

A Novel Accelerated Parallel Shrinking Projection Regularized Splitting Algorithm for Solving Generalized Split Monotone Variational Inclusion Problems

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Abstract: This paper introduces a novel parallel hybrid algorithm, based on an accelerated shrinking projection method, for solving the Split Monotone Variational Inclusion problems with multiple output sets in Hilbert spaces. Under an appropriate set of conditions, we establish the strong convergence of the proposed algorithm. Its practical viability is further demonstrated through a numerical experiment. The results obtained herein generalize and improve upon several existing findings in the literature.

Keywords: Parallel Shrinking projection Algorithm, Acceleration, Tikhonov Regularization, Strong Convergence, Split Monotone Variational Inclusion problem.

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1. Introduction

Fixed point theory, particularly metric fixed point theory, is a powerful tool for solving practical problems in numerous scientific fields. Originating from the Banach Contraction Principle (BCP) which ensures a unique fixed point for contractions in complete metric spaces this theory provides critical insights for nonlinear analysis. When the direct existence of a fixed point is difficult to establish, its approximation becomes essential (see [1]). The BCP is pivotal here, as it offers both an existence proof and an iterative approximation algorithm. This approach has successfully addressed problems in integral equations, differential equations, and image processing by framing them as fixed point problems.

The Split Convex Feasibility Problem (SCFP) is to find a point x^+ that satisfies:

$$x^+ \in C \quad \text{and} \quad (x^+) \in Q, \quad (1)$$

where $C \subset \mathbb{H}_1$ and $Q \subset \mathbb{H}_2$ are nonempty, closed, convex subsets of real Hilbert spaces, and $T : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ is a bounded linear operator with the adjoint operator $T^* : \mathbb{H}_2 \rightarrow \mathbb{H}_1$. This problem was originally formulated by Censor and Elfving [15]. The SCFP has diverse applications across numerous fields, including inverse problems, phase retrieval, medical image reconstruction, signal processing, and radiation therapy treatment planning [13–15, 17]. Consequently, several iterative methods for solving (1) have been developed and analyzed [3, 4, 6–11, 14, 39, 40]. A prominent example is Byrne's CQ algorithm [14], which, from an arbitrary initial point x_1 generates a sequence of approximations defined by the following iteration:

$$x_{n+1} = P_C^{\mathbb{H}_1} (Id - \delta^* (Id - P_Q^{\mathbb{H}_2}) x_n), \quad n \geq 1, \quad (2)$$

where, δ is a step-size parameter satisfying $\delta \in \frac{2}{\|T\|^2}$, Id is the identity operator and $P^{\mathbb{H}_1}$ and $Q^{\mathbb{H}_2}$ are the metric projections onto C and Q , respectively. Byrne [14] established that the sequence generated by Algorithm (2) converges weakly to a solution of the problem (1). Another important generalization of the SCFP is Split Variational Inequality Problem (SVIP), which is introduced and studied by Censor et. al. [19]. In the

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wake of the work of Censor et al. [19], Moudafi [29] introduced the Split Monotone Variational Inclusion Problem (SMVIP), which he formulated as follows:

$$\text{Find } x^\dagger \in \mathbb{H}_1 \text{ that solves } 0 \in A_1(x^\dagger) + \mathcal{B}_1(x^\dagger) \quad (3)$$

such that

$$y^\dagger = (x^\dagger) \in \mathbb{H}_2 \text{ solves } 0 \in A_2(y^\dagger) + \mathcal{B}_2(y^\dagger), \quad (4)$$

where $\mathcal{B}_1 : \mathbb{H}_1 \rightarrow \mathbb{H}_1$ and $\mathcal{B}_2 : \mathbb{H}_2 \rightarrow \mathbb{H}_2$ are single-valued operators, $A_1 : \mathbb{H}_1 \rightarrow 2^{\mathbb{H}_1}$ and $A_2 : \mathbb{H}_2 \rightarrow 2^{\mathbb{H}_2}$ are set-valued operators. In this paper, we study the concept of split monotone variational inclusion problem with multiple output sets. Let $\mathbb{H}, \mathbb{H}_j, j = 1, 2, \dots, M$, be real Hilbert spaces and let $j : \mathbb{H} \rightarrow \mathbb{H}_j, j = 1, 2, \dots, M$, be bounded linear operators. Let $A_1 : \mathbb{H} \rightarrow 2^{\mathbb{H}}, A_{2,j} : \mathbb{H}_j \rightarrow 2^{\mathbb{H}_j}, j = 1, 2, \dots, M$, be set valued or multivalued operators, and $\mathcal{B}_1 : \mathbb{H} \rightarrow \mathbb{H}, \mathcal{B}_{2,j} : \mathbb{H}_j \rightarrow \mathbb{H}_j, j = 1, 2, \dots, M$, be single-valued operators. The split monotone variational inclusion problem with multiple output sets (SMVIPMOS) is formulated as finding a point $x^\dagger \in \mathbb{H}$ such that

$$x^\dagger \in \Gamma_1 := (A_1 + \mathcal{B}_1)^{-1}(0) \cap \left(\bigcap_{j=1}^M j^{-1}(A_{2,j} + \mathcal{B}_{2,j})^{-1}(0) \right) \neq \emptyset. \quad (5)$$

Γ_1 is the solution set of SMVIPMOS. The significance of the problems involving multiple output sets, in particular, the importance of SMVIPMOS lies in the modeling of complex systems arising in optimization, signal processing, and related applied fields (see for instance [35–37]). It is clear that the SMVIPMOS (5) is quite general; all the above optimization problems discussed so far in this paper could be viewed as special cases of the SMVIPMOS (5). In particular, if we set $\mathcal{B}_1 = 0, \mathcal{B}_{2,j} = 0, j = 1, 2, \dots, M$, we obtain the Split Null Point Problems (SNPPMOS) (see [32]).

It is remarked that [14] is analyzed with some well-known iterative algorithms to establish weak as well as strong convergence results associated with the problem (1). These algorithms have been improved by many authors to compute the common solution of (1), see [16, 18, 19, 22, 41–43] and the references cited therein.

It is worth mentioning that the concept of SNPP has been extended to the concept of generalized split null point problem (SNPPMOS) in Hilbert spaces [32, 33]. A variant of the classical CQ-algorithm, essentially due to Byrne [14], is employed in [33] whereas a shrinking projection approximants is analyzed in to obtain the strong convergence results in Hilbert spaces. In 2017, the split null point problem (SNPP) is studied in [21] by employing the hybrid shrinking projection method. Currently, hybrid iterative algorithms, such as the one defined in [34], are often preferred over their non-hybrid counterparts. A key advantage of these hybrid methods is their ability to guarantee strong convergence under mild assumptions. Beyond convergence guarantees, computational performance is a critical consideration. Recent literature reflects various modifications aimed at this improvement. For instance, Hieu et al. [25] introduced a parallel hybrid algorithm in 2017 to reduce computational cost (see also [24]). Separately, the inertial extrapolation technique, pioneered by Polyak [31], is renowned for accelerating convergence. This inertial technique has been successfully integrated with numerous classical algorithms, as demonstrated in [2, 28] and related works. Recently [30] studied the regularization technique in strong convergence analysis for split variational inclusion problems in Hilbert spaces.

Inspired by the recent works of [5, 21, 25, 30, 32, 33], this paper introduces an accelerated parallel hybrid algorithm, based on the shrinking projection method, for solving the Split Monotone Variational Inclusion Problem with multiple output sets (SMVIPMOS) in Hilbert spaces. The paper is structured as follows: Section 2 compiles the necessary preliminaries and foundational results. Section 3 is dedicated to the convergence analysis of the proposed algorithm, Finally, Section 4 demonstrates its numerical viability through a pertinent example.

