

Editorial Material

Foreword: A Collection (28 Articles, pp. 193–617) Dedicated to Alexander Plakhov on the occasion of his 65th birthday

This volume of the *Journal of Convex Analysis* is dedicated to our colleague and friend Alexander Plakhov, Associate Professor at the Department of Mathematics of the University of Aveiro, Portugal, on the occasion of his 65th birthday. Plakhov's research connects geometry, dynamics, and optimization, particularly Newton's problem of least resistance, billiards and scattering, optimal transport, and the Kakeya problem. His results encompass the singular structure of optimal bodies, the single-impact assumption, invisible bodies and retroreflectors, and the Magnus effect for spinning bodies in rarefied media [12, 17, 19, 20, 29, 31]. He is the author of *Exterior Billiards: Systems with Impacts outside Bounded Domains* (Springer, 2012) [14] and more than 80 peer-reviewed articles.



This Collection grew out of the International Workshop on Optimization, Dynamics and Convex Analysis (WODCA 2025), held at the University of Aveiro on 11–14 June 2025 (<https://sites.google.com/view/wodca2025>), and brings together twenty-eight contributions:

1. Alexander Plakhov, *The Magic and Mystery of Newton's least resistance problem.*
2. Giuseppe Buttazzo, *The Problem of Minimal Resistance, Old and New.*
3. Yasir Arfat, *A Novel Accelerated Parallel Shrinking Projection Regularized Splitting Algorithm for Solving Generalized Split Monotone Variational Inclusion Problems.*
4. Genrich Belitskii and Victoria Rayskin, *Functional Equations in α -Hölder Class on Banach Spaces.*
5. Priyanka Bharati and Vivek Laha, *On robust nonsmooth semi-infinite multiobjective optimization problems involving intersection of closed sets.*
6. Andrzej Biś, *Variational Principles for Nonautonomous Dynamical Systems.*
7. Danila Cherkashin, *On branching points in the Gilbert–Steiner problem.*
8. Nguyen Phung Hai Chung, Pham Thi Hoai, and Hoang Van Chung, *New efficient accelerated and stochastic gradient descent algorithms based on locally Lipschitz gradient constants.*
9. Maria Stella Gelli, *Asymptotics of nonlocal energies with a bilateral constraint.*
10. Valentin V. Gorokhovik, *On Proper Preordering of Convex Sets.*
11. Pham Viet Hai and Trinh Ngoc Hai, *Regularized dynamical systems for finding zeros of generalized monotone operators.*
12. Aleksandar Jović, *Duality for Smooth Vector Continuous-Time Programming Problems.*
13. Rinat Kamalov and Vladimir Protasov, *Chebyshev approximation by non-Chebyshev systems.*

14. Olga I. Kostyukova, *On Parametric Copositive Programming Problems*.
15. Victor A. Kovtunenکو, *A snowpiercer model: dynamic beam solution for highly nonlinear reaction, energy inequality, and granular foundation*.
16. Sergey Kryzhevich and Eugene Stepanov, *Minimal points and non-holonomic controllability on compact manifolds*.
17. Dylan Langharst, *Some comments on the m th-order Projection Bodies*.
18. Ana P. Lemos-Paião, Sofia O. Lopes, and Maria do Rosário de Pinho, *Irrigation perturbed optimal linear-control problem*.
19. Gonçalo Penelas, Tiago Pinto, Luís Barbosa, Arsénio Reis, and João Barroso, *Autonomous navigation in realistic urban environments using machine learning and path planning*.
20. Alexey Piunovskiy and Yi Zhang, *Primal-Dual Programs for the Constrained Optimal Impulse Control: Discounted Model*.
21. Alexander Plakhov and Pedro Cruz, *Rotating rod and ball revisited: comparison of billiard and sliding motions*.
22. Vincenzo Recupero and Federico Stra, *Excess-continuous prox-regular sweeping processes*.
23. Bismark Singh, *On the Resolution of Ties in Fair Convex Allocation Problems*.
24. Paulo Tavares and João Luís Soares, *There is no mni clutter whose core is the projective plane of order 3*.
25. Yana Teplitskaya, *Open problems on Steiner trees and maximal distance minimizers*.
26. Delfim F. M. Torres, *Variational Formulations for Fractional Integral and Differential Equations*.
27. Van Nam Tran, *A relaxed proximal point algorithm with double-inertial effects for nonconvex equilibrium problems*.
28. Gerd Wachsmuth, *New numerical solutions to Newton's problem of least resistance via a convex hull approach*.

