

On Robust Nonsmooth Semi-Infinite Multiobjective Optimization Problems Involving Intersection of Closed Sets

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Abstract: This paper investigates robust optimality and mixed-type duality for robust semi-infinite multiobjective optimization problems involving the intersection of closed sets within the robust optimization framework (UP), using limiting subdifferentials, where uncertainties are incorporated into both the objective and inequality constraint functions. Using the concepts of advanced variational analysis techniques, we first derive the robust Karush-Kuhn-Tucker (KKT) necessary optimality conditions for its robust counterpart (RP) under the assumption of robust limiting constraint qualification (RLCQ). We further establish robust sufficient optimality conditions using the notions of generalized convexity and generalized pseudoquasiconvexity. Additionally, we formulate a mixed-type robust dual model corresponding to the primal problem and analyze robust weak and strong duality. To validate our theoretical findings, we present numerical examples that demonstrate the applicability and effectiveness of the proposed approach.

Keywords: Nonsmooth analysis, robust optimization, Mordukhovich subdifferential, limiting subdifferential, multiobjective optimization, semi-infinite programming, closed set, mixed-type duality results.

2020 Mathematics Subject Classification: 49J52, 58E17, 90C17, 90C26, 90C30, 90C34, 90C46.

1. Introduction

Recently, Hung et al. [26] derived some useful results for a nonsmooth and nonconvex robust optimization problem involving inequality and geometric constraints, utilizing advanced tools of variational analysis [45] and robust optimization [6]. The problem considered by Hung et al. [26] extends and generalizes some problems previously studied by Chieu et al. [13], Ghafari and Mohebi [22], and Allevi et al. [2]. Motivated by these findings, we investigate the following uncertain semi-infinite multiobjective optimization problem:

$$\begin{aligned}
 & \min_{x \in \mathbb{R}^n} (f_1(x, u_1), \dots, f_p(x, u_p)) \\
 \text{(UP)} \quad & \text{s.t. } x \in \bigcap_{j=1}^k C_j, \\
 & g_t(x, v_t) \leq 0, \quad \forall t \in \mathcal{T},
 \end{aligned}$$

where $x := (x_1, \dots, x_n) \in \mathbb{R}^n$ is the vector of decision variables, $u_i \in \mathcal{U}_i$, where $i \in \mathcal{P} := \{1, \dots, p\}$, $v_t \in \mathcal{V}_t$, $t \in \mathcal{T}$, are uncertain parameters, $\mathcal{U}_i$, $i \in \mathcal{P}$, and $\mathcal{V}_t$, $t \in \mathcal{T}$, are nonempty compact convex subsets of $\mathbb{R}^{n_i}$, $i \in \mathcal{P}$, and $\mathbb{R}^{n_t}$, $t \in \mathcal{T}$, respectively, $C_j \subset \mathbb{R}^n$, $j \in \mathcal{K} := \{1, \dots, k\}$, are nonempty closed subsets. The functions $f_i : \mathbb{R}^n \times \mathcal{U}_i \rightarrow \mathbb{R}$, $i \in \mathcal{P}$, and $g_t : \mathbb{R}^n \times \mathcal{V}_t \rightarrow \mathbb{R}$, $t \in \mathcal{T}$, are objective and constraint functions, respectively. Here, $\mathcal{T}$ is an arbitrary nonempty set that may be infinite.

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<https://doi.org/10.68381/jca34016>

Received date: October 24, 2025; Accepted date: June 23, 2026

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The following represents the robust counterpart of the (UP):

$$\begin{aligned}
 \min_{x \in \mathbb{R}^n} \quad & \left(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \dots, \max_{u_p \in \mathcal{U}_p} f_p(x, u_p) \right) \\
 \text{s.t.} \quad & x \in \bigcap_{j=1}^k C_j, \\
 & g_t(x, v_t) \leq 0, \quad \forall v_t \in \mathcal{V}_t, \quad \forall t \in \mathcal{T}.
 \end{aligned}
 \tag{RP}$$

The feasible set corresponding to the (RP) is defined as follows:

$$\Omega := \left\{ x \in \bigcap_{j=1}^k C_j : g_t(x, v_t) \leq 0, \quad \forall v_t \in \mathcal{V}_t, \quad \forall t \in \mathcal{T} \right\}.$$

Our problem is closely related to several existing studies with geometric constraints as follows: when $p = 1$ and for finite set $\mathcal{T}$, our problem reduces to the one studied by Hung et al. [26]. If $p = 1$, $\mathcal{T}$ is a finite set, objective and constraint functions are deterministic smooth, then Jalilian and Pirbazari [29] established optimality conditions for a convex optimization problem over not necessarily convex sets. If the objective functions and the constraints are deterministic along with a finite set $\mathcal{T}$, then our problem reduces to the problem investigated by Pham [48]. If $p = 1, k = 1$, $\mathcal{T}$ is finite, then for any closed convex set $C_1 \subseteq \mathbb{R}^n$, the problem was investigated by Chieu et al. [13] to establish constraint qualifications involving continuous convex objective function and continuously differentiable inequality constraints without uncertainty. Similarly, under the same conditions, Ghafari and Mohebi [22] formulated optimality conditions for a convex function over a constraint set with cone constraints. Furthermore, for $p = 1, k = 1$, and $C_1 \subseteq \mathbb{R}^n$ as a closed convex set, Liu et al. [43] characterized the set of robust optimal solutions using the new constraint qualification of robust type in terms of tangential subdifferentials.

