

Functional Equations in α -Hölder Class on Banach Spaces

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Abstract: We prove existence and uniqueness of global α -Hölder solutions for a special class of functional equations on Banach spaces. As an application of this theory, we suggest an estimate of α for the α -Hölder linearization theorem.

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1. Introduction

We consider functional equations on Banach spaces arising in various studies of dynamical systems. For this class of equations, we prove that each equation has a unique global solution in the space of α -Hölder functions. The proof is based on the contraction mapping theorem and on the existence of some closed subset of α -Hölder functions, which is invariant under the mapping. This allows us to use Banach fixed point Theorem.

We then apply the above result to the linearization problem, and extend the well-known Grobman-Hartman Theorem to the case of α -Hölder linearization on Banach spaces. We suggest an estimate of α for the α -Hölder solutions. The extension is valuable, in particular because it applies to a wide class of differential equations. Unlike smooth conjugation on Banach spaces, which requires the Banach spaces to satisfy a certain "smoothness" condition (see [3], [10] for details), α -Hölder linearization does not need this "smoothness" assumption.

A result similar to the linearization theorem proved in this paper was available in pre-print format ([4]). It found various applications and was cited by other authors (see [2], [5], [7], [12]). Recently an interest to [4] arose, which motivated us to present this more general proof. We also note that the discussed functional equations approach allows a short and simple proof of linearization results.

We would like to thank L.Barreira and C.Valls for offering another method to prove α -Hölder linearization and for bringing this proof to our attention.

2. Functional equations.

Let E be a real Banach space, and

$$E = E_1 \oplus \dots \oplus E_n$$

be a direct decomposition. Consider the system of equations

$$\varphi_i(x) = \Lambda_i \varphi_i(G_i x) + h_i(x, \varphi(H_i x)), \quad x \in E, \quad i \in \overline{1, n}. \quad (1)$$

Here $\Lambda_i : E_i \rightarrow E_i$ are bounded linear operators, while $G_i, H_i : E \rightarrow E$ are maps satisfying the Lipschitz conditions

$$\|G_i(x') - G_i(x'')\| \leq L_i \|x' - x''\|$$

and

$$\|H_i(x') - H_i(x'')\| \leq L \|x' - x''\|.$$

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Also, maps $h_i : E \times E \rightarrow E_i$ are “small” in the sense

$$\sup \|h_i(u)\| \leq \delta, \quad \|h_i(u') - h_i(u'')\| \leq \delta \|u' - u''\|$$

for all $u, u', u'' \in E \times E$ and some $\delta \geq 0$.

We consider (1) as a system with respect to unknown maps $\varphi = (\varphi_1, \dots, \varphi_n)$ with $\varphi_i : E \rightarrow E_i$, $i \in \overline{1, n}$.

Theorem 1. Assume that

$$\max_i \|\Lambda_i\|_i + \delta < 1 \quad (2)$$

and for some $\alpha \in (0, 1]$

$$\max_i \|\Lambda_i\|_i L_i^\alpha + \delta L^\alpha < 1. \quad (3)$$

Then, (1) has a unique bounded continuous solution $\varphi : E \rightarrow E$. This solution is an α -Hölder map, i.e., $\|\varphi(x') - \varphi(x'')\| \leq C \|x' - x''\|^\alpha$ for all $x', x'' \in E$ and some $C > 0$.

Proof. Denote by $C_b^0(E)$ the Banach space of bounded continuous maps $\varphi : E \rightarrow E$ endowed with the norm

$$\|\varphi\| = \sup_x \|\varphi(x)\|.$$

Denote by T the map defined by right side of (1). It acts in the space $C_b^0(E)$. Moreover,

$$\|T\varphi - T\psi\| \leq (\max_i \|\Lambda_i\|_i + \delta) \|\varphi - \psi\|.$$

Inequality (2) implies that the map T is a contraction in $C_b^0(E)$. Therefore, (1) has a unique solution $\varphi \in C_b^0(E)$.