2. Preliminaries

Throughout this section, we assume familiarity with concepts from monotone operator theory, particularly as presented in the monograph of Bauschke and Combettes [12]. Let $P_C^{\mathbb{H}_1}$ denote the metric projection onto a nonempty, closed, and convex subset. This operator is well-known to be firmly nonexpansive [23] and

$$\langle x - P_C^{\mathbb{H}_1} x, P_C^{\mathbb{H}_1} x - x^\dagger \rangle \geq 0, \text{ for all } x \in \mathbb{H}_1, x \in C.$$

A set-valued operator $A_1 \subseteq \mathbb{H}_1 \times \mathbb{H}_1$, with the usual definitions of $\text{dom}(A_1)$ and $\text{zer}(A_1)$, is said to be monotone if $\langle x - x^\dagger, v - x \rangle \geq 0$, for all $(x, v), (x^\dagger, x) \in \text{gra}(A_1)$, where $\text{gra}(A_1) := \{(x^\dagger, v) \in \mathbb{H}_1 \times \mathbb{H}_1 : x^\dagger \in \text{dom}(A_1), v \in A_1(x^\dagger)\}$. A monotone operator A_1 is called as maximal monotone operator if there is no proper monotone extension of A_1 , equivalently if $\text{ran}(Id + rA_1) = \mathbb{H}_1$ for all $r > 0$, where $\text{ran}(A_1) = \{A_1(x) : x \in \text{dom}(A_1)\}$. An important notion associated with the monotone operator A_1 is the resolvent operator $J_r^{A_1} = (Id + rA_1)^{-1}$, where $(\cdot)^{-1}$ denotes the inverse operator and Id stands for the identity operator. The resolvent operator $J_r^{A_1}$ is a well-defined, single-valued, nonexpansive operator satisfying $\text{Fix}(J_r^{A_1}) = A_1^{-1}(0)$ for all $r > 0$.

Recall that the minimization problem associated with a proper, lower semicontinuous and convex function $f : \mathbb{H}_1 \rightarrow (-\infty, +\infty]$ is connected with the zero point problem via the subgradient operator of f defined by

$$\partial f : \mathbb{H}_1 \rightarrow 2^{\mathbb{H}_1} : x \mapsto \{e \in \mathbb{H}_1 | f(x^\dagger) \geq f(x) + \langle e, x^\dagger - x \rangle, \forall x^\dagger \in \mathbb{H}_1\}.$$

Observe the following variant of the convex minimization problem:

$$\min_{x \in \mathbb{H}_1} \{f(x) + g(x)\}, \quad (6)$$

associated with a proper, lower semicontinuous and convex function f and a convex, differentiable, Lipschitz continuous gradient function g . Assume that $\mathbb{k}_0(\mathbb{H}_1)$ is a class of proper, lower semicontinuous and convex functions then the operator

$$\text{prox}_{rf} : \mathbb{H}_1 \rightarrow \mathbb{H}_1 : x \mapsto \text{argmin}_{x^\dagger \in \mathbb{H}_1} \left\{ \frac{1}{2} \|x^\dagger - x\|^2 + rf(x^\dagger) \right\},$$

is called the proximity operator of f with the index $r > 0$ [27]. It is remarked that the proximity operator is a natural extension of orthogonal projection operator $P_C^{\mathbb{H}_1}$.

In the sequel, an operator $\mathbb{T} : C \rightarrow C$ is called:

- (i) nonexpansive if $\|\mathbb{T}x - \mathbb{T}x^\dagger\| \leq \|x - x^\dagger\|$, $\forall x, x^\dagger \in C$ and
- (ii) quasi-nonexpansive if $\|\mathbb{T}x - x^\dagger\| \leq \|x - x^\dagger\|$, $\forall x \in C, x^\dagger \in \text{Fix}(\mathbb{T})$, where $\text{Fix}(\mathbb{T}) = \{x \in \mathbb{H} | x = \mathbb{T}x\}$ denotes the fixed point set of the operator $\mathbb{T}$
- (iii) Ξ -demicontractive if $\text{Fix}(\mathbb{T}) \neq \emptyset$ and there exists $\Xi \in [0, 1)$ such that $\|\mathbb{T}x - x^\dagger\|^2 \leq \|x - x^\dagger\|^2 + \Xi \|x - \mathbb{T}x^\dagger\|^2$, $\forall x \in C, x^\dagger \in \text{Fix}(\mathbb{T})$.

Next, we present some key results needed later.

Lemma 2.1 ([12]). Let $\mathbb{T} : C \rightarrow C$ be a nonexpansive operator on a nonempty, closed, convex subset C of a real Hilbert space $\mathbb{H}_1$. Then $Id - \mathbb{T}$ is demiclosed at 0. Specifically, if a sequence $(x_n \subset C)$ converges weakly to x and the sequence $(Id - \mathbb{T})x_n$ converges strongly to 0, then x is a fixed point of $\mathbb{T}$, i.e., $(Id - \mathbb{T})(x) = 0$.

Lemma 2.2 ([12]). Let $v, x^\dagger \in \mathbb{H}_1$ and $\hat{a} \in \mathbb{R}$ then

1. $\|v + x^\dagger\|^2 \leq \|v\|^2 + 2\langle x^\dagger, v + x^\dagger \rangle$;
2. $\|v - x^\dagger\|^2 \leq \|v\|^2 - \|x^\dagger\|^2 - 2\langle v - x^\dagger, x^\dagger \rangle$;
3. $\|\hat{a}v + (1 - \hat{a})x^\dagger\|^2 = \hat{a}\|v\|^2 + (1 - \hat{a})\|x^\dagger\|^2 - \hat{a}(1 - \hat{a})\|v - x^\dagger\|^2$.

Lemma 2.3 ([26]). Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}_1$. For every $v, x^\dagger, h \in \mathbb{H}_1$ and $\alpha \in \mathbb{R}$, then the following set is closed and convex:

$$\mathbb{U} = \{x \in C : \|x^\dagger - x\|^2 \leq \|v - x\|^2 + \langle h, x \rangle + \alpha\}.$$

Lemma 2.4 ([12]). Let $h_1 > 0, h_2 > 0$, two real Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, the single-valued maximally monotone operators $\mathcal{B}_1 : \mathbb{H}_1 \rightarrow \mathbb{H}_1$ and $\mathcal{B}_2 : \mathbb{H}_2 \rightarrow \mathbb{H}_2$, and the two set-valued monotone operators $A_1 : \mathbb{H}_1 \rightarrow 2^{\mathbb{H}_1}$ and $A_2 : \mathbb{H}_2 \rightarrow 2^{\mathbb{H}_2}$. Then, x^* solves 3 and 4 if and only if $x^\dagger \in \text{Fix}(J_{h_1}^{A_1}(Id - h_1\mathcal{B}_1))$ and $x^\dagger \in \text{Fix}(J_{h_2}^{A_2}(Id - h_2\mathcal{B}_2))$.

Lemma 2.5 ([29]). Let $r > 0$, $\mathcal{B}$ be a $\hat{u}$ -inverse strongly monotone and A be a maximal monotone operator, respectively. Let C be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}_1$. If $r \in (0, 2\hat{u})$, then the operator $J_r^A(Id - r\mathcal{B})$ is averaged.

