-parallel lines. The latter may be considered a limit case of the former. To see this consider, for example, the set $\{(x_1,x_2)\colon |x_1^2-x_2^2|\le \frac{1}{n}\}$.

On the other hand, for convex $\mathrm{Lin}(X)$-convex sets there exists a family $\mathcal{M}$ consisting only of convex sublevel sets $S_d(\varphi_A)$. In the two dimensional case, these represent regions bounded either by an ellipse or by two parallel lines. The family $\mathcal{M}$ can be reduced to consist only of sublevel sets of the second type. These sublevel sets may again be regarded as a limiting case of hyperbolas. To see this, consider the set $\{(x_1, x_2) \colon |x_1^2 - \frac{1}{n}x_2^2| \leq 1\}$.

In any case, if G is abstract convex then $G \cap S_d(\varphi_A)$ is abstract convex. Example 4.6 shows that if the resulting set is not convex then it must contain points lying on a certain hyperbola. Hence, it can be supposed that the missing condition is of the form: any two points can be joined by an arc of a certain hyperbola, possibly degenerated to a segment.

For $\mathrm{Aff}(X)$-convex subsets of $X \times \mathbb{R}$ necessary conditions are given in Theorems 4.8, 4.9 and 4.10. The condition in (Γ) stated in Theorem 4.10 is certainly a part of an inner definition of $\mathrm{Aff}(X)$-convexity. However, since it addresses only radial directions, it is clearly not sufficient.

It can be supposed that the missing condition involves a description of the geometry in directions other than radial. An examination of the geometry of $\operatorname{epi} f$ and $\operatorname{epi} g$ from Examples 4.11 and 4.12 suggests that it may involve parabolas, this time opening both downward and upward, as well as the number of linearly independent directions in which the epigraph is an halfplane.

Some further remarks on the condition (Γ) are in order. Since E is X-balanced, the points $\overline{P}_0 = (-t_0 v, d_0)$ and $\overline{P}_1 = (-t_1 v, d_1)$ also belong to E. Consequently, the four points $\overline{P}_1, \overline{P}_0, P_0$ and P_1 lie on the parabola described by (Γ), and the arcs $\gamma([-t_1, -t_0]), \gamma([t_0, t_1])$ are contained in E.

However, the entire arc $\gamma([-t_1, t_1])$ may not be contained in E. This depends on the specific configuration of P_0 and P_1. Consider, for example, the epigraph of $f(x) = \max\{0, \|x\|^2 - 1\}$.

On the other hand, the segment $[-1, 1]\{P_0\}$ is contained in E, due to X-balancedness. Thus, the quadruple of points $\{\overline{P}_1, \overline{P}_0, P_0, P_1\}$ can be joined by the curve

$$\overline{\gamma}(t) = \begin{cases} \gamma(t) & t \in [-t_1, -t_0] \cup [t_0, t_1] \\ (t, d_0) & t \in [-t_0, t_0] \end{cases}$$

which is entirely contained in E.

If $d_0 = d_1$ then γ degenerates to a horizontal line passing through all four points, so that $\gamma([-t_1, t_1]) \subset E$. This situation is also covered by the fact that E is X-balanced. Finally, if $x_0 = x_1$ then by epigraphicity points P_0 and P_1 can also be joined by a segment contained in E.

Independent of the present discussion, we propose the following conjecture. Its validity is suggested by the argument used in the proof of Theorem 5.3.

Proposition 4.35. (Conjecture) *Let E be a proper subset of $X \times \mathbb{R}$. Assume that*

- *E is epigraphic, closed, X-balanced and bounded below and satisfies the condition (Γ),*
- *E is convex,*

then $E \in \mathcal{K}(X \times \mathbb{R})$.

For $\mathrm{Lin}(\mathcal{L})$- and $\mathrm{Aff}(\mathcal{L})$-convex sets the situation is different. Since sets in both classes are convex, the curve in question is just a segment.

For $\mathrm{Lin}(\mathcal{L})$-convex subsets of $\mathcal{L}$ necessary conditions are given in Theorem 4.17. For $\mathrm{Aff}(\mathcal{L})$-convex subsets of $\mathcal{L} \times \mathbb{R}$ necessary conditions are given in Theorem 4.28. These conditions are not sufficient as the examples in this section show.

One might assume that the missing condition should be geometric in nature, taking into account not only convexity but also the specific metric geometry introduced by both the operator norm and the Frobenius inner product. Examining examples, it seems that a key obstacle is the presence of a face perpendicular to a direction that does not take the form T_x. Therefore, the condition should also take into account a special form of the normal vector T_x, although how remains unclear. Unless these relations are better understood, the description of $\mathrm{Lin}(\mathcal{L})$-convex sets will remain difficult.

Let us also point out that there is an intriguing duality between $\mathrm{Lin}(X)$-convex sets and vectors T_x, which deserves further investigation.

To conclude this section, we compare the abstract convexity of subsets of $\mathbb{R}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$. Set

- $X = \mathbb{R}^n, \mathcal{L}_1 = \mathcal{L}^2_{sym}(X, \mathbb{R})$ and $\varphi_1 \colon X \times \mathcal{L}_1 \to [0, \infty)$ be given by (φ);
- $Y = \mathbb{R}^{n+1}, \mathcal{L}_2 = \mathcal{L}^2_{sym}(Y, \mathbb{R})$ and $\varphi_2 \colon Y \times \mathcal{L}_2 \to [0, \infty)$ be given by (φ).

Convex subsets of $X \times \mathbb{R}$ and Y coincide, since both classes are defined as the intersection of some family of halfspaces in $\mathbb{R}^{n+1}$. In particular, epigraphs of convex functions $f \colon X \to \mathbb{R}$ are convex epigraphic subsets of Y.

In our setting, classes $\mathcal{K}(X \times \mathbb{R})$ and $\mathcal{K}(Y)$ are distinct. Abstract convexity with respect to $\mathrm{Aff}(X)$ is founded on functions $x \mapsto |A[x]^2| + r$, but abstract convexity with respect to $\mathrm{Lin}(Y)$ is founded on functions of the form $y = (x, d) \mapsto |A[(x, d)]^2|$.

From a geometric perspective, as noted in Section 3, the classes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are defined by geometric objects of qualitatively different types. In particular, an $\mathrm{Aff}(X)$-convex subset of $X \times \mathbb{R}$ does not need to contain $(0, 0)$ but it must be X-balanced, bounded from below, and unbounded from above. In contrast, $\mathrm{Lin}(Y)$-convex subsets must contain $(0, 0)$ and be balanced. That is, if they are unbounded in the direction of some vector v, they must also be unbounded in the direction $-v$.

It turns out that the only sets that are abstract convex with respect to both families are $\emptyset$ and $X \times \mathbb{R} = Y$. Indeed, assume that $G \in \mathcal{K}(X \times \mathbb{R}) \cap \mathcal{K}(Y)$ is a proper subset of $X \times \mathbb{R} = Y$.