In order to show that φ is α -Hölder, denote by $K(M)$ the closed subset of maps $\varphi \in C_b^0(E)$ satisfying

$$\|\varphi(x') - \varphi(x'')\| \leq M \|x' - x''\|^\alpha.$$

We want to show that $K(M)$ is T -invariant for an appropriate choice of M . Indeed, let $\varphi \in K(M)$. Then

$$\begin{aligned} \|T\varphi(x') - T\varphi(x'')\| &\leq (\max_i \|\Lambda_i\|_i \cdot L_i^\alpha M) \|x' - x''\|^\alpha + \\ &\quad + \max_i \|h_i(x', \varphi(H_i x')) - h_i(x'', \varphi(H_i x''))\|. \end{aligned}$$

It follows that if $\|x' - x''\| \geq 1$, then

$$\|T\varphi(x') - T\varphi(x'')\| \leq (\max_i \|\Lambda_i\|_i \cdot L_i^\alpha M + 2\delta) \|x' - x''\|^\alpha.$$

If $\|x' - x''\| \leq 1$, then

$$\begin{aligned} \|T\varphi(x') - T\varphi(x'')\| &\leq \max_i (\|\Lambda_i\|_i L_i^\alpha M) \|x' - x''\|^\alpha \\ &\quad + \delta \|x' - x''\|^\alpha + \delta L^\alpha M \|x' - x''\|^\alpha. \end{aligned}$$

Inequality 3 allows to choose $M > 0$, such that

$$M > 2\delta / \left(1 - \delta L^\alpha - \max_i \|\Lambda_i\|_i L_i^\alpha\right).$$

Then

$$\varphi \in K(M) \Rightarrow T\varphi \in K(M).$$

Being closed, the set $K(M)$ contains the solution φ . □

3. Application: α -Hölder linearization

In this section we discuss application of Theorem 1 to linearization in Banach spaces. We extend the well-known Grobman-Hartman Theorem (see [8] and [9]) for topological linearization to α -Hölder linearization. The technique of linearization in Banach spaces was developed in [3]. This technique helps to study various aspects of linearization. For instance, it was used in the proof of existence of smooth conjugation for non-resonant maps ([10]) in Banach spaces. However, in the case of smooth conjugation, the Banach space itself has to satisfy certain “smoothness” condition. For α -Hölder linearization we do not need this condition. We also note that linearization of finite smoothness in Banach spaces is qualitatively different from the one in $\mathbb{R}^n$. It is known that if the spectrum of linear part Λ is not separated by a unit sphere, the diffeomorphism is C^1 -linearizable in $\mathbb{R}^n$. (see [6] or [1]). However, it was shown in the work of [11] that the non-resonance assumption in Banach space is essential even for contractions. The authors construct an example of a contraction diffeomorphism in infinite dimensions that is not C^1 linearizable. In our paper, we define a constant $\alpha < 1$ and prove the existence of α -Hölder linearization for contractions (expansions) in Banach spaces without non-resonance assumption. We prove the following result. Given a diffeomorphism with a hyperbolic fixed point, the Theorem 1 defines $\alpha < \sigma(\Lambda) \in (0, 1]$ and a local conjugation in the class of α -Hölder functions, such that the diffeomorphism can be conjugated to its linear part Λ .

A local homeomorphism $\Phi : E \rightarrow E$ is called an α -Hölder homeomorphism if it is an α -Hölder map together with inverse, i.e.,

$$\|\Phi(x') - \Phi(x'')\| \leq C\|x' - x''\|^\alpha$$

and

$$\|\Phi^{-1}(x') - \Phi^{-1}(x'')\| \leq C\|x' - x''\|^\alpha$$

in a neighborhood of the origin.

Let $\Lambda : E \rightarrow E$ be a hyperbolic linear operator. It divides E into the direct sum $E = E_- \oplus E_+$ of Λ -invariant sub-spaces such that the restrictions $\Lambda_\mp = \Lambda|_{E_\mp}$ is a contraction and an expansion, respectively. We denote by $r(\Lambda)$ the spectral radius of an operator Λ . Further, we set

$$\sigma(\Lambda) = \min(-\ln r(\Lambda_-)/\ln r(\Lambda^{-1}), -\ln r(\Lambda_+^{-1})/\ln r(\Lambda)).$$

We assume

$$\sigma(\Lambda) = -\ln r(\Lambda)/\ln r(\Lambda^{-1})$$

if Λ is a contraction, i.e. $E_- = E$, and

$$\sigma(\Lambda) = -\ln r(\Lambda^{-1})/\ln r(\Lambda)$$

if Λ is an expansion, i.e. $E_+ = E$.

In all three cases $\sigma(\Lambda) \in (0, 1]$.