The contributions range from geometric optimization and convex analysis to nonsmooth dynamics, optimal control, numerical methods, and applications. Some engage directly with Plakhov's work; others develop related variational, geometric, dynamical, or algorithmic questions. The issue opens with Plakhov's paper, followed by Giuseppe Buttazzo's survey. The remaining papers appear alphabetically by the first author's surname. We introduce each contribution below, grouping related subjects together for discussion.

In "*The magic and mystery of Newton's least resistance problem*," Alexander Plakhov gives a personal and mathematical account of recent developments in the subject. The article ranges from convex and nonconvex bodies to translational and rotational motion and the free dynamics of bodies in Newtonian aerodynamics, emphasizing connections with billiards, optimal mass transport, the Kakeya problem, and the Magnus effect [5, 10, 22, 29].

"*The problem of minimal resistance, old and new*," by Giuseppe Buttazzo, surveys the classical problem from Newton's original formulation to modern variational treatments for general convex shapes, while also pointing to new directions. The paper provides a broad framework for much of Plakhov's work on least resistance, including his studies of convex and nonconvex admissible bodies, the single-impact assumption, and generalized formulations [1, 8, 18, 20].

The computational side of the same classical problem is developed by Gerd Wachsmuth in "*New numerical solutions to Newton's problem of least resistance via a convex hull approach*". A convex-hull based numerical method reveals striking structure in highly accurate computed minimizers, including apparent symmetries and organization of extremal points along curves. This contribution complements Plakhov's analytical investigations of the geometry and singular structure of minimizers, notably his result that a solution can be characterized by its singular set [9].

In "*Some comments on the m th-order projection bodies*", Dylan Langharst develops the higher-order theory of projection bodies, considering mixed projection bodies and stability questions related to Petty's projection inequality. Although arising from a different branch of convex geometry, this work shares with A.

Plakhov's research a concern with geometric functionals of convex bodies and with the fine geometry of their boundaries and surface measures [15].

The geometric-variational theme extends beyond aerodynamic resistance. *Danila Cherkashin*, in "On branching points in the Gilbert–Steiner problem", determines the pairs of dimension and exponent for which the Gilbert–Steiner branched optimal-transport problem can possess branching points of degree three. The contribution resonates with the way how A. Plakhov applies the optimal-transport ideas to mechanics and billiard scattering [11, 13, 29].

Yana Teplitskaya, in her short survey "Open problems on Steiner trees and maximal distance minimizers", collects several open questions on two families of one-dimensional geometric optimization problems.

The scope of geometric optimization is further developed by *Maria Stella Gelli* in "Asymptotics of nonlocal energies with a bilateral constraint", which examines a variational problem for nonlocal energies subject to bilateral constraints.

Billiards and impact systems provide another direct connection to research of A. Plakhov. In "Rotating rod and ball revisited: comparison of billiard and sliding motions", *Alexander Plakhov and Pedro Cruz* study a ball interacting with a uniformly rotating rod and show that billiard and sliding motions become asymptotically almost indistinguishable, with the billiard trajectory retaining only high-frequency, low-amplitude oscillations. The paper extends Plakhov's recent work on billiards in a rotating half-plane and on the rotating rod-and-ball system [3, 4].

Related themes of nonlinear mechanical evolution and variational analysis arise in *Victor A. Kovtunen*'s "A snowpiercer model: dynamic beam solution for highly nonlinear reaction, energy inequality, and granular foundation". The paper develops a variational treatment of a dynamic Euler–Bernoulli beam on a granular foundation, combining nonlinear evolution, energy inequalities, and Galerkin approximation. These themes are closely related to the impact and resistance models that motivate much of A. Plakhov's applied mathematics.