3. Convergence Analysis

In this section, we are interested in computing the solution of the problem (3) and (4). Let $H_j, j = 1, 2, \dots, M$ be real Hilbert spaces. Let $A_1 : \mathbb{H}_j \rightarrow 2^{\mathbb{H}_j}, A_{2,j} : \mathbb{H}_{2,j} \rightarrow 2^{\mathbb{H}_{2,j}}$ for $j = \{1, 2, \dots, M\}$ two finite families of maximal monotone operators and $\mathcal{B}_1 : \mathbb{H}_j \rightarrow \mathbb{H}_j, \mathcal{B}_{2,j}$ be finite families of single valued mappings, respectively. Let $j : \mathbb{H} \rightarrow \mathbb{H}_j$ be a finite family of bounded linear operators with its adjoint j^* . Let $\mathbb{T}_i : \mathbb{H} \rightarrow \mathbb{H}, i = 1, 2, \dots, K$ be a finite family of Ξ -demicontractive operators. Assume that $\Gamma := \Gamma_1 \cap (\bigcap_{i=1}^K \text{Fix}(\mathbb{T}_i)) \neq \emptyset$. Now we employing the following accelerated projection based parallel hybrid algorithm: The main result

Algorithm 1 A Novel Accelerated Parallel Shrinking Projection Regularized Splitting Algorithm

Initialization: Let $C_0 = \mathbb{H}_1$ and choose $x_0, x_1 \in \mathbb{H}_1$. Set $n = 1$ and nonincreasing sequence $(\theta_n), (\zeta_n), (\gamma_n) \subset (0, 1), (\Psi_n) \subset [0, 1], r_{n,j} \in (0, \infty), j = 1, 2, \dots, M$ and $\delta_n \in (0, \frac{2}{\|\bar{h}\|^2})$, where $\bar{h} = \max\{\bar{h}_1, \bar{h}_2, \dots, \bar{h}_M\}$ such that $\|j\|^2 = \bar{h}_j$, is the spectral radius of j_j^* .

Recursive Steps: Choose the accelerated parameter

$$\widehat{\Psi}_n = \begin{cases} \min\{\frac{\ell_n}{\|x_n - x_{n-1}\|}, \Psi\} & \text{if } x_n \neq x_{n-1}; \\ \Psi & \text{otherwise,} \end{cases}$$

where $\{\ell_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} \ell_n < \infty$ and $\Psi \in [0, 1)$. Further given $x_n \in \mathbb{H}_1$, calculate $p_n, \bar{q}_n$ and $\bar{z}_n$ as follows:

Step 1. Compute

$$\begin{cases} p_n = x_n + \Psi_n(x_n - x_{n-1}); \\ q_n^i = (1 - \theta_n)p_n + \theta_n \mathbb{T}_i p_n; \\ i_n = \arg \max\{\|q_n^i - x_n\| : i = 1, 2, 3, \dots, K\}, \bar{q}_n = q_n^{i_n}; \\ z_n^j = (1 - \zeta_n)\bar{q}_n + \zeta_n(J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)(\bar{q}_n - \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j})))_j \bar{q}_n) - \delta_n \gamma_n \bar{q}_n; \\ j_n = \arg \max\{\|z_n^j - x_n\| : j = 1, 2, \dots, M\}, \bar{z}_n = z_n^{j_n}; \end{cases}$$

If $\bar{z}_n = \bar{q}_n = p_n = x_n$ then stop and x_n is the solution of problem Γ . Otherwise proceed step 2,

Step 2. Compute

$$\begin{aligned} C_{n+1} &= \{x^\dagger \in C_n : \|\bar{z}_n - x^\dagger\|^2 \leq \|x_n - x^\dagger\|^2 + \Psi_n^2 \|x_n - x_{n-1}\|^2 \\ &\quad + 2\Psi_n \langle x_n - x^\dagger, x_n - x_{n-1} \rangle + 2\gamma_n \delta_n G\}, \\ x_{n+1} &= P_{C_{n+1}}^{\mathbb{H}_1} x_1, \forall n \geq 1, \end{aligned}$$

Put $n = n + 1$ and perform **Step 1**.

of this section is as follows:

Theorem 3.1. Assume that $\Gamma \neq \emptyset$ then the Algorithm 1 converges strongly to an element $x^\dagger \in \Gamma$ if the following conditions are satisfied:

- (C1) $\sum_{n=1}^{\infty} \Psi_n \|x_n - x_{n-1}\| < \infty$;
- (C2) $0 \leq \tilde{a} \leq \theta_n \leq \tilde{b} < 1$ and $\liminf_{n \rightarrow \infty} \theta_n > 0$;
- (C3) $\lim_{n \rightarrow \infty} \zeta_n = 0$ and $\sum_{n=0}^{\infty} \zeta_n = \infty$;
- (C4) $\liminf_{n \rightarrow \infty} r_{n,j} > 0, \forall j \in 1, 2, \dots, M$;
- (C5) $\gamma_n \in [0, 1)$ with $\sum_{n=1}^{\infty} \gamma_n < \infty$;
- (C6) $0 < \bar{c} \leq \delta_n \leq \bar{d} < \frac{2}{\|\bar{h}\|^2}$;
- (C7) $\lim_{n \rightarrow \infty} \frac{\delta_n \gamma_n}{\zeta_n} = 0$.

Proof. We divided the proof of our main result into the following steps:

Step 1. The Algorithm 1 is well-defined.

It is well-known that Γ is closed and convex. Moreover, the convexity and closedness of the set C_{n+1} for each $n \geq 1$ follows from Lemma 2.3. From these observations, we conclude that the projection $P_{C_{n+1}}^{\mathbb{H}_1} x_1$ is well defined. From Algorithm 1, the mapping $Y : \mathbb{H} \rightarrow \mathbb{H}$, defined by $Y := Id - \delta_n^* J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j - \delta_n \gamma_n Id$, is a strict contraction with Lipschitz constant $(1 - \delta_n \gamma_n)$. Let $x^\dagger, x^* \in \mathbb{H}$ then we have

$$\begin{aligned}
& \langle x^\dagger - x^*, {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* \rangle \\
&= \langle j x^\dagger - j x^*, (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* \rangle \\
&= \|j x^\dagger - j x^*\|^2 - \langle j x^\dagger - j x^*, J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j x^\dagger - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j x^* \rangle \\
&\geq \|j x^\dagger - j x^*\|^2 - \|j x^\dagger - j x^*\| \|J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j x^\dagger - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j x^*\| \\
&\geq \|j x^\dagger - j x^*\|^2 - \|j x^\dagger - j x^*\|^2 \\
&= 0.
\end{aligned} \tag{7}$$

Utilizing (7), we get

$$\begin{aligned}
& \|Y(x^\dagger) - Y(x^*)\|^2 \\
&= \|(x^\dagger - \delta_n^* (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - \gamma_n x^\dagger) - \\
&\quad (x^* - \delta_n^* (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* - \gamma_n x^*)\|^2 \\
&= \|x^\dagger - x^*\|^2 + \delta_n^2 \|{}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger + \gamma_n x^\dagger - \\
&\quad ({}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* + \gamma_n x^*)\|^2 - 2\delta_n \langle x^\dagger - x^*, {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger + \\
&\quad \gamma_n x^\dagger - ({}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* + \gamma_n x^*) \rangle \\
&= \|x^\dagger - x^*\|^2 + \delta_n^2 \|{}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger + \gamma_n x^\dagger - \\
&\quad ({}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* + \gamma_n x^*)\|^2 - 2\delta_n \gamma_n \|x^\dagger - x^*\|^2 - \\
&\quad 2\delta_n \langle x^\dagger - x^*, {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* \rangle \\
&= \|x^\dagger - x^*\|^2 + \delta_n^2 (\|{}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^*\|^2 \\
&\quad + 2\gamma_n \langle x^\dagger - x^*, {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger \\
&\quad - {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* \rangle + \gamma_n^2 \|x^\dagger - x^*\|^2) \\
&\quad - 2\delta_n \gamma_n \|x^\dagger - x^*\|^2 \\
&\quad - 2\delta_n \langle j x^\dagger - j x^*, {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* \rangle \\
&\leq (1 - 2\delta_n \gamma_n + \delta_n^2 \gamma_n^2) \|x^\dagger - x^*\|^2 - (2\delta_n - 2\delta_n^2 \gamma_n - 2\delta_n^2 \|j\|^2) \\
&\quad \langle x^\dagger - x^*, {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^\dagger - {}^*_j (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j x^* \rangle \\
&\leq (1 - 2\delta_n \gamma_n + \delta_n^2 \gamma_n^2) \|x^\dagger - x^*\|^2 \\
&= (1 - \delta_n \gamma_n)^2 \|x^\dagger - x^*\|^2.
\end{aligned} \tag{8}$$