- Since $G \in \mathcal{K}(X \times \mathbb{R})$, it is epigraphic. In particular, together with any point (x, d), it contains the halfline $\{x\} \times [d, \infty)$. Moreover, it is contained in some halfspace $X \times [D, \infty)$, hence it does not contain any vertical line.
- Since $G \in \mathcal{K}(Y)$ it contains $(0, 0)$ and is symmetric.

Hence, G at the same time contains $\{0\} \times \mathbb{R}$ and is a subset of upper halfspace. A contradiction.

The same observation holds for $\mathcal{K}(\mathcal{L})$ and $\mathcal{K}(\mathcal{L} \times \mathbb{R})$.

5. Special case: $X = \mathbb{R}^1$

The case $X = \mathbb{R}^1$ is distinctive because a bilinear map in this case reduces to a multiplication operator on X. Every element $A \in \mathcal{L}^2_{sym}(X, \mathbb{R}) \simeq \mathbb{R}$ is represented by the unique real number. Therefore, we can write:

$$\varphi(x, A) = |A|x^2$$

We keep the symbol X instead of using just $\mathbb{R}^1$. We also keep using notation $\mathcal{L} = \mathcal{L}^2_{sym}(X, \mathbb{R})$, despite the fact that $\mathcal{L} \simeq \mathbb{R}$.

Consequently, we have the following characterization of abstract linear and abstract affine functions:

$$\begin{aligned} \mathrm{Lin}X &= \{x \mapsto \alpha x^2 \colon \alpha \geq 0\}, \quad \mathrm{Aff}X = \{x \mapsto \alpha x^2 + \beta \colon \alpha \geq 0, \beta \in \mathbb{R}\}, \\ \mathrm{Lin}(\mathcal{L}) &= \{A \mapsto |A|\alpha \colon \alpha \geq 0\}, \quad \mathrm{Aff}(\mathcal{L}) = \{A \mapsto |A|\alpha + \beta \colon \alpha \geq 0, \beta \in \mathbb{R}\}. \end{aligned}$$

5.1. Subsets of X and $X \times \mathbb{R}$

Theorem 4.2 provides full description of Lin$X$-convex sets in the case $X = \mathbb{R}^1$: they are closed intervals centered at zero.

Proposition 5.1. *Let $\varphi_A \in \mathrm{Lin}X$ and $d \in \mathbb{R}$. We have*

$$S_d(\varphi_A) = \begin{cases} \emptyset & d < 0, A \in \mathcal{L}, \\ \mathbb{R} & d \geq 0, A = 0, \\ \{0\} & d = 0, A \neq 0, \\ \left[-\sqrt{\frac{d}{|A|}}, \sqrt{\frac{d}{|A|}}\ \right] & d > 0, A \neq 0. \end{cases}$$

Moreover, $$\mathcal{K}(X) = \{\emptyset, \mathbb{R}\} \cup \{[-\alpha, \alpha] \colon \alpha \geq 0\}.$$

The proof is strightforward. Note that $\mathcal{K}(X)$ is a proper subset of the family of all closed convex subsets of $\mathbb{R}$.

Epigraphs of X-affine functions divide into two classes.

Proposition 5.2. *Let $A \in \mathcal{L}$ and $d \in \mathbb{R}$. If $A = 0$ then we have*

$$\mathrm{epi}(\varphi_0 + r) = \{(x, d) \in X \times \mathbb{R} \colon r \leq d\} = X \times [r, \infty)$$

that is, an upper halfplane. For $A \neq 0$ we have

$$\mathrm{epi}(\varphi_A + r) = \{(x, d) \in X \times \mathbb{R} \colon |A|x^2 + r \leq d\}$$

the region bounded by an upward opening parabola with axis $x = 0$.

As a consequence, an Aff$X$-convex set $E \subset X \times \mathbb{R}$ is convex, as it is the intersection of regions bounded by a family of upward opening parabolas and halfplanes.

Convexity of Aff$X$-convex subsets is a distinctive property in the case $X = \mathbb{R}^1$. Moreover, such sets are also epigraphic, X-balanced and closed. These properties alone do not characterize Aff$X$-convex sets. However, by additionally assuming that E satisfies condition (Γ), we obtain a full description of the class $\mathcal{K}(X \times \mathbb{R})$ and an inner characterization of Aff$X$-convex subsets of $X \times \mathbb{R}$.

Theorem 5.3. *Let $X = \mathbb{R}$ and $E \subset X \times \mathbb{R}$. If $E = \emptyset$ or $E = X \times \mathbb{R}$ then $E \in \mathcal{K}(X \times \mathbb{R})$. Assume that E is a proper subset. The following conditions are equivalent:*

(1) $E \in \mathcal{K}(X \times \mathbb{R})$

(2) *E satisfies the following conditions:*

- $(\mathcal{E}_1^X)$ *E is closed,*
- $(\mathcal{E}_2^X)$ *E is bounded below,*
- $(\mathcal{E}_3^X)$ *if $(x,d) \in E$ then $\{x\} \times [d,\infty) \subset E$,*
- $(\mathcal{E}_4^X)$ *E is X-balanced,*
- $(\mathcal{E}_5^X)$ *for any two points (x_0,d_0), $(x_1,d_1) \in E$ satisfying $0 \le x_0 < x_1$ and $d_0 < d_1$, we have*

$$\left(x, \frac{d_1 - d_0}{x_1^2 - x_0^2}\,(x^2 - x_0^2) + d_0\right) \in E, \text{ for all } x \in [x_0, x_1]$$

Proof. Assume that $E \subset X \times \mathbb{R}$ is a proper subset. The implication (1) $\implies$ (2) follows from the results in Section 4.

Now we prove that (2) $\implies$ (1). To begin with, note that if $E = \{0\} \times [d_E, \infty)$, $d_E \in \mathbb{R}$ then the theorem holds. The separate treatment of this case allows us to avoid technical inconveniences in the proof.

From now on, we assume that

$$\text{there exists } (x,d) \in E \text{ such that } x > 0. \tag{5}$$

We first show that E is convex. Indeed, let $P_0 = (x_0,d_0)$ and $P_1 = (x_1,d_1)$ be two points in E. Let us denote by S the segment joining P_0 and P_1. We will show that, regardless of the position of P_0 and P_1 the segment S is contained in E.

- If $x_0 = x_1$ then $S \subset E$ by $(\mathcal{E}_3^X)$. If $d_0 = d_1$ then $S \subset E$ by $(\mathcal{E}_4^X)$.
- Assume $0 \le x_0 < x_1$ and $d_0 < d_1$. By $(\mathcal{E}_5^X)$, points P_0 and P_1 can be joined by an arc of an upward opening parabola. Therefore $S \subset E$ by convexity of the parabola and $(\mathcal{E}_3^X)$.
- Assume $0 \le x_0 < x_1$ and $d_0 > d_1$. The segment joining $P = (x_0,d_1)$ and P_1 is contained in E. Hence,$S \subset E$ by $(\mathcal{E}_3^X)$, since S lies above this segment.
- If $x_1 < x_0 \le 0$ then $S \subset E$ by symmetry and two previous cases.
- Assume $x_0 < 0 < x_1$ and $d_0 > d_1$. Then $P = (-x_0,d_0) \in E$ and $S \subset E$, as it lies above the polyline $P_0PP_1 \subset E$.