Theorem 2. *Let*

$$F(x) = \Lambda x + f(x), \quad f(0) = 0, \quad f'(0) = 0$$

be a local hyperbolic diffeomorphism on E . Then for every $\alpha < \sigma(\Lambda)$ there is a local α -Hölder homeomorphism conjugating F with its linear part Λ .

The proof of this theorem uses arguments proving an existence of a topological linearization from the book of Nitecki (see [8], Theorem 2, p. 80) up to our Theorem 1, which guaranties α -Hölder solutions of corresponding equations.

Proof. By Proposition (p.69) and Lemma (p.79) from Nitecki [8], given $\delta > 0$, we can assume that $\tilde{F}$ is a homeomorphism defined on E , which coincides with F in a neighborhood of the origin and such that the residues

$$f(x) = \tilde{F}(x) - \Lambda x, \quad f_1(x) = \tilde{F}^{-1}(x) - \Lambda^{-1}x$$

are $\mathcal{Lip}1$ -maps with a Lipschitz constant $\leq \delta$. (To simplify notations, we will drop tilde sign.)

Let us look for a linearization transformation in the form $\Phi(x) = x + \varphi(x)$. Then we have the equation

$$\varphi(Fx) = \Lambda\varphi(x) - f(x).$$

Taking into account the decomposition $E = E_- \oplus E_+$, we write this equation in the form (1):

$$\begin{aligned}\varphi_-(x) &= \Lambda_- \varphi_-(F^{-1}x) - f_-(F^{-1}x) \\ \varphi_+(x) &= \Lambda_+^{-1} \varphi_+(Fx) + \Lambda_+^{-1} f_+(x).\end{aligned}\quad (4)$$

Here $f_{\mp} : E \rightarrow E_{\mp}$, $f = (f_-, f_+)$ and $\varphi_{\mp} : E \rightarrow E_{\mp}$. Respectively, we look for a solution $\varphi = (\varphi_-, \varphi_+)$. We can choose a norm in E such that

$$\max(\|\Lambda_-\| \cdot \|\Lambda^{-1}\|^\alpha, \|\Lambda_+^{-1}\| \cdot \|\Lambda\|^\alpha) < 1.$$

We can also choose $\delta > 0$ to be small enough so that Theorem 1 can be applied to (4). By the Theorem 1, the system has a unique bounded continuous solution φ . The solution belongs to α -Hölder class of functions. The map $\Phi(x) = x + \varphi(x)$ satisfies

$$\Phi(Fx) = \Lambda\Phi(x).$$

In order to show that Φ is an α -Hölder homeomorphism, consider the equations

$$\begin{aligned}\psi_-(x) &= \Lambda_- \psi_-(\Lambda^{-1}x) + f_-(\Lambda^{-1}x + \psi(\Lambda^{-1}x)) \\ \psi_+(x) &= \Lambda_+^{-1} \psi_+(\Lambda x) - \Lambda_+^{-1} f_+(x + \psi(x)).\end{aligned}\quad (5)$$

Under an appropriate choice of $\delta > 0$ the Theorem 1 is applicable to (5). As a consequence it has an α -Hölder solution $\psi = (\psi_-, \psi_+)$, that is unique in $C_b^0(E)$.

Now we will construct the map $\Psi(x)$ and show that it is the inverse of $\Phi(x)$. The map $\Psi(x) = x + \psi(x)$ satisfies

$$F(\Psi(x)) = \Psi(\Lambda x).$$

Further, the map $H(x)$ defined by $H(x) = \Phi\Psi(x) = x + h(x)$ is α -Hölder and the bounded continuous map h satisfies the equation $h(\Lambda x) = \Lambda h(x)$. By uniqueness, $h = 0$. Similarly, the map $\tilde{H}(x) = \Psi(\Phi(x)) = x + \tilde{h}(x)$ satisfies

$$\begin{aligned}\tilde{h}_-(x) &= \Lambda_- \tilde{h}_-(F^{-1}x) + f_-(F^{-1}x + \tilde{h}(F^{-1}x)) - f_-(F^{-1}x) \\ \tilde{h}_+(x) &= \Lambda_+^{-1} \tilde{h}_+(Fx) - \Lambda_+^{-1} f_+(F^{-1}x + \tilde{h}(F^{-1}x)) + \Lambda_+^{-1} f_+(F^{-1}x).\end{aligned}$$

By the same arguments, $\tilde{H}$ is the identity map. Hence, Φ is an α -Hölder homeomorphism, and Ψ is its inverse. $\square$

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