Without geometric constraints, our problem also aligns with several established studies as follows: without uncertainty, Chuong and Kim [16] demonstrated optimality and duality conditions for semi-infinite multiobjective optimization problems via the limiting subdifferential, while Jaisawal et al. [27] explored sufficiency and duality conditions for semi-infinite multiobjective optimization problems under V -invexity using Clarke subdifferentials. For $p = 1$ and without uncertainty in the objective function, our problem reduces to that of Lee and Lee [41], where approximate solutions were obtained using ϵ -subdifferentials. If the functions $f_i, i \in \mathcal{P}$, and $g_t, t \in \mathcal{T}$, are continuously differentiable, our problem aligns with the one studied by Lee and Lee [40] with deterministic objective functions and uncertain semi-infinite constraints to derive optimality results under the extended Mangasarian-Fromovitz constraint qualification (EMFCQ) with smooth functions. If there is no uncertainty in the objective function, then our problem reduces to the one studied by Tung [62] and Pham [47, 49]. If $\mathcal{T}$ is finite, there is no uncertainty in the objective function, Chen et al. [12] studied robust optimality and duality via convexificators. By using Clarke subdifferentials, Sun et al. [59] characterized approximate solutions for robust semi-infinite multiobjective optimization problems, while Tung [61] further established robust optimality conditions using the minimax robustness approach. For the finite set $\mathcal{T}$, our problem aligns with the one examined by Kuroiwa and Lee [36].

Robust optimization offers a deterministic framework for real-world optimization problems that are impacted by uncertainty due to factors such as measurement inaccuracies, imperfect data, unpredictable future occurrences, variations, and external disruptions. The concept of robust optimization was initially introduced by Soyster [54], and further extensively studied in [6, 7, 8, 9, 39].

The semi-infinite optimization problem deals with finitely many variables that are subject to infinitely many inequality constraints. This class of problems has attracted significant attention from researchers [24, 28]. However, the presence of uncertain parameters in these infinite constraints introduces additional complexity, leading to the formulation of robust semi-infinite optimization problems, which aim to ensure optimal solutions remain feasible under uncertainty [35, 41]. The simultaneous minimization of a finite number of objective functions subject to infinitely many inequality constraints across the entire feasible region is commonly referred to as semi-infinite multiobjective optimization. Over the years, numerous studies have explored semi-infinite multiobjective optimization; see e.g., nonconvex and nonsmooth semi-infinite multiobjective optimization [16, 17, 33, 38, 50], robust semi-infinite multiobjective optimization [1, 23, 40].

While our approach is inspired by the work of Chuong [14], Sun et al. [59], and Kerdkaew and Wangkeeree [34], it differs in several significant ways. In contrast to [14], where local robust efficient (or weakly

efficient) solutions are obtained for uncertain multiobjective optimization problems without uncertainty in the objective functions, under finitely many constraints, and in the absence of geometric constraints using generalized convexity, our framework incorporates uncertainty in a more general setting and explicitly includes geometric constraints using generalized pseudoquasiconvexity. Moreover, [59] established robust optimality and mixed-type robust duality conditions for uncertain semi-infinite multiobjective optimization problem in the absence of geometric constraints using Clarke subdifferential. In contrast, our results extend this framework by incorporating geometric constraints and employing the limiting subdifferential, thereby yielding a broader class of robust optimality and Mixed-type duality results. Furthermore, [34] derives robust optimality conditions for uncertain convex programming problems with a single closed convex set as a geometric constraint. In comparison, the present work generalizes these results to nonconvex programming problems involving multiple closed sets as geometric constraints.

Dealing with multiobjective optimization problems with geometric constraints $C_j, j \in \mathcal{K}$, is particularly challenging in the robust optimization framework, especially when the feasible region is nonconvex and includes an infinite number of inequality constraints. To the best of our knowledge, no existing research focuses on nonsmooth, nonconvex robust semi-infinite multiobjective optimization problems involving the intersection of closed sets, analyzed through the limiting subdifferentials, where uncertainty is incorporated in both the objective and constraint functions. Therefore, it is essential to examine robust optimality and mixed-type duality for robust semi-infinite multiobjective optimization problems involving the intersection of closed sets. This motivates us to the following key questions:

- How can one employ robust optimization techniques to reformulate semi-infinite multiobjective optimization problems characterized by intersections of closed sets and uncertainty in both the objective function and the feasible region?
- How can robust Karush-Kuhn-Tucker (KKT) necessary optimality conditions be developed to characterize optimal solutions in the presence of uncertainty and nonsmoothness using limiting subdifferentials?
- Under what assumptions can robust sufficient optimality conditions be established?
- How can a robust dual model be formulated for the uncertain nonsmooth, nonconvex primal problem (UP)?
- How can the duality gap between the primal and dual problems be characterized under uncertainty?