The remaining cases can be treated in similar manner and are left to the reader.

Using the same argument as in the proof of Theorem 4.9 we get that

$$d_E \overset{df}{=} \inf\{d\colon (x,d) \in E\} = \min\{d\colon (0,d) \in E\}.$$

Let $\overline{P} = (\overline{x}, \overline{d}) \notin E$. We show that E can be separated from $\overline{P}$ by an X-affine function $\varphi_A + r$. Namely, we show that there exist $A \in \mathcal{L} \simeq \mathbb{R}$ and $r \in \mathbb{R}$ such that

$$\forall_{(x,d)\in E}\ |A|x^2 + r \le d \wedge |A|\overline{x}^2 + r > \overline{d}.$$

Observe that:

- If $\overline{x} < 0$ and $d \in \mathbb{R}$ then any $\varphi_A + r$ separating E form $\overline{P}$ also separates E from $(-\overline{x}, \overline{d})$ by evenness.
- If $\overline{d} < d_E$ and $\overline{x} \in X$ (i.e. $\overline{P}$ lies strictly below E) then E can be separated from $\overline{P}$ by $\varphi_0 + d_E$.

Hence,we may assume that $\quad 0 < \overline{x}$ and $\overline{d} \geq d_E. \qquad (6)$

That means that the point $\overline{P}$ is placed below and to the right of E. Note that $\overline{P} \neq (0, d_E)$. Since E is closed and convex subset of $X \times \mathbb{R} \simeq \mathbb{R}^2$, there exists the unique point $P_0 = (x_0, d_0) \in E$ satisfying

$$(\overline{x} - x_0)(x - x_0) + (\overline{d} - d_0)(d - d_0) \leq 0 \quad \text{for all } (x, d) \in E. \tag{7}$$

The point P_0 is the point in E that is closest to $\overline{P}$ (see [6, Theorem 3.1.1]). Clearly, P_0 is a boundary point of E. Set

$$L(x, d) = (\overline{x} - x_0)(x - x_0) + (\overline{d} - d_0)(d - d_0). \tag{8}$$

The equation $L(x, d) = 0$ represents a line, which we will also denote by L, that separates E from $\overline{P}$. Indeed, we have $L(x, d) \leq 0$ for all $(x, d) \in E$ and

$$L(\overline{x}, \overline{d}) = (\overline{x} - x_0)^2 + (\overline{d} - d_0)^2 > 0.$$

Moreover, L is supporting E at P_0, since $L(x_0, d_0) = 0$.

We claim that $0 < x_0 < \overline{x}$ and $\overline{d} \leq d_0$. Suppose, on the contrary, that this claim is false. We analyze each possible case and, in each, we show that there exists a point P in E such that $L(P) > 0$, contradicting (7).

- $x_0 \in \mathbb{R}$, $d_0 < \overline{d}$: By assumption $(\mathcal{E}_3^X)$, $(x_0, \overline{d}) \in E$. Substituting into (8) we get $(\overline{x} - x_0)(x_0 - x_0) + (\overline{d} - d_0)(\overline{d} - d_0) = (\overline{d} - d_0)^2 > 0$;
- $x_0 < 0$, $d_0 \geq \overline{d}$: Since E is X-symmetric, $(-x_0, d_0) \in E$. Hence $(\overline{x} - x_0)(-x_0 - x_0) + (\overline{d} - d_0)(d_0 - d_0) = (\overline{x} - x_0)(-2x_0) > 0$ since we assumed $0 < \overline{x}$;
- $x_0 = 0$, $d_0 > \overline{d}$: From (6) and $(\mathcal{E}_3^X)$ we obtain $(0, \overline{d}) \in E$. This yields $(\overline{x} - 0)(0 - 0) + (\overline{d} - d_0)(\overline{d} - d_0) = (\overline{d} - d_0)^2 > 0$;
- $x_0 = 0$, $d_0 = \overline{d}$: We assumed in (5) that there exists $(x, d) \in E$ with $x > 0$. Hence, $(\overline{x} - 0)(x - 0) + (\overline{d} - \overline{d})(d - \overline{d}) = \overline{x}x > 0$;
- $x_0 = \overline{x}$, $d_0 > \overline{d}$: Since $(0, \overline{d}) \in E$, we obtain $(\overline{x} - \overline{x})(0 - \overline{x}) + (\overline{d} - d_0)(\overline{d} - d_0) = (\overline{d} - d_0)^2 > 0$;
- $x_0 > \overline{x}$, $d_0 \geq \overline{d}$: By $(\mathcal{E}_4^X)$ $(\overline{x}, d_0) \in E$. This gives $(\overline{x} - x_0)(\overline{x} - x_0) + (\overline{d} - d_0)(d_0 - d_0) = (\overline{x} - x_0)^2 > 0$.

Thus the point $\overline{P}$ satisfies $\quad 0 < x_0 < \overline{x}$ and $\overline{d} \leq d_0$. $\qquad (9)$

Having completed these preliminary steps, we can now proceed with the main part of the proof. It will be divided into two steps.

Step 1: Assume $0 < x_0 < \overline{x}$ and $\overline{d} = d_0$. Since the equation of the line L is now $(\overline{x} - x_0)(x - x_0) = 0$, the set E lies to the left of this line and, by symmetry, to the right of the line $x = -x_0$. Hence, E it is contained in $S = [-x_0, x_0] \times [d_E, \infty)$.

By Proposition 4.14, the set S is Aff$X$-convex. Since $(\overline{x}, \overline{d}) \notin S$, the set S can be separated from $(\overline{x}, \overline{d})$ by an X-affine function, and the same holds for E.

Step 2: Assume $0 < x_0 < \overline{x}$ and $\overline{d} < d_0$. Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = -\frac{\overline{x} - x_0}{2x_0(\overline{d} - d_0)}(x^2 - x_0^2) + d_0.$$

Note that $\varphi(x_0) = d_0$ and φ is an X-affine function, since

$$-\frac{\overline{x} - x_0}{2x_0(\overline{d} - d_0)} > 0.$$

It is easy to check that L is tangent to φ at x_0. The function φ is convex, therefore L lies below its graph. It follows that $\varphi(\overline{x}) > \overline{d}$.

Let $P_1 = (x_1, d_1) \in E$. We show that $P_1 \in \operatorname{epi}\varphi$, i.e. it satisfies $q(x_1) \le d_1$. We may assume that $x_1 \ge 0$, due to symmetry.