For any $x^\dagger \in \Gamma$, and Set: $w_n = \bar{q}_n - \delta_n^* (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \gamma_n \bar{q}_n$, where $n \geq 1$ then we have $Y(w_n) = \bar{q}_n - \delta_n^* (Id - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \gamma_n \bar{q}_n$. Implies that, by using (8), we also have $x^\dagger = J_{r_n}^{A_1} (Id - r_n \mathcal{B}_1) x^\dagger$ and $j x^\dagger = J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j}) x^\dagger (x^\dagger)$, the triangle inequality, and Y is nonexpansive, we obtain

$$\begin{aligned}
\|J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j (x_n) - j(x^\dagger)\| &= \|J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j (x_n) - J_{r_{n,j}}^{A_{2,j}} (Id - r_{n,j} \mathcal{B}_{2,j})_j (x^\dagger)\| \\
&\leq \|j(x_n) - j(x^\dagger)\|.
\end{aligned} \tag{9}$$

Therefore, it follows from Algorithm 1 that

$$\begin{aligned}
\|p_n - x^\dagger\|^2 &= \|(x_n - x^\dagger) + \Psi_n(x_n - x_{n-1})\|^2 \\
&\leq \|x_n - x^\dagger\|^2 + \Psi_n^2 \|x_n - x_{n-1}\|^2 + 2\Psi_n \langle x_n - x^\dagger, x_n - x_{n-1} \rangle.
\end{aligned} \tag{10}$$

Further, we have

$$\begin{aligned}
\|q_n^i - x^\dagger\|^2 &= \|\bar{q}_n - x^\dagger\|^2 \\
&= \|(1 - \theta_n)p_n + \theta_n \mathbb{T}_i p_n - x^\dagger\|^2 \\
&\leq (1 - \theta_n)\|p_n - x^\dagger\|^2 + \theta_n \|\mathbb{T}_i p_n - x^\dagger\|^2 - \theta_n(1 - \theta_n)\|(Id - \mathbb{T})p_n\|^2 \\
&\leq (1 - \theta_n)\|p_n - x^\dagger\|^2 + \theta_n \|p_n - x^\dagger\|^2 + \theta_n \Xi \|\mathbb{T}_i p_n - x^\dagger\|^2 - \theta_n(1 - \theta_n)\|(Id - \mathbb{T})p_n\|^2 \\
&\leq \|p_n - x^\dagger\|^2 - \theta_n(1 - \Xi - \theta_n)\|(Id - \mathbb{T})p_n\|^2 \\
&\leq \|p_n - x^\dagger\|^2 \\
&\leq \|x_n - x^\dagger\|^2 + \Psi_n^2 \|x_n - x_{n-1}\|^2 + 2\Psi_n \langle x_n - x^\dagger, x_n - x_{n-1} \rangle.
\end{aligned} \tag{11}$$

Furthermore, we obtain

$$\begin{aligned}
\|z_n^j - x^\dagger\|^2 &= \|\bar{z}_n - x^\dagger\|^2 \\
&= \|(1 - \zeta_n)(\bar{q}_n - x^\dagger) + \zeta_n(J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)(\bar{q}_n - \\
&\quad \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \gamma_n \bar{q}_n) - x^\dagger)\|^2 \\
&\leq \|(1 - \zeta_n)\|\bar{q}_n - x^\dagger\|^2 + \zeta_n \|J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)(\bar{q}_n - \\
&\quad \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \gamma_n \bar{q}_n) - x^\dagger\|^2
\end{aligned} \tag{12}$$

Since $J_{r_n,j}^{A_{1,i}}(Id - r_n \mathcal{B}_1)$ is nonexpansive and by using Lemma 2.4, therefore, we have

$$\begin{aligned}
&\|J_{r_n,j}^{A_{1,i}}(Id - r_{n,j} \mathcal{B}_{n,j})(\bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \gamma_n \bar{q}_n) - J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)x^\dagger\|^2 \\
&= \|(\bar{q}_n - x^\dagger) - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \gamma_n \bar{q}_n\|^2 \\
&= \|(\bar{q}_n - x^\dagger) - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n\|^2 + \delta_n^2 \gamma_n^2 \|\bar{q}_n\|^2 \\
&\quad - 2\gamma_n \langle (\bar{q}_n - x^\dagger) - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n, \bar{q}_n \rangle \\
&= \|\bar{q}_n - x^\dagger\|^2 + \delta_n^2 \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n\|^2 \\
&\quad - 2\delta_n \langle j \bar{q}_n - j x^\dagger, (Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n \rangle \\
&\quad + \delta_n^2 \gamma_n^2 \|\bar{q}_n\|^2 - 2\gamma_n \langle (\bar{q}_n - x^\dagger) - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n, \bar{q}_n \rangle \\
&= \|\bar{q}_n - x^\dagger\|^2 + \delta_n^2 \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n\|^2 \\
&\quad - 2\delta_n \langle j \bar{q}_n - j x^\dagger, (Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n \rangle \\
&\quad + \delta_n^2 \gamma_n^2 \|\bar{q}_n\|^2 - 2\gamma_n \delta_n \|\bar{q}_n\|^2 + 2\gamma_n \langle x^\dagger - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n, \bar{q}_n \rangle \\
&= \|\bar{q}_n - x^\dagger\|^2 + \delta_n^2 \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n\|^2 \\
&\quad - 2\delta_n \langle j \bar{q}_n - j x^\dagger, (Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n \rangle \\
&\quad - \gamma_n \delta_n \|\bar{q}_n\|^2 (2 - \gamma_n \delta_n) + 2\gamma_n \delta_n \langle x^\dagger - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n, \bar{q}_n \rangle \\
&= \|\bar{q}_n - x^\dagger\|^2 + \delta_n^2 \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n\|^2 \\
&\quad - 2\delta_n \langle j \bar{q}_n - j x^\dagger, (Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n \rangle + 2\gamma_n \delta_n G
\end{aligned} \tag{13}$$

where $G := \sup_n \{ \langle x^\dagger - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j (\bar{q}_n), \bar{q}_n \rangle \}$ and since $2 - \delta_n \gamma_n > 0$ because $\delta_n \gamma_n < 1$.