Vincenzo Recupero and Federico Stra, in "Excess-continuous prox-regular sweeping processes", study Moreau sweeping processes driven by time-dependent prox-regular sets and establish existence and uniqueness under substantially weaker continuity assumptions. Sweeping processes belong to the broader theory of nonsmooth constrained dynamics, naturally neighbouring the collision and impact systems present in Plakhov's billiard work [14].

In "Variational principles for nonautonomous dynamical systems", *Andrzej Biś* uses convex-analysis methods to derive variational principles for pressure functions in discrete nonautonomous dynamical systems. This paper illustrates the interplay between convexity and dynamics, also prominent in the work of Alexander Plakhov on billiards, scattering, and mechanical systems.

Sergey Kryzhevich and Eugene Stepanov, in "Minimal points and non-holonomic controllability on compact manifolds", study controllability of systems with drift under recurrence and Hörmander-type bracket-generating conditions. Their results are particularly relevant to A. Plakhov's collaboration with Kryzhevich on rotating billiard systems [3, 4].

Genrich Belitskii and Victoria Rayskin, in "Functional Equations in α -Hölder Class on Banach Spaces", establish existence and uniqueness of global Hölder solutions and apply the results to Hölder linearization, contributing to the qualitative theory of dynamical systems.

Several papers address the geometry and structure of convex optimization problems. *Olga Kostyukova*, in "On Parametric Copositive Programming Problems", studies sensitivity of one-parameter copositive programs. By exploiting the complementary structure of the copositive cone and its relation to a block semi-infinite program, the author derives a finite system characterizing nearby primal and dual optima. The focus on convex cones, perturbations, and qualitative structure fits naturally within the convex-analytic framework underlying many optimal-shape and control problems.

Valentin V. Gorokhovik, in "On Proper Preordering of Convex Sets" introduces a geometrically defined dominance pre-order on convex sets in infinite-dimensional spaces and studies its implications for their faces. The work highlights the facial and order structure of convex sets, complementing the more concrete convex-body geometry represented elsewhere in the issue.

In "On the Resolution of Ties in Fair Convex Allocation Problems", *Bismark Singh* studies observationally indistinguishable outcomes in fairness-oriented convex allocation. Using the KKT conditions, the author classifies such outcomes and proposes smooth perturbations to select among them while preserving the fairness objective, combining convex optimization, perturbation, and constrained modelling – themes that echo longstanding interest of A. Plakhov in optimization under physically or structurally meaningful constraints.

Paulo Tavares and João Luís Soares in "There is no mni clutter whose core is the projective plane of order 3", settle

a conjecture on minimally nonideal (mni) clutters by proving that a clutter with the ternary projective plane as core would have to coincide with its blocker, yielding nonexistence. This broadens the issue toward polyhedral and combinatorial structures underlying discrete optimization.

Optimal control provides one of the clearest links between the issue and another major strand of Plakhov's work. *Alexey Piunovskiy and Yi Zhang*, in *"Primal-Dual Programs for the Constrained Optimal Impulse Control: Discounted Model"*, formulate constrained deterministic impulse-control problems as primal convex programs in measures and associated dual programs, establish duality, and derive a procedure for constructing optimal mixed strategies. This paper directly extends a field in which both authors have collaborated with A. Plakhov [2, 6, 7].

In *"Irrigation perturbed optimal linear-control problem"*, *Ana P. Lemos-Paião, Sofia O. Lopes, and Maria do Rosário de Pinho* formulate irrigation management as a parametric optimal-control problem with state constraints. After reducing it to finite-dimensional nonlinear programming, they derive optimality conditions and perform sensitivity analysis with respect to water loss by deep percolation. The paper exemplifies the interplay between modeling, variational analysis, and optimization also present in Plakhov's work on mechanically motivated control problems [32].

Delfim F. M. Torres, in *"Variational Formulations for Fractional Integral and Differential Equations"*, develops variational formulations for fractional integral and differential equations by constructing suitable symmetric bilinear forms. The resulting perspective connects fractional dynamics with the calculus of variations and least-squares methods, recalling D. Torres's joint work with A. Plakhov of Newton-type optimal-control and aerodynamic problems [28, 32].