The point P_1 lies above L. We divide the halfplane $L(x, d) \le 0$ into three regions using the vertical and horizontal lines passing through P_0.

Assume $0 \le x_1 \le x_0$ and $d_1 \ge d_0$. This case is strightforward:

$$\varphi(x_1) = \underbrace{-\frac{\overline{x} - x_0}{2x_0(\overline{d} - d_0)}}_{>0}\underbrace{(x_1^2 - x_0^2)}_{<0} + d_0 \le d_1$$

Hence, $P_1 \in \operatorname{epi}\varphi$. Assume $x_0 < x_1$ and $d_0 < d_1$. Set

$$q(x) = \frac{d_1 - d_0}{x_1^2 - x_0^2}(x^2 - x_0^2) + d_0.$$

By assumption $(\mathcal{E}_5^X)$, we obtain $(x, q(x)) \in E$ for $x \in [x_0, x_1]$. Substituting $(x, q(x))$ into (7) we get

$$\begin{aligned} 0 &\ge (\overline{x} - x_0)(x - x_0) + (\overline{d} - d_0)(q(x) - d_0) \\ &= (\overline{x} - x_0)(x - x_0) + (\overline{d} - d_0)\frac{d_1 - d_0}{x_1^2 - x_0^2}(x^2 - x_0^2) \\ &= \Big((\overline{x} - x_0) + (\overline{d} - d_0)\frac{d_1 - d_0}{x_1^2 - x_0^2}(x + x_0)\Big)(x - x_0). \end{aligned}$$

Assume $x_0 < x$, then we have $(\overline{x} - x_0) + (\overline{d} - d_0)\dfrac{d_1 - d_0}{x_1^2 - x_0^2}(x + x_0) \le 0.$

Letting $x \to x_0^+$ we get $(\overline{x} - x_0) + 2x_0(\overline{d} - d_0)\frac{d_1 - d_0}{x_1^2 - x_0^2} \le 0.$

From $x_0 > 0$, $\overline{d} - d_0 < 0$ and the above inequality we obtain

$$-\frac{\overline{x} - x_0}{2x_0(\overline{d} - d_0)} \le \frac{d_1 - d_0}{x_1^2 - x_0^2}.$$

Since $x_1^2 - x_0^2 > 0$, we obtain

$$\varphi(x_1) = -\frac{\overline{x} - x_0}{2x_0(\overline{d} - d_0)}(x_1^2 - x_0^2) + d_0 \leq \frac{d_1 - d_0}{x_1^2 - x_0^2}(x_1^2 - x_0^2) + d_0 = q(x_1).$$

Hence, $P_1 \in \operatorname{epi}\varphi$. Assume $0 \leq x_1 < x_0$ and $d_1 < d_0$. Set

$$q(x) = \frac{d_0 - d_1}{x_0^2 - x_1^2}(x^2 - x_1^2) + d_1 = \frac{d_0 - d_1}{x_0^2 - x_1^2}(x^2 - x_0^2) + d_0.$$

Then $(x, q(x)) \in E$, for $x \in [x_1, x_0]$, by assumption $(\mathcal{E}_5^X)$. Substituting $(x, q(x))$ into (7) we get

$$\begin{aligned}
0 &\geq (\overline{x} - x_0)(x - x_0) + (\overline{d} - d_0)(q(x) - d_0) \\
&= (\overline{x} - x_0)(x - x_0) + (\overline{d} - d_0)\frac{d_0 - d_1}{x_0^2 - x_1^2}(x^2 - x_0^2) \\
&= \Big((\overline{x} - x_0) + (\overline{d} - d_0)\frac{d_0 - d_1}{x_0^2 - x_1^2}(x + x_0)\Big)(x - x_0).
\end{aligned}$$

Assume $x < x_0$, then we have $(\overline{x} - x_0) + (\overline{d} - d_0)\dfrac{d_0 - d_1}{x_0^2 - x_1^2}(x + x_0) \geq 0.$

Letting $x \to x_0^-$ we get $(\overline{x} - x_0) + 2x_0(\overline{d} - d_0)\frac{d_1 - d_0}{x_1^2 - x_0^2} \geq 0.$

From $x_0 > 0$, $\overline{d} - d_0 < 0$ and the above inequality we obtain

$$\frac{d_1 - d_0}{x_1^2 - x_0^2} \leq -\frac{(\overline{x} - x_0)}{2x_0(\overline{d} - d_0)}.$$

Since $x_1^2 - x_0^2 < 0$, we obtain

$$\varphi(x_1) = -\frac{\overline{x} - x_0}{2x_0(\overline{d} - d_0)}(x_1^2 - x_0^2) + d_0 \leq \frac{d_1 - d_0}{x_1^2 - x_0^2}(x_1^2 - x_0^2) + d_0 = q(x_1).$$

Hence, $P_1 \in \operatorname{epi}\varphi$. The proof is finished. □

5.2. Subsets of $\mathcal{L}$ and $\mathcal{L} \times \mathbb{R}$

In the case $X = \mathbb{R}$, Theorem 4.28 provides a complete description of $\mathrm{Lin}(\mathcal{L})$-convex sets: they are closed intervals centered at zero.

Proposition 5.4. *Let $\psi_x \in \mathrm{Lin}(\mathcal{L})$ and $d \in \mathbb{R}$. We have*

$$S_d(\psi_x) = \begin{cases} \emptyset & d < 0, x \in X, \\ \mathcal{L} & d \geq 0, x = 0, \\ \{0\} & d = 0, x \neq 0, \\ \left[-\frac{d}{x^2}, \frac{d}{x^2}\right] & d > 0, x \neq 0. \end{cases}$$

Moreover, $$\mathcal{K}(\mathcal{L}) = \{\emptyset, \mathcal{L}\} \cup \{[-\alpha, \alpha] \colon \alpha \geq 0\}.$$

Here, we identify A with a real number. The proof is straightforward. Note that $\mathcal{K}(\mathcal{L})$ is a proper subset of the family of all closed convex subsets of $\mathcal{L}$.

Epigraphs of $\mathcal{L}$-affine functions divide into two classes.

Proposition 5.5. *Let $x \in X$ and $d \in \mathbb{R}$. If $x = 0$ then we have*

$$\operatorname{epi}(\varphi_0 + r) = \{(A, d) \in \mathcal{L} \times \mathbb{R} \colon r \le d\} = \mathcal{L} \times [-d, \infty)$$

that is, an upper halfplane. For $x \neq 0$ we have

$$\operatorname{epi}(\psi_x + r) = \{(A, d) \in \mathcal{L} \times \mathbb{R} \colon |A|x^2 + r \le d\}$$

the region bounded below by the graph of function $A \mapsto \alpha|A| + \beta$.

We already know that $\operatorname{Aff}(\mathcal{L})$-convex sets are epigraphic, $\mathcal{L}$-balanced, closed and convex. In the general case, these properties do not fully characterize abstract convexity. As the examples in Section 4 show, orientation in space also plays a role. This issue disappears in the one dimensional case.