To address these questions, the main contributions of this work are summarized as follows:

- This paper generalizes the results of [26] in the setting of multiobjective functions and infinitely many constraints. Additionally, the constraint qualification employed here differs from that used in [26].
- We employ advanced tools from variational analysis, including the Mordukhovich (or limiting) subdifferential sum rule, the maximum function rule, and the nonsmooth version of Fermat's rule, to derive robust KKT necessary optimality conditions for problem (UP) using robust limiting constraint qualification (RLCQ).
- We present the notions of generalized convexity and generalized pseudoquasiconvexity in terms of limiting subdifferentials, which enable the establishment of robust sufficient optimality conditions. Furthermore, we provide an example that does not satisfy generalized convexity but satisfies generalized pseudoquasiconvexity.
- We formulate a mixed-type robust dual model (constructed by combining both Wolfe type and Mond-Weir type dual models) based on [65] which differs from the version presented in [26].
- We investigate robust weak and strong duality results, thereby characterizing the relationship between the primal and dual problems under uncertainty.

The paper is structured as follows: In Section 2, we recall some fundamental definitions and calculus rules related to the variational analysis tools. Section 3 establishes robust KKT necessary and sufficient optimality conditions for the (UP). Section 4 presents the mixed-type robust dual model and explores robust weak and strong duality. Finally, the last Section 5 summarizes the key findings.

2. Preliminaries

In this paper, we use the standard notation of variational analysis [45, 46]. The inner product and the norm in n -dimensional Euclidean space $\mathbb{R}^n$ are represented by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}_+^n$. Let $\mathcal{T}$ be any index set, possibly infinite. Let $\mathbb{R}^{|\mathcal{T}|}$ be the set of all functions $\lambda : \mathcal{T} \rightarrow \mathbb{R}$ such that λ is nonzero at only finitely many elements of $\mathcal{T}$ and zero elsewhere [24], i.e.

$$\mathbb{R}^{|\mathcal{T}|} := \{\lambda = (\lambda_t)_{t \in \mathcal{T}} : \lambda_t = 0 \text{ for all } t \in \mathcal{T} \text{ but only finitely many } \lambda_t \neq 0\}.$$

The nonnegative cone associated with $\mathbb{R}^{|\mathcal{T}|}$ is defined as

$$\mathbb{R}_+^{|\mathcal{T}|} := \{\lambda = (\lambda_t)_{t \in \mathcal{T}} \in \mathbb{R}^{|\mathcal{T}|} : \lambda_t \geq 0\}.$$

For any nonempty subset $S \subseteq \mathbb{R}^n$, we denote its interior, boundary, and convex hull, by $\text{int}S$, $\text{bd}S$, and $\text{co}S$, respectively. Accordingly, the polar cone of a set S is defined as:

$$S^\circ := \{\xi \in \mathbb{R}^n : \langle \xi, x \rangle \leq 0, \quad \forall x \in S\}.$$

Given a multifunction $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, the *sequential Painlevé-Kuratowski upper/outer limit* of $\mathcal{H}$ at $\bar{x} \in \text{dom}\mathcal{H}$, we denote by

$$\limsup_{x \xrightarrow{S} \bar{x}} \mathcal{H}(x) := \left\{ \xi \in \mathbb{R}^n : \begin{array}{l} \exists \text{ sequences } x_n \xrightarrow{S} \bar{x} \text{ and } \xi_n \rightarrow \xi \\ \text{with } \xi_n \in \mathcal{H}(x_n) \text{ for all } n \in \mathbb{N} \end{array} \right\}, \quad (1)$$

where notation $x \xrightarrow{S} \bar{x}$ means that $x \rightarrow \bar{x}$ and $x \in S$ and

$$\text{dom}\mathcal{H} := \{x \in \mathbb{R}^n : \mathcal{H}(x) \neq \emptyset\}.$$

The *regular/Fréchet normal cone* $\widehat{\mathcal{N}}(\bar{x}; S)$ to S at $\bar{x} \in S$ is expressed as

$$\widehat{\mathcal{N}}(\bar{x}; S) := \left\{ \xi \in \mathbb{R}^n : \limsup_{x \xrightarrow{S} \bar{x}} \frac{\langle \xi, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

For any $\bar{x} \notin S$, we set $\widehat{\mathcal{N}}(\bar{x}; S) := \emptyset$. The *Mordukhovich/limiting normal cone* $\mathcal{N}(\bar{x}; S)$ to S at $\bar{x} \in S$ is characterized by

$$\mathcal{N}(\bar{x}; S) := \limsup_{x \xrightarrow{S} \bar{x}} \widehat{\mathcal{N}}(\bar{x}; S).$$