In *"On robust nonsmooth semi-infinite multiobjective optimization problems involving intersection of closed sets"*, *Priyanka Bharati and Vivek Laha* derive robust KKT and sufficient optimality conditions, together with mixed-type duality results, using limiting subdifferentials and generalized convexity. The paper extends the optimization theme toward nonsmooth and uncertainty-aware formulations.

In *"Duality for Smooth Vector Continuous-Time Programming Problems"*, *Aleksandar Jović* develops Mond-Weir and Wolfe dual models for vector continuous-time programming with almost-everywhere phase constraints. Under suitable regularity, concavity, and constraint-qualification assumptions, weak and strong duality are established in an infinite-dimensional setting.

The algorithmic contributions are equally diverse. *Van Nam Tran* in *"A relaxed proximal point algorithm with double-inertial effects for nonconvex equilibrium problems"* proposes a relaxed proximal scheme with two-step inertial extrapolation, proves global convergence for nonconvex equilibrium problems, and applies it to mixed variational inequalities. The work contributes to the iterative optimization perspective, also present in A. Plakhov's early research on adaptive stochastic approximation [25, 26].

In *"Regularized dynamical systems for finding zeros of generalized monotone operators"*, *Pham Viet Hai and Trinh Ngoc Hai* introduce continuous- and discrete-time regularized dynamics for generalized monotone zero-finding problems and establish strong convergence, illustrating connection between optimization algorithms and dynamical systems.

Nguyen Phung Hai Chung, Pham Thi Hoai, and Hoang Van Chung, in *"New efficient accelerated and stochastic gradient descent algorithms based on locally Lipschitz gradient constants"*, combine a recently proposed adaptive gradient step-size with Heavy Ball and Nesterov acceleration, and also develop stochastic variants. Convergence results and machine-learning experiments show the effectiveness of using locally estimated rather than globally prescribed Lipschitz information. The adaptive character of these methods provides a modern counterpart to Plakhov's earlier research on step-size adaptation in stochastic optimization [25, 26].

In *"A Novel Accelerated Parallel Shrinking Projection Regularized Splitting Algorithm for Solving Generalized Split Monotone Variational Inclusion Problems"*, *Yasir Arfat* proposes an accelerated parallel hybrid projection algorithm for split monotone variational inclusions with multiple output sets, establishes strong convergence, and gives a numerical illustration. The paper reinforces the importance of themes like monotonicity, regularization, and constructive approximation.

Rinat Kamalov and Vladimir Protasov, in *"Chebyshev approximation by non-Chebyshev systems"*, extend uniform approximation theory beyond the classical Haar setting. They formulate a modified alternance principle, characterize the full set of best approximants, and regularize a generalized Remez method with linear convergence, with applications ranging from exponential and Gaussian systems to signal processing and switching dynamics.

A particularly contemporary application is presented by *Gonçalo Penelas, Tiago Pinto, Luís Barbosa, Arsénio Reis, and João Barroso*, in *"Autonomous navigation in realistic urban environments using machine learning and path planning"*. The authors compare deep-reinforcement-learning approaches, notably Proximal Policy

Optimization and Soft Actor-Critic, for autonomous navigation in a realistic simulated urban environment based on Vila Real, Portugal. This contribution brings the issue's themes of optimization, geometry, and dynamics into the contemporary setting of learning-based autonomous navigation.

Together, these twenty-eight papers reflect the range of questions and methods associated with Plakhov's scientific interests, from Newtonian resistance and billiard scattering to convex geometry, control, and contemporary optimization algorithms. We thank the authors for their contributions, the referees for their careful work, and everyone who took part in the celebration at WODCA 2025. We are grateful to the Editors of the *Journal of Convex Analysis*—in particular Norbert Heldermaun, Giuseppe Buttazzo, and Lionel Thibault—for their support throughout the preparation of this Collection. We hope that the volume will inspire further work on the problems and connections that Plakhov's research has brought to light.

This foreword was prepared by the Guest Editors: Tatiana Tchemisova, Delffm F. M. Torres, and João Pedro Cruz.

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