Theorem 5.6. *Let $X = \mathbb{R}$ and let $E \subset X \times \mathbb{R}$. If $E = \emptyset$ or $E = \mathcal{L} \times \mathbb{R}$ then $E \in \mathcal{K}(\mathcal{L} \times \mathbb{R})$. Assume that E is a proper subset. The following conditions are equivalent:*

(1) $E \in \mathcal{K}(\mathcal{L} \times \mathbb{R}))$

(2) *E satisfies the following conditions:*

- $\left(\mathcal{E}_1^{\mathcal{L}}\right)$ *E is closed,*
- $\left(\mathcal{E}_2^{\mathcal{L}}\right)$ *E is bounded below,*
- $\left(\mathcal{E}_3^{\mathcal{L}}\right)$ *if $(A, d) \in E$ then $\{A\} \times [d, \infty) \subset E$,*
- $\left(\mathcal{E}_4^{\mathcal{L}}\right)$ *E is $\mathcal{L}$-balanced,*
- $\left(\mathcal{E}_5^{\mathcal{L}}\right)$ *E is convex.*

Proof. Assume that $E \subset \mathcal{L} \times \mathbb{R}$ is a proper subset. The implication (1) $\implies$ (2) follows from the results in Section 4.

Now we prove that (2) $\implies$ (1). To begin with, note that if $E = \{0\} \times [d_E, \infty)$, $d_E \in \mathbb{R}$ then the theorem holds. The separate treatment of this case allows us to avoid technical inconveniences in the proof.

From now on, we assume that

$$\text{there exists } (A, d) \in E \text{ such that } A > 0. \tag{10}$$

Using the same argument as in the proof of Theorem 4.29 we obtain that

$$d_E \overset{df}{=} \inf\{d \colon (A, d) \in E\} = \min\{d \colon (0, d) \in E\}.$$

Let $\overline{P} = (\overline{A}, \overline{d}) \notin E$. We show that E can be separated from $\overline{P}$ by $\mathcal{L}$-affine function $\psi_x + r$. Namely, we show that there exist $x \in X$ and $r \in \mathbb{R}$ such that

$$\forall_{(A,d)\in E}\ x^2|A| + r \le d \ \text{ and } \ x^2|\overline{A}| + r > \overline{d}.$$

We make two observations:

- If $\overline{A} < 0$ and $d \in \mathbb{R}$ then any $\psi_x + r$ separating E from $\overline{P}$ also separates E from $(-\overline{A}, \overline{d})$ by evenness.
- If $\overline{d} < d_E$ and $\overline{A} \in \mathcal{L}$ then E can be separated from $\overline{P}$ by $\psi_0 + d_E$.

Hence, we may assume that $\quad 0 < \overline{A}$ and $d_E \leq \overline{d}$. (11)

That means the point $\overline{P}$ is placed below and to the right of E. Note that $\overline{P} \neq (0, d_E)$. Since $E \subset \mathbb{R}^2$ is closed and convex, there exists a unique point $P_0 = (A_0, d_0) \in E$ satisfying

$$(\overline{A} - A_0)(A - A_0) + (\overline{d} - d_0)(d - d_0) \leq 0 \quad \text{for all } (A, d) \in E. \tag{12}$$

The point P_0 is the point in E that is closest $\overline{P}$ (see [6, Theorem 3.1.1]). Clearly, P_0 is a boundary point of E.

An analysis similar to that in the proof of Theorem 5.3, shows that

$$0 \leq A_0 < \overline{A} \text{ and } \overline{d} \leq d_0. \tag{13}$$

The main part of the proof will be divided into two steps.

Step 1: Assume $0 \leq A_0 < \overline{A}$ and $\overline{d} = d_0$. Substituting into (12) we obtain that if $(A, d) \in E$, then

$$(\overline{A} - A_0)(A - A_0) \leq 0 \implies A \leq A_0.$$

Hence, the set E lies to the left of the line $A = A_0$ and, by symmetry, to the right of the line $A = -A_0$.

Consequently, E is contained in $S = [-A_0, A_0] \times [d_E, \infty)$. By Proposition 4.32, the set S is $\mathrm{Aff}(\mathcal{L})$ convex. Since $(\overline{A}, \overline{d}) \notin S$, the set S can be separated from $(\overline{A}, \overline{d})$ by an $\mathcal{L}$-affine function, and the same holds for E.

Step 2: Assume $0 \leq A_0 < \overline{A}$ and $\overline{d} < d_0$. Define $\psi\colon \mathcal{L} \to \mathbb{R}$ by

$$\psi(A) = -\frac{\overline{A} - A_0}{\overline{d} - d_0}(|A| - A_0) + d_0.$$

Note that $\psi(A_0) = d_0$ and $-\dfrac{\overline{A} - A_0}{\overline{d} - d_0} > 0$ since $A_0 < \overline{A}$ and $\overline{d} < d_0$. Clearly, $\psi \in \mathrm{Aff}(\mathcal{L})$. It follows that

$$\psi(A) - d = -\frac{(\overline{A} - A_0)(A - A_0)}{\overline{d} - d_0} + d_0 - d = -\frac{(\overline{A} - A_0)(A - A_0) + (\overline{d} - d_0)(d - d_0)}{\overline{d} - d_0} \leq 0$$

for any $(A, d) \in E$, and

$$\psi(\overline{A}) - \overline{d} = -\frac{(\overline{A} - A_0)^2}{\overline{d} - d_0} + d_0 - \overline{d} = \frac{(\overline{A} - A_0)^2 + (d_0 - \overline{d})^2}{d_0 - \overline{d}} > 0.$$

Thus ψ separates E from $\overline{P}$. This completes the proof. □

A. Abstract convexity of subsets

All results presented in this paper are based on the terminology and results on abstract convexity as presented in [11]. This appendix provides the necessary definitions and theorems, referring the reader to [11] for the proofs and for a more comprehensive treatment of the subject, including historical context.

The notation used here generally follows that of [11], with some modifications made for clarity. Additionally, the level of generality has been reduced to better adapt the abstract approach to the situation considered in the present paper.

It is interesting on its own that all the concepts of abstract convexity can be derived from a single abstract notion – abstract convexity in complete lattices. This covers abstract convexity of sets considered in this paper and abstract convexity of functions, abstract conjugate functions and abstract subdifferential, as well. For this reason, an abstract approach is presented first to provide the reader with a broader context.

A.1. Abstract convexity in complete lattices

Let $(\mathbb{E}, \leq)$ be a complete lattice i.e., a poset such that every subset has the least and the greatest element. We denote the lattice operations in $\mathbb{E}$ by $\inf^{\mathbb{E}}$ and $\sup^{\mathbb{E}}$. The least (resp. the greatest) element of $\mathbb{E}$ is denoted by $-\infty^{\mathbb{E}}$ (resp. by $+\infty^{\mathbb{E}}$). We adopt the conventions

$$\inf^{\mathbb{E}} \emptyset = +\infty^{\mathbb{E}}, \quad \sup^{\mathbb{E}} \emptyset = -\infty^{\mathbb{E}}.$$

The abstract convexity in $\mathbb{E}$ is built upon a distinguished (nonempty) subset $\mathcal{M}$ of $\mathbb{E}$. The fundamental concept in this theory is an $\mathcal{M}$-convex element.