If $\bar{x} \notin S$, then $\mathcal{N}(\bar{x}; S) := \emptyset$. Given a function $\Phi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we define $\text{epi}\Phi := \{(x, \beta) \in \mathbb{R}^n \times \mathbb{R} : \Phi(x) \leq \beta\}$. The *Fréchet subdifferential* and *limiting/Mordukhovich subdifferential* of Φ at $\bar{x} \in \mathbb{R}^n$ with $|\Phi(\bar{x})| < \infty$ are defined respectively as follows:

$$\widehat{\partial}\Phi(\bar{x}) := \{\xi \in \mathbb{R}^n : (\xi, -1) \in \widehat{\mathcal{N}}((\bar{x}, \Phi(\bar{x})); \text{epi}\Phi)\};$$

$$\partial\Phi(\bar{x}) := \{\xi \in \mathbb{R}^n : (\xi, -1) \in \mathcal{N}((\bar{x}, \Phi(\bar{x})); \text{epi}\Phi)\}.$$

If $|\Phi(\bar{x})| = \infty$, then the above subdifferentials are empty i.e. $\widehat{\partial}\Phi(\bar{x}) = \partial\Phi(\bar{x}) = \emptyset$. It is well known that the above-defined subdifferentials and normal cones coincide with those in convex analysis, provided that S is a convex set and Φ is a convex function.

For a given set $S \subseteq \mathbb{R}^n$, we consider the indicator function $\delta(\cdot; S)$ defined by

$$\delta(\bar{x}; S) := \begin{cases} 0, & \text{if } \bar{x} \in S, \\ \infty, & \text{if } \bar{x} \notin S. \end{cases} \quad (2)$$

By [45, Proposition 1.79], for $\bar{x} \in S$, it follows that

$$\mathcal{N}(\bar{x}, S) = \partial\delta(\bar{x}; S). \quad (3)$$

If a function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz at a point $\bar{x} \in \mathbb{R}^n$, i.e., there exist constant $L > 0$ and $\epsilon > 0$ such that

$$|\Phi(x) - \Phi(y)| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{B}(\bar{x}, \epsilon) := \{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq \epsilon\},$$

then it follows from [45, Corollary 1.81] that $\|\xi\| \leq L$ for every $\xi \in \partial\Phi(\bar{x})$.

Lemma 2.1. [45, Theorem 3.36] Consider functions $\Phi_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, i = 1, \dots, p, p \geq 2$, which are lower semicontinuous around $\bar{x} \in \mathbb{R}^n$, with all but one being locally Lipschitz at $\bar{x}$. Then we have

$$\partial(\Phi_1 + \dots + \Phi_p)(\bar{x}) \subset \partial\Phi_1(\bar{x}) + \dots + \partial\Phi_p(\bar{x}). \quad (4)$$

Lemma 2.2. [45, Theorems 3.36 and 3.46] Suppose the functions $\Phi_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, i \in \mathcal{P} := \{1, \dots, p\}, p \geq 2$, are locally Lipschitz at $\bar{x} \in \mathbb{R}^n$, and define the maximum function as $\Phi(x) := \max_{i \in \mathcal{P}} \Phi_i(x)$ for $x \in \mathbb{R}^n$. Then, the following holds:

$$\partial\Phi(\bar{x}) \subset \bigcup \left\{ \sum_{i=1}^p \mu_i \partial\Phi_i(\bar{x}) : \sum_{i=1}^p \mu_i = 1, \mu_i \geq 0, \mu_i(\Phi_i(\bar{x}) - \Phi(\bar{x})) = 0, i \in \mathcal{P} \right\}. \quad (5)$$

3. Robust optimality conditions

This section presents robust necessary and sufficient optimality conditions for a robust multiobjective problem with uncertainty in both objective functions and infinite inequality constraints involving the intersection of closed sets.

We use the notation $F_i(x) := \max_{u_i \in \mathcal{U}_i} f_i(x, u_i), i \in \mathcal{P}, F := (F_1, \dots, F_p)$, and $G_t(x) := \max_{v_t \in \mathcal{V}_t} g_t(x, v_t), t \in \mathcal{T}$. Define

$$\begin{aligned} \mathcal{U}_i(\bar{x}) &:= \{u_i \in \mathcal{U}_i : F_i(\bar{x}) = f_i(\bar{x}, u_i)\}, \quad i \in \mathcal{P}, \\ \mathcal{V}_t(\bar{x}) &:= \{v_t \in \mathcal{V}_t : G_t(\bar{x}) = g_t(\bar{x}, v_t)\}, \quad t \in \mathcal{T}, \\ \Delta(\bar{x}) &:= \{\lambda \in \mathbb{R}_+^{|\mathcal{T}|} : \lambda_t G_t(\bar{x}) = 0, \forall t \in \mathcal{T}\}, \\ \mathcal{T}(\bar{x}) &:= \{t \in \mathcal{T} : G_t(\bar{x}) = 0\}. \end{aligned}$$