From equation (13), observe that

$$\begin{aligned}
&\langle j \bar{q}_n - j x^\dagger, J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j})_j (\bar{q}_n) - j (\bar{q}_n) \rangle \\
&= \langle J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j})_j (\bar{q}_n) - j (x^\dagger), J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j})_j (\bar{q}_n) - j (\bar{q}_n) \rangle \\
&\quad - \|J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j})_j (\bar{q}_n) - j (\bar{q}_n)\|^2.
\end{aligned} \tag{14}$$

By using Lemma 2.4(i), we get

$$\begin{aligned}
& \langle J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j(\bar{q}_n) - j(x^\dagger), J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j(\bar{q}_n) - j(\bar{q}_n) \rangle \\
&= \frac{1}{2} (\| [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(x^\dagger) \|^2 - \| j\bar{q}_n - j(x^\dagger) \|^2 \\
&+ \| [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(\bar{q}_n) \|^2). \tag{15}
\end{aligned}$$

Using (15) in (14) and (10), we have

$$\begin{aligned}
& \langle j\bar{q}_n - jx^\dagger, [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(\bar{q}_n) \rangle \\
&= \frac{1}{2} (\| [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(x^\dagger) \|^2 - \| j\bar{q}_n - j(x^\dagger) \|^2 \\
&+ \| [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(\bar{q}_n) \|^2) - \| [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(\bar{q}_n) \|^2 \\
&\leq -\frac{1}{2} \| [J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})]_j(\bar{q}_n) - j(\bar{q}_n) \|^2.
\end{aligned}$$

By using the above result, C(6) and equation (13) implies that utilizing (10) and (11), we then obtain from (12) that

$$\begin{aligned}
\|z_n^j - x^\dagger\|^2 &\leq (1 - \zeta_n) \|\bar{q}_n - x^\dagger\|^2 + \zeta_n (\|\bar{q}_n - x^\dagger\|^2 \\
&\quad + \delta_n^2 \|j\|^2 \|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j(\bar{q}_n) - j\bar{q}_n\|^2 \\
&\quad - 2\delta_n \|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j(\bar{q}_n) - j\bar{q}_n\|^2 + 2Y_n \delta_n G) \\
&\leq (1 - \zeta_n) \|\bar{q}_n - x^\dagger\|^2 + \zeta_n (\|\bar{q}_n - x^\dagger\|^2 \\
&\quad - \delta_n (2 - \delta_n \|j\|^2) \|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j\bar{q}_n - j\bar{q}_n\|^2 + 2Y_n \delta_n G) \\
&\leq (1 - \zeta_n) \|\bar{q}_n - x^\dagger\|^2 + \zeta_n \|\bar{q}_n - x^\dagger\|^2 + 2Y_n \delta_n G \\
&\leq \|p_n - x^\dagger\|^2 + 2Y_n \delta_n G \\
&\leq \|x_n - x^\dagger\|^2 + \Psi_n^2 \|x_n - x_{n-1}\|^2 + 2\Psi_n \langle x_n - x^\dagger, x_n - x_{n-1} \rangle + 2Y_n \delta_n G. \tag{16}
\end{aligned}$$

It is evident from the estimate (16) that $\Gamma \subset C_{k+1}$. As a consequence, we have that the Algorithm 1 is well-defined

Step 2. Prove that the limit of the sequence $(\|x_n - x_1\|)$ exists.

Observe that $\Gamma \subset \mathbb{H}_1$ being a nonempty closed convex subset implies the existence of a unique $x^* \in \Gamma$ such that $x^* = P_\Gamma^{\mathbb{H}_1} x_1$. From $x_{n+1} = P_{C_{n+1}}^{\mathbb{H}_1} x_1$, we have $\|x_{n+1} - x_1\| \leq \|x^* - x_1\|$, for all $x^* \in \Gamma \subset C_{n+1}$. In particular $\|x_{n+1} - x_1\| \leq \|P_\Gamma^{\mathbb{H}_1} x_1 - x_1\|$. This establishes the fact $(\|x_n - x_1\|)$ is bounded. Also, observe from the fact that $x_n = P_{C_n}^{\mathbb{H}_1} x_1$ and $x_{n+1} = P_{C_{n+1}}^{\mathbb{H}_1} x_1 \in C_{n+1}$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|.$$

The above estimate refers that the sequence $(\|x_n - x_1\|)$ is nondecreasing. As a consequence, we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - x_1\| \text{ exists.} \tag{17}$$

Step 3. Show that $x^\dagger \in \Gamma$.

We first compute some preliminary estimates for our claim. Note that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2 \langle x_n - x_1, x_{n+1} - x_1 \rangle \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2 \langle x_n - x_1, x_{n+1} - x_n + x_n - x_1 \rangle \\
&= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2 \langle x_n - x_1, x_{n+1} - x_k \rangle \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.
\end{aligned}$$

As a result of (17), it follows from the above that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{18}$$

Using (C1) and the sequence (p_n) from the Algorithm 1, we have

$$\lim_{n \rightarrow \infty} \|p_n - x_n\| = \lim_{n \rightarrow \infty} \Psi_n \|x_n - x_{n-1}\| = 0. \quad (19)$$

Utilizing (19) and the following triangle inequality:

$$\|p_n - x_{n+1}\| \leq \|p_n - x_n\| + \|x_n - x_{n+1}\|,$$

we have

$$\lim_{n \rightarrow \infty} \|p_n - x_{n+1}\| = 0. \quad (20)$$

Note that $x_{n+1} \in C_{n+1}$, therefore, we have

$$\|\bar{z}_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + 2\Psi_n \|x_n - x_{n-1}\| + 2\langle x_n - x_{n+1}, x_n - x_{n-1} \rangle.$$

Using (18) and (C1), we derive from the above estimate that

$$\lim_{n \rightarrow \infty} \|\bar{z}_n - x_{n+1}\| = 0. \quad (21)$$

Similarly, from (18), (21) and the following triangular inequality:

$$\|\bar{z}_n - x_n\| \leq \|\bar{z}_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we get

$$\lim_{n \rightarrow \infty} \|\bar{z}_n - x_n\| = 0. \quad (22)$$

Since $\|z_n^j - x_n\| \leq \|\bar{z}_n - x_n\|$, from definition of j and (22), we have

$$\lim_{n \rightarrow \infty} \|z_n^j - x_n\| = 0. \quad (23)$$

Consider the following variant of the estimate (11)

$$\|\bar{q}_n - x^\dagger\|^2 \leq \|x_n - x^\dagger\|^2 + \Psi_n^2 \|x_n - x_{n-1}\|^2 + 2\Psi_n \langle x_n - x^\dagger, x_n - x_{n-1} \rangle - \theta_n(1 - \Xi - \theta_n) \|(Id - \mathbb{T}_i)p_n\|^2.$$

Re-arranging the above estimate and using condition (C2), we have

$$\begin{aligned} \theta_n(1 - \Xi - \theta_n) \|(Id - \mathbb{T}_i)p_n\|^2 &\leq (\|x_n - x^\dagger\| + \|\bar{q}_n - x^\dagger\|) \|x_n - \bar{z}_n\| + \Psi_n^2 \|x_n - x_{n-1}\|^2 \\ &\quad + 2\Psi_n \|x_n - x^\dagger\| \|x_n - x_{n-1}\|. \end{aligned}$$

Letting $n \rightarrow \infty$ and utilizing (C1), (C2) and (22), we have

$$\lim_{n \rightarrow \infty} \|(Id - \mathbb{T}_i)p_n\| = 0. \quad (24)$$

This implies that,

$$\lim_{n \rightarrow \infty} \|\bar{q}_n - p_n\| = \lim_{n \rightarrow \infty} \theta_n \|(Id - \mathbb{T}_i)p_n\| = 0. \quad (25)$$