Definition A.1. ([11, Definition 1.1]) An element $e \in \mathbb{E}$ is said to be $\mathcal{M}$-*convex* if

$$e = \sup^{\mathbb{E}}\{m \in \mathcal{M} \colon m \leq e\}.$$

We will denote by $\mathcal{C}(\mathcal{M})$ the set of all $\mathcal{M}$-convex elements. □

In some situations, it is more convenient to use the following simple observation:

$$e \in \mathcal{C}(\mathcal{M}) \iff \exists_{\mathcal{M}_e \subset \mathcal{M}} \; e = \sup^{\mathbb{E}} \mathcal{M}_e.$$

It is clear that $\mathcal{M}_e \subset \{m \in \mathcal{M} \colon m \leq e\}$. The main advantage is that the set $\mathcal{M}_e$ may contain significantly fewer elements.

Basic properties of $\mathcal{M}$-convex elements are contained in [11, Propositions 1.1 and 1.2].

Proposition A.2.

(a) $\mathcal{M} \subset \mathcal{C}(\mathcal{M})$

(b) *For any index set I we have*

$$\{e_i\}_{i\in I} \subset \mathcal{C}(\mathcal{M}) \implies \sup^{\mathbb{E}}\{e_i \colon i \in I\} \in \mathcal{C}(\mathcal{M})$$

(c) $-\infty^{\mathbb{E}} \in \mathcal{C}(\mathcal{M})$

One of the fundamental tools in (classical) convex analysis is the separation theorem. It turns out to be a general property that is also present in abstract convexity.

A subset Y of $\mathbb{E}$ is called an *infimal generator* of $\mathbb{E}$ if for each $e \in \mathbb{E}$ there exists a subset Y_e of Y such that

$$e = \inf^{\mathbb{E}} Y_e.$$

Let $e, \overline{e} \in \mathbb{E}$. We say that the element $m \in \mathcal{M}$ separates e from $\overline{e}$ if $m \leq e$ and $m \not\leq \overline{e}$. Every $\mathcal{M}$-convex element can be characterized by separation from elements of an infimal generator.

Proposition A.3. ([11, Theorem 1.3]) *If Y is an infimal generator of $\mathbb{E}$, then*

$$\mathcal{C}(\mathcal{M}) = \{e \in \mathbb{E} \colon \forall_{\substack{\overline{e} \in Y \\ e \not\leq \overline{e}}} \exists_{m \in \mathcal{M}}\, m \leq e \wedge m \not\leq \overline{e}\}.$$

As demonstrated in [11], this approach allows one to recognize a wide range of situations as instances of abstract convexity. In particular, for a fixed set $\mathcal{Z}$, the abstract convexity of subsets is formulated in $(2^{\mathcal{Z}}, \subseteq)$, while the abstract convexity of functions is formulated in $(\overline{\mathbb{R}}^{\mathcal{Z}}, \leq)$. We will focus on the abstract convexity of subsets.

A.2. Abstract convexity with respect to a family $\mathcal{M} \subset 2^{\mathcal{Z}}$

Fix a nonempty set $\mathcal{Z}$ and consider the complete lattice $(\mathbb{E}, \leq) = (2^{\mathcal{Z}}, \supset)$. The power set is a complete lattice ordered by inclusion:

$$G_1 \leq G_2 \iff G_1 \supset G_2 \quad G_1, G_2 \in 2^{\mathcal{Z}}.$$

Note the reversed order in the definition of $\leq$. The corresponding lattice operations, the least and the greatest element of $2^{\mathcal{Z}}$ are given by

$$\inf{}^{\mathbb{E}} = \bigcup, \ \sup{}^{\mathbb{E}} = \bigcap, \quad -\infty^{\mathbb{E}} = \mathcal{Z}, \ +\infty^{\mathbb{E}} = \emptyset.$$

The conventions introduced above take the following form:

$$\bigcup \emptyset = \emptyset, \ \bigcap \emptyset = \mathcal{Z}.$$

We will write $G \subset \mathcal{Z}$ rather than $G \in 2^{\mathcal{Z}}$. For the reader's convenience, the definitions provided above will now be restated using notation adapted to this specific setting.

Let $\mathcal{M} \subset 2^{\mathcal{Z}}$ be a nonempty family of subsets of $\mathcal{Z}$. In the current setting, an $\mathcal{M}$-convex subset ($\mathcal{M}$-convex element) is the intersection of all elements of $\mathcal{M}$ that contain it.

Definition A.4. ([11, Definition 2.1]) A set $G \subset \mathcal{Z}$ is said to be $\mathcal{M}$-*convex* if

$$G = \bigcap \{M \in \mathcal{M} \colon G \subset M\}$$

We will denote by $\mathcal{C}(\mathcal{M})$ the family of all $\mathcal{M}$-convex subsets of $\mathcal{Z}$. □

We will call an $\mathcal{M}$-convex element of $2^{\mathcal{Z}}$ simply an abstract convex subset of $\mathcal{Z}$. We have,

$$G \in \mathcal{C}(\mathcal{M}) \iff \exists_{\mathcal{M}_G \subset \mathcal{M}}\, G = \bigcap \mathcal{M}_G.$$

Basic properties of $\mathcal{M}$-convex sets follow from Proposition A.2.

Proposition A.5. (cf. [11, Proposition 2.1])

(a) *$\mathcal{M} \subset \mathcal{C}(\mathcal{M})$;*

(b) *For any index set I we have*

$$\{G_i\}_{i \in I} \subset \mathcal{C}(\mathcal{M}) \implies \bigcap_{i \in I} G_i \in \mathcal{C}(\mathcal{M});$$

(c) *$\mathcal{Z} \in \mathcal{C}(\mathcal{M})$.*

Clearly, the set of all singletons

$$Y = \big\{\{z\} \in 2^{\mathcal{Z}} \colon z \in \mathcal{Z}\big\}$$

is an infimal generator of $2^{\mathcal{Z}}$. Hence we say that the set $M \in \mathcal{M}$ separates $G \subset \mathcal{Z}$ from $\overline{z} \in \mathcal{Z}$ if

$$G \subset M \text{ and } \overline{z} \notin M.$$

The $\mathcal{M}$-convexity of subsets can be characterized by separation of a subset G from a singleton $\{z\}$ by an element of $\mathcal{M}$ (cf. [11, Theorem 2.3]).