In what follows, for $w_1, w_2 \in \mathbb{R}^p$, we use the following order relations for convenience.

$$\begin{aligned} w_1 < w_2 &\iff w_1 - w_2 \in -\text{int}\mathbb{R}_+^p, & w_1 \not< w_2 &\iff w_1 - w_2 \notin -\text{int}\mathbb{R}_+^p, \\ w_1 \leq w_2 &\iff w_1 - w_2 \in -\mathbb{R}_+^p \setminus \{0\}, & w_1 \not\leq w_2 &\iff w_1 - w_2 \notin -\mathbb{R}_+^p \setminus \{0\}. \end{aligned}$$

Definition 3.1. A point $\bar{x} \in \Omega$ is a *local robust weakly efficient solution* of the (UP) iff there exists a neighbourhood $\mathcal{B}$ of $\bar{x}$ such that

$$F(x) \not\leq F(\bar{x}), \quad \forall x \in \Omega \cap \mathcal{B}.$$

The set of all local robust weakly efficient solutions of the (UP) is denoted by Ω_{UP}^{lrwe} .

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$$F(x) \not\leq F(\bar{x}), \quad \forall x \in \Omega \cap \mathcal{B}.$$

We denote the set of all local robust efficient solutions of the (UP) by Ω_{UP}^{lre} .

If $\mathcal{B} = \Omega$, then we have the concepts of robust weakly efficient solution and robust efficient solution, respectively. We denote the set of all robust weakly efficient solutions and robust efficient solutions by Ω_{UP}^{rwe} and Ω_{UP}^{re} , respectively. It is easy to observe that Ω_{UP}^{lre} is a subset of Ω_{UP}^{lrwe} . $\square$

In what follows, we assume that the objective functions $f_i, i \in \mathcal{P}$ and constraint functions $g_t, t \in \mathcal{T}$, of the problem (UP) satisfy the following assumptions:

- (B1) Given a fixed $\bar{x} \in \mathbb{R}^n$, there exist $\alpha_i > 0, \beta_t > 0$ such that the functions $u_i \in \mathcal{U}_i \mapsto f_i(x, u_i), i \in \mathcal{P}$ and $v_t \in \mathcal{V}_t \mapsto g_t(x, v_t), t \in \mathcal{T}$, are upper semicontinuous for each x in the neighborhood $\mathcal{B}_i(\bar{x}, \alpha_i), i \in \mathcal{P}$ and $\mathcal{B}_t(\bar{x}, \beta_t), t \in \mathcal{T}$, respectively, and the functions $f_i(\cdot, u_i), u_i \in \mathcal{U}_i, i \in \mathcal{P}$ and $g_t(\cdot, v_t), v_t \in \mathcal{V}_t, t \in \mathcal{T}$ are Lipschitz of ranks $\mathcal{L}_i > 0$ and $\widehat{\mathcal{L}}_t > 0$ on $\mathcal{B}_i(\bar{x}, \alpha_i)$ and $\mathcal{B}_t(\bar{x}, \beta_t)$, respectively, i.e.

$$\begin{aligned} |f_i(x, u_i) - f_i(y, u_i)| &\leq \mathcal{L}_i \|x - y\|, \quad \forall x, y \in \mathcal{B}_i(\bar{x}, \alpha_i), u_i \in \mathcal{U}_i, i \in \mathcal{P}, \\ |g_t(x, v_t) - g_t(y, v_t)| &\leq \widehat{\mathcal{L}}_t \|x - y\|, \quad \forall x, y \in \mathcal{B}_t(\bar{x}, \beta_t), v_t \in \mathcal{V}_t, t \in \mathcal{T}. \end{aligned}$$

(B2) For the above $\bar{x} \in \mathbb{R}^n$, the multivalued functions

$$(x, u_i) \in \mathcal{B}_i(\bar{x}, \alpha_i) \times \mathcal{U}_i \rightrightarrows \partial_x f_i(x, u_i)$$

are closed at $(\bar{x}, \bar{u}_i)$ for each $\bar{u}_i \in \mathcal{U}_i(\bar{x})$, $i \in \mathcal{P}$, and the multivalued functions $(x, v_t) \in \mathcal{B}_t(\bar{x}, \beta_t) \times \mathcal{V}_t \rightrightarrows \partial_x g_t(x, v_t)$ are closed at $(\bar{x}, \bar{v}_t)$ for each $\bar{v}_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$, where ∂_x represents the limiting subdifferential operation wrt the first variable.