From (19), (25) and the following triangular inequality:

$$\|\bar{q}_n - x_n\| \leq \|\bar{q}_n - p_n\| + \|p_n - x_n\|,$$

we get

$$\lim_{n \rightarrow \infty} \|\bar{q}_n - x_n\| = 0. \quad (26)$$

Since $\|q_n^i - x_n\| \leq \|\bar{q}_n - x_n\|$, from the definition of i and (20), we have

$$\lim_{n \rightarrow \infty} \|q_n^i - x_n\| = 0. \quad (27)$$

From (22), (26) and the following triangular inequality:

$$\|\bar{z}_n - \bar{q}_n\| = \|\bar{z}_n - x_n\| + \|x_n - \bar{q}_n\|$$

we get,

$$\lim_{n \rightarrow \infty} \|\bar{z}_n - \bar{q}_n\| = 0. \quad (28)$$

From Lemma 2.2 and in view of (15),(16), we have

$$\begin{aligned} \|z_n^j - x^\dagger\|^2 &= \|(1 - \zeta_n)\bar{q}_n + \zeta_n(J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)(\bar{q}_n - \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n \\ &\quad - \delta_n\Upsilon_n\bar{q}_n) - x^\dagger\|^2 \\ &= \|(1 - \zeta_n)(\bar{q}_n - x^\dagger) + \zeta_n(J_{r_n}^{A_1}(\bar{q}_n - \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n \\ &\quad - \delta_n\Upsilon_n\bar{q}_n - x^\dagger)\|^2 \\ &\leq (1 - \zeta_n)\|\bar{q}_n - x^\dagger\|^2 + \zeta_n(\|\bar{q}_n - x^\dagger\|^2 \\ &\quad - \delta_n(2 - \delta_n\|j\|^2)\|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j\bar{q}_n - j\bar{q}_n\|^2 + 2\delta_n\Upsilon_n G) \\ &\leq \|\bar{q}_n - x^\dagger\|^2 - \zeta_n\delta_n(2 - \delta_n\|j\|^2)\|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j\bar{q}_n - j\bar{q}_n\|^2 + 2\delta_n\Upsilon_n G \\ &\leq \|x_n - x^\dagger\|^2 + 2\Psi_n\langle x_n - x_{n-1}, \bar{q}_n - x^\dagger \rangle \\ &\quad - \zeta_n\delta_n(2 - \delta_n\|j\|^2)\|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j\bar{q}_n - j\bar{q}_n\|^2 + 2\delta_n\Upsilon_n G. \end{aligned} \quad (29)$$

Rearranging the above estimate, we have

$$\begin{aligned} &\zeta_n\delta_n(2 - \delta_n\|j\|^2)\|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j\bar{q}_n - j\bar{q}_n\|^2 \\ &\leq \|x_n - x^\dagger\|^2 - \|z_n^j - x^\dagger\|^2 + 2\Psi_n\langle x_n - x_{n-1}, p_n - x^\dagger \rangle + 2\delta_n\Upsilon_n G \\ &\leq \left(\|x_n - x^\dagger\| + \|p_n - x^\dagger\|\right)\|x_n - p_n\| + 2\Psi_n\langle x_n - x_{n-1}, p_n - x^\dagger \rangle + 2\delta_n\Upsilon_n G. \end{aligned} \quad (30)$$

By using (C1)-(C3), (C6), (19) and $\delta_n \in (0, \frac{2}{\|h\|^2})$, the estimate (30) implies that

$$\lim_{n \rightarrow \infty} \|J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})_j\bar{q}_n - j\bar{q}_n\| = 0. \quad (31)$$

Note that $J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)$ is firmly nonexpansive, it follows that

$$\begin{aligned} \|z_n^j - x^\dagger\|^2 &= \|(1 - \zeta_n)\bar{q}_n + \zeta_n(J_{r_{n,j}}^{A_{1,n}}(Id - r_n\mathcal{B}_1)(\bar{q}_n - \\ &\quad \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n - \delta_n\Upsilon_n\bar{q}_n) - x^\dagger\|^2 \\ &\leq (1 - \zeta_n)\|\bar{q}_n - x^\dagger\|^2 + \zeta_n\|J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)(\bar{q}_n - \\ &\quad \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n - \delta_n\Upsilon_n\bar{q}_n) - x^\dagger\|^2. \end{aligned} \quad (32)$$

Observe that, using (14) and (15), we have the following simplification for

$$J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)(\bar{q}_n - \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n - \delta_n\Upsilon_n\bar{q}_n):$$

$$\begin{aligned} &\|J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)(\bar{q}_n - \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n - \delta_n\Upsilon_n\bar{q}_n) - J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)x^\dagger\|^2 \leq \\ &\|(\bar{q}_n - \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n - \delta_n\Upsilon_n\bar{q}_n) - x^\dagger\|^2 \leq \|\bar{q}_n - x^\dagger\|^2. \end{aligned} \quad (33)$$

To make simplification, we set $\Delta_n = J_{r_n}^{A_1}(Id - r_n\mathcal{B}_1)(\bar{q}_n + \delta_n^*(Id - J_{r_{n,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{2,j})))_j\bar{q}_n - \delta_n\Upsilon_n\bar{q}_n$ and utilizing

C(7),(31) and Lemma 2.5, it follows that

$$\begin{aligned}
\|\Delta_n - x^\dagger\|^2 &= \|J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)(\bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n) - J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)x^\dagger\|^2 \\
&\leq \langle J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)(\bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n) - J_{r_n}^{A_1}(Id - r_n \mathcal{B}_1)x^\dagger, \bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n - x^\dagger \rangle \\
&= \langle \Delta_n - x^\dagger, \bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n - x^\dagger \rangle \\
&= \frac{1}{2} \left(\|\Delta_n - x^\dagger\|^2 + \|\bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n - x^\dagger\|^2 \right. \\
&\quad \left. - \|\Delta_n - \bar{q}_n - \delta_n^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\|^2 \right) \\
&\leq \frac{1}{2} \left(\|\Delta_n - x^\dagger\|^2 + \|\bar{q}_n - x^\dagger\|^2 - \|\Delta_n - \bar{q}_n - \delta_n^* \right. \\
&\quad \left. (Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\|^2 \right) \\
&= \frac{1}{2} \left(\|\Delta_n - x^\dagger\|^2 + \|\bar{q}_n - x^\dagger\|^2 - \|\Delta_n - \bar{q}_n\|^2 - \delta_n^2 \right. \\
&\quad \left. \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\|^2 \right. \\
&\quad \left. + 2\delta_n \langle \Delta_n - \bar{q}_n, {}^*_j(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n \rangle \right) \\
&\leq \frac{1}{2} \left(\|\Delta_n - x^\dagger\|^2 + \|\bar{q}_n - x^\dagger\|^2 - \|\Delta_n - \bar{q}_n\|^2 - \delta_n^2 \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n \right. \\
&\quad \left. - \delta_n \Upsilon_n \bar{q}_n\|^2 + 2\delta_n \|\Delta_n - \bar{q}_n\| \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\| \right). \tag{34}
\end{aligned}$$

The estimate (34) then implies that

$$\|\Delta_n - x^\dagger\|^2 \leq \|\bar{q}_n - x^\dagger\|^2 - \|\Delta_n - \bar{q}_n\|^2 + 2\delta_n \|\Delta_n - \bar{q}_n\| \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\| \tag{35}$$

Hence

$$\begin{aligned}
\|z_n^j - x^\dagger\|^2 &\leq (1 - \zeta_n) \|\bar{q}_n - x^\dagger\|^2 + \zeta_n \|\Delta_n - x^\dagger\|^2 \\
&\leq (1 - \zeta_n) \|\bar{q}_n - x^\dagger\|^2 + \zeta_n (\|\bar{q}_n - x^\dagger\|^2 - \|\Delta_n - \bar{q}_n\|^2 \\
&\quad + 2\delta_n \|\Delta_n - \bar{q}_n\| \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\|). \tag{36}
\end{aligned}$$