Proposition A.6. (separation property for subsets)

$$\mathcal{C}(\mathcal{M}) = \{G \subset \mathcal{Z} \colon \forall_{\overline{z} \notin G}\ \exists_{M \in \mathcal{M}}\ G \subset M \wedge \overline{z} \notin M\}.$$

Within this theory, one can define further concepts such as the abstract convex hull and the abstract support function. However, we do not introduce them in this paper, as they are not necessary for obtaining our results. It is likely that they will prove useful in further developing and deeper understanding the results presented in this article.

The family $\mathcal{M}$ can be given explicitly or derived from other objects. Several methods for constructing $\mathcal{M}$ are presented in [11, Chapter 2]. The method used in this paper is to introduce abstract convexity in the set $\mathcal{Z}$ using a family of functions $\mathcal{W} \subset \mathbb{R}^{\mathcal{Z}}$.

A.3. Abstract convexity with respect to a family $\mathcal{W} \subset \mathbb{R}^{\mathcal{Z}}$

Let $\mathcal{Z}$ be a nonempty set. Consider a family of functions $\emptyset \neq \mathcal{W} \subset \mathbb{R}^{\mathcal{Z}}$ and define

$$\mathcal{W} + \mathbb{R} = \{w + r \colon w \in \mathcal{W}, r \in \mathbb{R}\} \subset \mathbb{R}^{\mathcal{Z}}.$$

Using these two families of functions, we will define abstract convexity of subsets of $\mathcal{Z}$ and $\mathcal{Z} \times \mathbb{R}$. We begin with subsets of $\mathcal{Z}$. For any $w \in \mathcal{W}$ and any $d \in \mathbb{R}$, define

$$S_d(w) = \{z \in \mathcal{Z} \colon w(z) \leq d\},$$
$$\mathcal{S}(\mathcal{W} \times \mathbb{R}) = \{S_d(w) \colon (w, d) \in \mathcal{W} \times \mathbb{R}\} \subset 2^{\mathcal{Z}}.$$

Substituting $\mathcal{M} = \mathcal{S}(\mathcal{W} \times \mathbb{R})$, we obtain an instance of the abstract convexity defined in the previous section. In this case, the family $\mathcal{M}$ consists of all sublevel sets of functions from $\mathcal{W}$. It is convenient to introduce the following notion.

Definition A.7. ([11, Definition 2.9]) A set $G \subset \mathcal{Z}$ is *$\mathcal{W}$-convex* if it is $\mathcal{S}(\mathcal{W} \times \mathbb{R})$-convex. We will denote by $\mathcal{K}(\mathcal{Z}; \mathcal{W})$ the family of all $\mathcal{W}$-convex subsets of $\mathcal{Z}$. □

Note that $G \subset S_d(w) \iff \forall_{g \in G}\ w(g) \leq d$.

All statements for $\mathcal{C}(\mathcal{M})$ listed in the preceding section can be reformulated replacing $M \in \mathcal{M}$ by $S_d(w) \in \mathcal{S}(\mathcal{W} \times \mathbb{R})$. In this way, one can obtain results on $\mathcal{W}$-convexity without referring to the sets $S_d(w)$.

By [11, Proposition 2.9], we have $\emptyset \in \mathcal{K}(\mathcal{X};\mathcal{W})$, this is due to the fact that all functions in $\mathcal{W}$ are finite valued. In [11] the set $\mathcal{W}$ is arbitrary subset of $\overline{\mathbb{R}}^{\mathcal{X}}$, i.e. functions assuming $\pm\infty$ are allowed. However, for our purposes this level of generality is not necessary.

We have (cf. [11, Corollary 2.3]) that

$$G \in \mathcal{K}(\mathcal{Z};\mathcal{W}) \text{ if and only if } G = \bigcap_{w \in \mathcal{W}} \{z \in \mathcal{Z} \colon \forall_{g \in G}\ w(z) \leq w(g)\}.$$

The separation property (i.e. Proposition A.6) can be expressed purely in the terms of functions in $\mathcal{W}$.

Proposition A.8. ([11, Theorems 2.6 and 2.7])

$$\mathcal{K}(\mathcal{Z};\mathcal{W}) = \{G \subset \mathcal{Z} \colon \forall_{\overline{z} \notin G}\ \exists_{w \in \mathcal{W}}\ \exists_{d \in \mathbb{R}}\ \forall_{g \in G}\ w(g) \leq d \wedge w(\overline{z}) > d\}.$$

It is worth noting that for every family $\mathcal{M} \subset 2^{\mathcal{Z}}$ there exists family of functions $\mathcal{W} \subset \overline{\mathbb{R}}^{\mathcal{Z}}$ such that $\mathcal{C}(\mathcal{M}) = \mathcal{C}\left(\mathcal{S}(\mathcal{W} \times \mathbb{R})\right)$ (see [11, Remark 2.28]). In this way, $\mathcal{M}$-convexity and $\mathcal{W}$-convexity are equivalent approaches to abstract convexity of subsets of a set $\mathcal{Z}$.

It turns out that there is no difference if we define abstract convexity of subsets with respect to $\mathcal{W}$ or $\mathcal{W} + \mathbb{R}$, see [11, Proposition 2.10 and Remark 2.27].

Now we introduce abstract convexity for subsets of $\mathcal{Z} \times \mathbb{R}$. For any $w \in \mathcal{W}$ and any $r \in \mathbb{R}$, define

$$\operatorname{epi}(w + r) = \{(z, d) \in \mathcal{Z} \times \mathbb{R} \colon w(z) + r \leq d\},$$
$$\mathcal{E}(\mathcal{W} + \mathbb{R}) = \{\operatorname{epi}(w + r) \colon w + r \in \mathcal{W} + \mathbb{R}\} \subset 2^{\mathcal{Z} \times \mathbb{R}}.$$

Substituting $\mathcal{M} = \mathcal{E}(\mathcal{W} + \mathbb{R})$, we obtain another instance of the abstract convexity of subsets. In this case, the family $\mathcal{M}$ consists of all epigraphs of functions from $\mathcal{W} + \mathbb{R}$.

Let us note that abstract convexity with respect to epigraphs is, in fact, an instance of abstract convexity with respect to the family of functions

$$\mathcal{W} \times \{-1\} = \{(w, -1) \in \mathbb{R}^{\mathcal{Z} \times \mathbb{R}} \colon (w, -1)(z, d) = w(z) - d,\ w \in \mathcal{W},\ (z, d) \in \mathcal{Z} \times \mathbb{R}\}.$$

It is easy to check that $S_r((w, -1)) = \operatorname{epi}(w - r)$. Hence, we have $\mathcal{E}(\mathcal{W} + \mathbb{R}) = \mathcal{S}\left(\left(\mathcal{W} \times \{-1\}\right) \times \mathbb{R}\right)$. See also [11, Theorem 3.4].

Despite the remark above, it is convenient for us to consider this case separately.