Remark 3.2. The above assumptions are motivated by the assumptions used in [14, 34, 58, 59]. If $\mathcal{U}_i$, $i \in \mathcal{P}$ are singleton sets and $\mathcal{T}$ is a finite index set, then the above assumptions (B1) and (B2) reduce to those given in [14]. Moreover, if $f_i(\cdot, u_i)$ for any $u_i \in \mathcal{U}_i$, $i \in \mathcal{P}$ and $g_t(\cdot, v_t)$, for any $v_t \in \mathcal{V}_t$, $t \in \mathcal{T}$ are convex functions, while $f_i(x, \cdot)$, $i \in \mathcal{P}$ and $g_t(x, \cdot)$, $t \in \mathcal{T}$, for any $x \in \mathbb{R}^n$ are concave functions, then the above assumptions are analogous to those in [34]. If the Clarke subdifferentials of objective functions f_i , $i \in \mathcal{P}$ and constraint functions g_t , $t \in \mathcal{T}$ exist, then the above assumptions are analogous to the assumptions given in [58, 59]. $\square$

Remark 3.3. The above assumptions are commonly employed in robust optimization and nonsmooth analysis, particularly for computing subdifferentials of supremum functions over infinite sets. Under assumption (B1), the functions f_i , $i \in \mathcal{P}$, g_t , $t \in \mathcal{T}$ are well-defined and locally Lipschitz in the context of robust optimization. Moreover, assumption (B2) concerns the closedness of the partial subdifferential with respect to its first variable which can be viewed as a relaxed property compared to the closedness of subdifferentials for a convex function. Furthermore, when the involved functions are smooth and the uncertainty sets are convex, both assumptions are automatically satisfied. $\square$

For each $v_t \in \mathcal{V}_t$, $t \in \mathcal{T}$, the limiting/Mordukhovich subdifferential of the function $g_t(\cdot, v_t)$ at $\bar{x} \in \mathbb{R}^n$ is defined as

$$\partial_x g_t(\bar{x}, v_t) := \{\xi \in \mathbb{R}^n : (\xi, -1) \in \mathcal{N}(\bar{x}, g_t(\bar{x}, v_t)); \text{epi}(g_t(\cdot, v_t))\}. \quad (6)$$

Motivated by the Limiting Constraint Qualification (LCQ) introduced in [16], we present the following robust variant, referred to as the Robust Limiting Constraint Qualification (RLCQ).

Definition 3.4. Let $\bar{x} \in \Omega$. We say that the RLCQ is satisfied at $\bar{x}$ iff

$$(RLCQ) \quad \mathcal{N}(\bar{x}; \Omega) \subset \bigcup_{\lambda \in \Delta(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \partial G_t(\bar{x}) \right] + \mathcal{N}\left(\bar{x}; \bigcap_{j=1}^k C_j\right).$$

Remark 3.5. It is worth observing that, when $k = 1$ and $\mathcal{V}_t$, $t \in \mathcal{T}$ are singleton sets, the RLCQ coincides with the LCQ introduced in [16]. Furthermore, when $C_j = \mathbb{R}^n$ for all $j \in \mathcal{K}$, the RLCQ is analogous to the LCQ introduced in [15]. Sufficient conditions ensuring the validity of (LCQ) in the case where $g_t(\cdot, v_t)$, $v_t \in \mathcal{V}_t$, is convex for all $t \in \mathcal{T}$, can be found in [18, 19] for the singleton set $\mathcal{V}_t$, $t \in \mathcal{T}$. The robust constraint qualification in terms of Clarke subdifferential given in [57, Definition 3.2] does not consider the presence of geometric constraints. $\square$

The following example validates the Definition 3.4.

Example 3.6. Consider the closed set $C_1 = [0, 1]$ and inequality constraint

$$g_t(x, v_t) = v_t x^2 - |x| \leq 0, \quad v_t \in [-t, t], \quad t \in [-1, 0], \quad x \in \mathbb{R}.$$

Then the feasible set is given by $\Omega = C_1 = [0, 1]$, and we define $G_t(\bar{x}) = t\bar{x}^2 - |\bar{x}|$. Let $\bar{x} = 0 \in \Omega$. The limiting subdifferential is $\partial G_t(\bar{x}) = \{-1, 1\}$. From condition RLCQ, we need to verify that

$$\mathcal{N}(\bar{x}; \Omega) \subset \bigcup_{\lambda \in \Delta(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \{-1, 1\} \right] + \mathcal{N}(\bar{x}; C_1).$$

Since $\Omega = C_1 = [0, 1]$, their normal cones at $\bar{x} = 0$ coincide:

$$\mathcal{N}(\bar{x}; \Omega) = \mathcal{N}(\bar{x}; C_1) = (-\infty, 0].$$

Thus, it suffices to show that there exists $\lambda \in \Delta(\bar{x})$ such that

$$0 \in \sum_{t \in \mathcal{T}} \lambda_t \{-1, 1\}.$$

By taking $\lambda = (1, 1, 0, 0, \dots)$, we get $0 \in \{-2, 0, 2\}$, which implies that condition RLCQ is satisfied at $\bar{x} = 0$. $\square$

To begin with, we establish a robust KKT-type necessary optimality condition for local robust weakly efficient solutions of the (UP), under the assumption that the RLCQ holds.