After simplification, we get

$$\begin{aligned}
\zeta_n \|\Delta_n - \bar{q}_n\|^2 &\leq \|\bar{q}_n - x^\dagger\|^2 - \|z_n^j - x^\dagger\|^2 - 2\zeta_n \delta_n \|\Delta_n - \bar{q}_n\| \|_j^*(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\| \\
&\leq \|\bar{q}_n - x^\dagger\|^2 - \|z_n^j - x^\dagger\|^2 + 2\zeta_n \delta_n \|\Delta_n - \bar{q}_n\| \\
&\quad (\|_j\|(Id - J_{r_n,j}^{A_{2,j}}(Id - r_{n,j} \mathcal{B}_{2,j}))_j \bar{q}_n - \delta_n \Upsilon_n \bar{q}_n\|) \tag{37}
\end{aligned}$$

Making use of (31), C(3) and C(7), we have the following estimate:

$$\lim_{n \rightarrow \infty} \|\Delta_n - \bar{q}_n\| = 0. \tag{38}$$

Reasoning as above, utilizing the definition of (p_n) from the Algorithm 1, (C1) and (38), we have

$$\lim_{n \rightarrow \infty} \|\Delta_n - x^\dagger\| = 0. \tag{39}$$

Note that the boundedness of (x_n) implies the existence of a subsequence (x_{n_k}) of (x_n) satisfying $x_{n_k} \rightharpoonup x^\top \in \mathbb{H}_1$ as $k \rightarrow \infty$. Therefore $\Delta_{n_k} \rightharpoonup x^\top$, $\bar{q}_{n_k} \rightharpoonup x^\top$ and $p_{n_k} \rightharpoonup x^\top$ as $k \rightarrow \infty$.

Next, we show that $x^\top \in \Gamma$. Let $(e, f) \in \text{gra}(A_1)$. Since $\Delta_{n_k} = J_{r_{n_k,j}}^{A_1}(\bar{q}_{n_k} + \delta_{n_j}^*(Id - J_{r_{n_j,j}}^{A_{2,j}}(Id - r_{n,j}\mathcal{B}_{n,j}))_j \bar{q}_{n_k})$, therefore, we have

$$\bar{q}_{n_k} + \delta_{n_j}^*(Id - J_{r_{n_k,j}}^{A_{2,j}}(Id - r_{n_k,j}\mathcal{B}_{2,j}))_j \bar{q}_{n_k} \in \Delta_{n_k} + r_{n_k,j}A_1(\Delta_{n_k}).$$

This implies that

$$\frac{1}{r_{n_k,j}}(\bar{q}_{n_k} - \Delta_{n_k}) + \frac{1}{r_{n_k,j}}\delta_{n_j}^*(J_{r_{n_k,j}}^{A_{2,j}}(Id - r_{n_k,j}\mathcal{B}_{2,j}))_j \bar{q}_{n_k} \in A_1(\Delta_{n_k}).$$

From the monotonicity of A_1 , we have

$$\langle e - \Delta_{n_k}, f - (\frac{1}{r_{n_k,j}}(\bar{q}_{n_k} - \Delta_{n_k}) + \frac{1}{r_{n_k,j}}(\delta_{n_j}^*(Id - J_{r_{n_k,j}}^{A_{2,j}}(Id - r_{n_k,j}\mathcal{B}_{2,j}))_j \bar{q}_{n_k})) \rangle \geq 0.$$

We also obtain from the above estimate that

$$\begin{aligned} \langle e - \Delta_{n_k}, f \rangle &\geq \langle e - \Delta_{n_k}, \frac{1}{r_{n_k,j}}(\bar{q}_{n_k} - \Delta_{n_k}) + \frac{1}{r_{n_k,j}}(\delta_{n_j}^*(Id - J_{r_{n_k,j}}^{A_{2,j}}(Id - r_{n_k,j}\mathcal{B}_{2,j}))_j r_{n_k}) \rangle \\ &= \langle e - \Delta_{n_k}, \frac{1}{r_{n_k,j}}(r_{n_k} - \Delta_{n_k}) \rangle \\ &\quad + \langle e - \Delta_{n_k}, \frac{1}{r_{n_k,j}}(\delta_{n_j}^*(Id - J_{r_{n_k,j}}^{A_{2,j}}(Id - r_{n_k,j}\mathcal{B}_{2,j}))_j \bar{q}_{n_k}) \rangle. \end{aligned} \quad (40)$$

Since $\Delta_{n_k} \rightharpoonup x^\top$, we obtain

$$\lim_{k \rightarrow \infty} \langle e - \Delta_{n_k}, f \rangle = \langle e - x^\top, f \rangle.$$

By making the use of (31), (38) and (40), it follows that

$$\langle e - x^\top, f \rangle \geq 0.$$

This implies that $0 \in (A_1 + \mathcal{B}_1)x^\top$ for each $j = 1, 2, \dots, M$.

Reasoning as above, it follows from the estimate from (31) while utilizing the nonexpansiveness and the corresponding demiclosedness of $J_{r_{n_j,j}}^{A_{2,j}}$, we conclude that $0 \in (A_{2,j} + \mathcal{B}_{2,j})x^\top$ for each $j = 1, 2, \dots, M$.

Therefore, $x^\top \in \Gamma$.

Step 4. Show that $x^\dagger = P_{\Gamma}^{\mathbb{H}_1} x_1$.

Let $x^\dagger = P_{\Gamma}^{\mathbb{H}_1} x_1$ and $x^\top \in \Gamma$. Then observe that

$$\begin{aligned} \|x^\dagger - x_1\| &\leq \|x^\top - x_1\| \leq \liminf_{n \rightarrow \infty} \|x_n - x_1\| \\ &\leq \limsup_{n \rightarrow \infty} \|x_n - x_1\| \leq \|x^\top - x_1\|. \end{aligned}$$

The uniqueness of metric projection implies that $x^\top = x^\dagger = P_{\Gamma}^{\mathbb{H}_1} x_1$. Therefore, the desired result $\lim_{n \rightarrow \infty} x_n = x^\dagger = P_{\Gamma}^{\mathbb{H}_1} x_1$ follows immediately. $\square$

The following result is a direct consequence of Theorem 3.1:

Corollary 3.2. Assume that $\Gamma = A_1^{-1}(0) \neq \emptyset$ then the Algorithm 1 converges strongly to an element $x^\dagger \in \Gamma$ provided that the conditions (C1)-(C7) are satisfied.

4. Performance Evaluation of the Algorithm

In this section, we provide numerical examples to evaluate the performance of the Algorithm 1 (also called Alg.1, $\Psi_n \neq 0$) and Classical Algorithm of split monotone variational inclusion problem [Algorithm (8), [29]].