Definition A.9. ([11, Definition 2.9]) A set $E \subset \mathcal{Z} \times \mathbb{R}$ is $(\mathcal{W} + \mathbb{R})$-*convex* if it is $\mathcal{E}(\mathcal{W} + \mathbb{R})$-convex. We will denote by $\mathcal{K}(\mathcal{X} \times \mathbb{R}; \mathcal{W} + \mathbb{R})$ the family of all $(\mathcal{W} + \mathbb{R})$-convex subsets of $\mathcal{Z} \times \mathbb{R}$. $\square$

For further reference, let us write explicitly the definition of $\mathcal{E}(\mathcal{W}+\mathbb{R})$-convex subsets and corresponding separation property.

$$\begin{aligned} E \in \mathcal{K}(\mathcal{Z}\times\mathbb{R};\mathcal{W}+\mathbb{R}) &\iff E = \bigcap\{\operatorname{epi}(w+r) \in \mathcal{E}(\mathcal{W}+\mathbb{R}) \colon E \subset \operatorname{epi}(w+r)\} \\ &\iff \forall_{(\overline{z},\overline{d})\notin E}\ \exists_{w\in\mathcal{W}}\ \exists_{r\in\mathbb{R}}\ E \subset \operatorname{epi}(w+r) \wedge (\overline{z},\overline{d}) \notin \operatorname{epi}(w+r) \\ &\iff \forall_{(\overline{z},\overline{d})\notin E}\ \exists_{w\in\mathcal{W}}\ \exists_{r\in\mathbb{R}}\ \forall_{(z,d)\in E}\ w(z)+r \le d \wedge w(\overline{z})+r > \overline{d}. \end{aligned}$$

For completeness, we note that a function $f\colon Z \to \overline{\mathbb{R}}$ is called $(\mathcal{W}+\mathbb{R})$-convex, if

$$f(z) = \sup\{w(z)+r \colon w+r \in \mathcal{W}+\mathbb{R}, w \le f\},\ z \in \mathcal{Z}.$$

Following [11, Theorem 3.4], the epigraphs of the $(\mathcal{W}+\mathbb{R})$-convex functions are the $\mathcal{E}(\mathcal{W}+\mathbb{R})$-convex subsets of $\mathcal{Z}\times\mathbb{R}$. In this case, unlike in the case of $\mathcal{K}(\mathcal{Z};\mathcal{W})$, the choice between $\mathcal{W}$ and $\mathcal{W}+\mathbb{R}$ matters. We only have the inclusion $\mathcal{K}(\mathcal{Z}\times\mathbb{R};\mathcal{W}) \subset \mathcal{K}(\mathcal{Z}\times\mathbb{R};\mathcal{W}+\mathbb{R})$, which may be strict in general.

A.4. Abstract convexity with respect to a coupling function $\varphi\colon \mathcal{X}\times\mathcal{V}\to\mathbb{R}$

Let $\mathcal{X}$ and $\mathcal{V}$ be two nonempty sets and let $\varphi\colon \mathcal{X}\times\mathcal{V}\to\mathbb{R}$ be a function. The function φ will be referred to as a coupling function. In [11], the coupling function is allowed to assume $\pm\infty$, for our purposes it is not necessary.

Define $$\mathcal{W}_{\mathcal{V},\varphi} = \{\varphi(\cdot,v) \colon v \in \mathcal{V}\} \subset \mathbb{R}^{\mathcal{X}}$$

and $$\mathcal{W}_{\mathcal{V},\varphi}+\mathbb{R} = \{\varphi(\cdot,v)+r \colon v\in\mathcal{V}, r\in\mathbb{R}\} \subset \mathbb{R}^{\mathcal{X}}.$$

Having a coupling function one can define the concepts of $\mathcal{W}_{\mathcal{V},\varphi}$-convexity of subsets of $\mathcal{X}$, $(\mathcal{W}_{\mathcal{V},\varphi}+\mathbb{R})$-convexity of subsets of $\mathcal{X}\times\mathbb{R}$ and $(\mathcal{W}_{\mathcal{V},\varphi}+\mathbb{R})$-convexity of functions in $\overline{\mathbb{R}}^{\mathcal{X}}$. All results on abstract convexity listed above can be easily reproduced by simply replacing $w(x)$ by $\varphi(x,w)$.

Following [11, Chapter 3.3], we introduce two additional notions that add some geometric interpretation to abstract convexity.

- The functions in $\mathcal{W}_{\mathcal{V},\varphi}$, that is, the functions of the form $\varphi(\cdot,v)$, $v\in\mathcal{V}$ are called *abstract linear functions.*
- The functions in $\mathcal{W}_{\mathcal{V},\varphi}+\mathbb{R}$, that is, the functions of the form $\varphi(\cdot,v)+r$, $v\in\mathcal{V}$, $r\in\mathbb{R}$ are called abstract affine functions.

Hence, we can say that

- a $\mathcal{W}_{\mathcal{V},\varphi}$-convex set is the intersection of sublevel sets of abstract linear functions,
- a $\mathcal{E}(\mathcal{W}_{\mathcal{V},\varphi}+\mathbb{R})$-convex set is the intersection of epigraphs of abstract affine functions,
- a $(\mathcal{W}_{\mathcal{V},\varphi}+\mathbb{R})$-convex function is the pointwise supremum of abstract affine functions.

This terminology can be motivated in the following way. Let $\mathcal{X}=\mathbb{R}^n$, $\mathcal{V}=\mathbb{R}^n$, and $\varphi(x,v)=\langle x,v\rangle$. Then $\varphi(\cdot,v)$ is a linear functional on $\mathcal{X}$ and $\mathcal{W}_{\mathcal{V},\varphi}$ is the dual space of $\mathbb{R}^n$, which can be identified with $\mathcal{V}$. Clearly, the functions $\varphi(\cdot,v)+r$ are affine functions on $\mathcal{X}$. Moreover, sets $S_d(\varphi(\cdot, v)) \subset \mathcal{X}$ and $\mathrm{epi}(\varphi(\cdot, v) + r) \subset \mathcal{X} \times \mathbb{R}$ are closed halfspaces. Consequently, by the well-known results from the convex analysis, abstract convexity reduces to the classical convexity.

A coupling function allows one to "dualize" the concept of abstract convexity defined for $\varphi\colon \mathcal{X}\times\mathcal{V} \to \mathbb{R}$. Define $\psi\colon \mathcal{V}\times\mathcal{X} \to \mathbb{R}$ by $\psi(v, x) = \varphi(x, v)$. The coupling function ψ is called *dual* to φ. Setting

$$\mathcal{W}_{\mathcal{X},\psi} = \{\psi(\cdot, x) \in \mathbb{R}^{\mathcal{V}} \colon x \in \mathcal{X}\}$$

we can obtain the notion of abstract convexity for subsets of $\mathcal{V}$, subsets of $\mathcal{V} \times \mathbb{R}$ and functions in $\overline{\mathbb{R}}^{\mathcal{V}}$. This concept of "dualization" (cf. [11, Chapters 5–7]) allows one to introduce the conjugacy and the subdifferential.

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