Theorem 3.7. (Robust KKT necessary optimality conditions) *Let $\bar{x} \in \Omega_{UP}^{lrwe}$. Assume that the conditions (B1) and (B2) are satisfied at $\bar{x}$. Suppose the equation $w_1 + \dots + w_k = 0$, where $w_j \in \mathcal{N}(\bar{x}; C_j)$, $j \in \mathcal{K}$, admits only trivial solution $w_j = 0$, $j \in \mathcal{K}$. If RLCQ is satisfied at $\bar{x}$, then there exist $\mu = (\mu_1, \dots, \mu_p) \in \mathbb{R}_+^p$ with $\sum_{i=1}^p \mu_i = 1$, $\lambda \in \mathbb{R}_+^{|\mathcal{T}|}$, such that*

$$\begin{aligned} 0 \in & \sum_{i=1}^p \mu_i \text{co} \left\{ \partial_x f_i(\bar{x}, u_i) : u_i \in \mathcal{U}_i(\bar{x}) \right\} \\ & + \sum_{t \in \mathcal{T}} \lambda_t \text{co} \left\{ \partial_x g_t(\bar{x}, v_t) : v_t \in \mathcal{V}_t(\bar{x}) \right\} + \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j), \\ & \lambda_t \max_{v_t \in \mathcal{V}_t} g_t(\bar{x}, v_t) = 0, \quad t \in \mathcal{T}. \end{aligned} \quad (7)$$

Proof. Let $\bar{x} \in \Omega_{UP}^{lrwe}$, and define a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ as follows:

$$\varphi(x) := \max\{F_i(x) - F_i(\bar{x})\}, \quad i \in \mathcal{P}.$$

It is straightforward to observe that $\varphi(\bar{x}) = 0$. Moreover, if $F(x) \neq F(\bar{x})$, then $\varphi(x) \geq 0$. Consequently, there exists an open neighborhood $\mathcal{B}$ of $\bar{x}$ such that

$$\varphi(\bar{x}) \leq \varphi(x), \quad \forall x \in \Omega \cap \mathcal{B}. \quad (8)$$

Otherwise, there exists $\bar{y} \in \Omega \cap \mathcal{B}$ such that $\varphi(\bar{y}) < \varphi(\bar{x})$. Since $\varphi(\bar{x}) = 0$, therefore

$$\varphi(\bar{y}) = \max\{F_i(\bar{y}) - F_i(\bar{x})\} < 0, \quad i \in \mathcal{P}.$$

Thus, $F(\bar{y}) < F(\bar{x})$, which contradicts the fact that $\bar{x} \in \Omega_{UP}^{lrwe}$. Consequently, from equation (8), it follows that $\bar{x}$ is a local minimizer of the scalar optimization problem:

$$\min_{x \in \Omega} \varphi(x),$$

which is equivalent to the unconstrained problem:

$$\min_{x \in \mathbb{R}^n} \varphi(x) + \delta(x; \Omega), \quad (9)$$

where δ denoting the indicator function defined in (2). Then, it follows that

$$0 \in \partial(\varphi + \delta(\cdot; \Omega))(\bar{x}).$$

Under assumption (B1), function φ is locally Lipschitz near $\bar{x}$. Moreover, the indicator function $\delta(\cdot; \Omega)$ is lower semicontinuous near $\bar{x}$.

Thus, from the formulation of the sum rule for limiting subdifferential (Lemma 2.1) together with the relation in (9), we have

$$0 \in \partial\varphi(\bar{x}) + \mathcal{N}(\bar{x}; \Omega). \quad (10)$$

From Lemma 2.2, we have

$$\partial\varphi_i(\bar{x}) \subset \bigcup \left\{ \sum_{i=1}^p \mu_i \partial F_i(\bar{x}) : \sum_{i=1}^p \mu_i = 1, \mu_i \geq 0, i \in \mathcal{P} \right\}. \quad (11)$$

Since RLCQ is satisfied at $\bar{x}$, therefore

$$\mathcal{N}(\bar{x}; \Omega) \subset \bigcup_{\lambda \in \Lambda(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \partial G_t(\bar{x}) \right] + \mathcal{N}(\bar{x}; \bigcap_{j=1}^k C_j). \quad (12)$$

Combining (10)–(12), we deduce

$$0 \in \bigcup \left\{ \sum_{i=1}^p \mu_i \partial F_i(\bar{x}) : \sum_{i=1}^p \mu_i = 1, \mu_i \geq 0, i \in \mathcal{P} \right\} \\ + \bigcup_{\lambda \in \Delta(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \partial_x G_t(\bar{x}) \right] + \mathcal{N} \left(\bar{x}; \bigcap_{j=1}^k C_j \right). \quad (13)$$