Example 4.1. Let $\mathbb{H}_1 = (\ell_2(\mathbb{R}, \|\cdot\|_{\ell_2})) = \mathbb{H}_2$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \dots, x_j, \dots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} \|x_j\|^2 < \infty\}$ and $\|x\|_{\ell_2} := (\sum_{j=1}^{\infty} \|x_j\|^2)^{\frac{1}{2}} \quad \forall x \in \ell_2(\mathbb{R})$. Let $j = : \ell_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{R})$ be defined by $x := \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{2}, \dots\right), \forall x \in$

$\ell_2(\mathbb{R})$. Then is a bounded linear operator on $\ell_2(\mathbb{R})$ with adjoint $^*y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{2}, \dots\right)$, $\forall y \in \ell_2(\mathbb{R})$. Define the mappings $\mathcal{B}_1, \mathcal{B}_2 : \ell_2 \rightarrow \ell_2$ by $\mathcal{B}_1x := \left(\frac{x_1 + \|x_1\|}{2}, \frac{x_2 + \|x_2\|}{2}, \dots, \frac{x_j + \|x_j\|}{2}, \dots\right)$, and $\mathcal{B}_2x = 3(x_1, x_2, \dots, x_j, \dots) \forall x \in \ell_2$. Let $A_1, A_2 : \ell_2 \rightarrow \ell_2$ be defined by $A_1(x) := (2x_1, 2x_2, \dots, 2x_j, \dots)$, $A_2(x) := (5x_1, 5x_2, \dots, 5x_j, \dots) \forall x \in \ell_2$. Then both A_1 and A_2 are maximal monotone operators. For each $i = 1, 2, \dots, K$, Let the operator $\mathbb{T}_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$\mathbb{T}_i = \begin{cases} \frac{-x}{i}, & \text{if } x \in [0, \infty); \\ x, & \text{if } x \in (-\infty, 0). \end{cases} \quad (41)$$

It is also clear that $\mathbb{T}_i$ defines a finite family of $\frac{1-i^2}{(1+i)^2}$ -demicontractive operators with $\bigcap_{i=1}^K \text{Fix}(\mathbb{T}_i) = \{0\}$. Hence $\Gamma_1 \cap (\bigcap_{i=1}^K \text{Fix}(\mathbb{T}_i)) = \{0\}$. In order to compute the numerical values of (x_{n+1}) , the coordinates of the initial point x_0 are randomly generated in the closed interval $[-2, 2]$. Choose $\theta_n = \frac{n}{75 \times n + 15}$ and $\zeta_n = \frac{n}{100 \times n + 15}$, $\Psi_n = \Psi = 0.57$, $\delta_n = \delta = \frac{1}{7}$, $r_{n,j} = r_n = 0.025$, $h = 3$ and $\ell_n = \frac{1}{n^2}$. The stopping criteria is defined as $\text{TOL} = \text{TOL}_n = \|x_n - x_{n-1}\| < 10^{-6}$.

The values of the Algorithm 1 and the classical Algorithm [(8), [29]] are computed for different choices of x_0, x_1 in the following table:

Table 1. Numerical Results for Example 4.1

	No. of Iter.		CPU(Sec)	
	Our Algo.3.1	Moudafi (8)	Our Algo. 3.1	Moudafi (8)
Choice 1. $x_0 = (3, 9)^T$	22	52	0.02424	0.045657
Choice 2. $x_0 = (1.04, 4.08)^T$	23	66	0.02757	0.065145
Choice 3. $x_0 = (-7, \sqrt{5})^T$	25	94	0.034965	0.079856

The tolerance plotting TOL and No. of Iterations against our Algorithm 1 and the classical algorithm [(8), [29]] for each choices in Table 1 has shown in Figure 1.

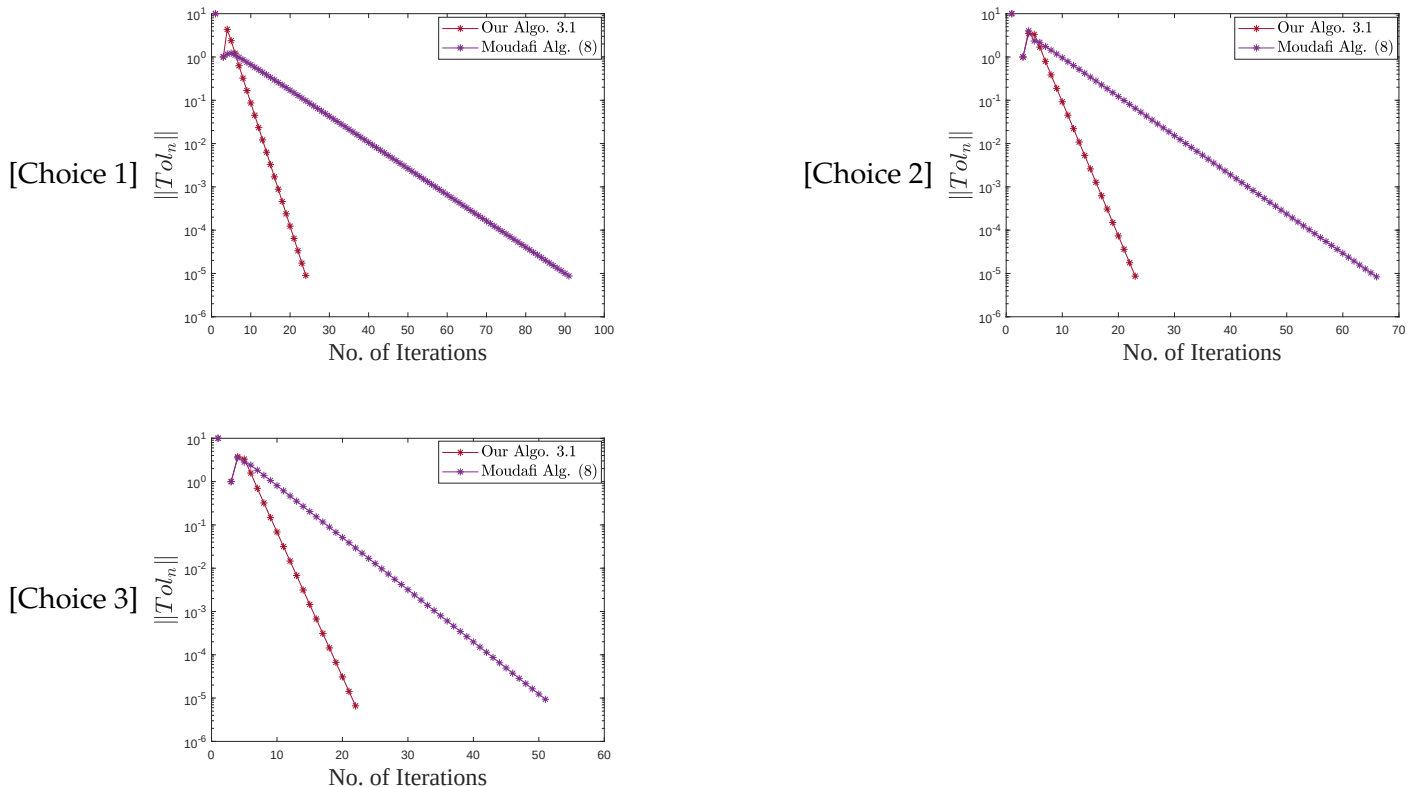


Figure 1. Comparison of our Algorithm 1 and the classical algorithm [(8), [29]]

Table 1 and Figure 1 show that Algorithm 1 achieves faster convergence compared to the classical Algorithm [(8), [29]]. Figure 1 also allows for a comparison of other evaluation parameters, such as the error, time consumption, and number of iterations needed for convergence.

5. Conclusions

We propose a parallel hybrid accelerated projection algorithm for solving the Split monotone variational inclusion problems with multiple output sets (SMVIPMOS) in Hilbert spaces. The algorithm's convergence is established theoretically and validated with numerical experiments. Future work will aim to create further iterative methods for this broadly relevant class of problems, such as monotone inclusion problems, equilibrium problems, variational inequality problems, feasibility problems, and minimization problems.

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Competing interests

The author declare that they have no competing interests.

Author's contributions

The author contributed equally and significantly in writing this article. The author read and approved the final manuscript.

Availability of Data and Material

No data-sets were generated or analyzed during the current study.

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