Moreover, since the system $w_1 + \cdots + w_k = 0$, where $w_j \in \mathcal{N}(\bar{x}; C_j)$, $j \in \mathcal{K}$, admits only trivial solution $w_j = 0$, $j \in \mathcal{K}$, we invoke the formula for normal cones to finite intersections of sets (cf. [46, Corollary 2.17]) to obtain:

$$\mathcal{N} \left(\bar{x}; \bigcap_{j=1}^k C_j \right) \subset \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j).$$

Substituting this into (13), we obtain:

$$0 \in \bigcup \left\{ \sum_{i=1}^p \mu_i \partial F_i(\bar{x}) : \sum_{i=1}^p \mu_i = 1, \mu_i \geq 0, i \in \mathcal{P} \right\} \\ + \bigcup_{\lambda \in \Delta(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \partial G_t(\bar{x}) \right] + \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j). \quad (14)$$

Therefore, there exist $\mu_i \geq 0$, $i \in \mathcal{P}$ with $\sum_{i=1}^p \mu_i = 1$, $\lambda \in \mathbb{R}_+^{|\mathcal{T}|}$, $u_i \in \mathcal{U}_i$, $i \in \mathcal{P}$, and $v_t \in \mathcal{V}_t$, $t \in \mathcal{T}$ such that

$$0 \in \sum_{i=1}^p \mu_i \partial F_i(\bar{x}) + \sum_{t \in \mathcal{T}} \lambda_t \partial G_t(\bar{x}) + \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j), \quad (15) \\ \lambda_t G_t(\bar{x}) = 0, \quad t \in \mathcal{T}.$$

Assuming that conditions (B1) and (B2) hold, and using an approach equivalent to that employed in the demonstration of [14, Theorem 3.3], we conclude that

$$\partial F_i(\bar{x}) \subset \text{co}\{\partial_x f_i(\bar{x}, u_i) : u_i \in \mathcal{U}_i(\bar{x})\}, i \in \mathcal{P}. \quad (16)$$

Since $\mathcal{V}_t(\bar{x})$ is nonempty, compact, and convex, therefore there exists $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$, such that $g_t(\bar{x}, v_t) = G_t(\bar{x})$. By applying (6) and argument similar to the proof of [14, Theorem 3.3] under the assumptions (B1) and (B2), we deduce that

$$\partial G_t(\bar{x}) \subset \text{co}\{\partial_x g_t(\bar{x}, v_t) : v_t \in \mathcal{V}_t(\bar{x})\}, t \in \mathcal{T}. \quad (17)$$

Now, substituting the inclusions (16) and (17) into (15), we get (7). This completes the proof. $\square$

Remark 3.8. In the case of a finite index set $\mathcal{T}$ and a scalar objective function, Theorem 3.7 is analogous to [26, Theorem 3.1]. Since Ω_{UP}^{lre} is contained in Ω_{UP}^{lrwe} , therefore the above necessary condition is also a necessary condition for a local robust efficient solution. $\square$

The following example illustrates the above results.

Example 3.9. Consider the following (UP) under data uncertainty:

$$\min_{x \in \mathbb{R}^2} (f_1(x, u_1), f_2(x, u_2)) \\ \text{s.t. } x \in C_1 \bigcap C_2, \\ g_t(x, v_t) \leq 0, \quad t \in \mathcal{T},$$

where

$$\begin{aligned} f_1(x, u_1) &= x_1 \sin u_1 + x_2 \cos u_1 - 1, \\ f_2(x, u_2) &= |x_1| - u_2, C_1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}, \\ C_2 &= [-1, 0] \times [-1, 1], g_t(x, v_t) = (v_t - 2t)x_1^2 - |x_2|, \\ u_1 &\in [\frac{\pi}{2}, 2\pi], u_2 \in [\frac{1}{2}, 1], \\ v_t &\in \mathcal{V}_t := [-t, t], t \in \mathcal{T} := (0, +\infty), x \in \mathbb{R}^2. \end{aligned}$$

The corresponding robust counterpart (RP) is given by:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \left(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2) \right) \\ \text{s.t.} \quad & x \in \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, x_1 \in [-1, 0]\}, \\ & g_t(x, v_t) := (v_t - 2t)x_1^2 - |x_2| \leq 0, \quad \forall v_t \in \mathcal{V}_t, t \in \mathcal{T}, \end{aligned}$$

where,

$$F_1(x) := \max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) = \begin{cases} \|x\| - 1, & \text{if } x_1 \leq 0 \text{ or } x_2 \leq 0, \\ x_2 - 1, & \text{if } 0 \leq x_1 \leq x_2, \\ x_1 - 1, & \text{if } 0 \leq x_2 \leq x_1, \end{cases}$$

and $F_2(x) := \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2) = |x_1| - \frac{1}{2}$. It is easy to see that $\bar{x} = (0, 0)$ is a local robust weakly efficient solution of the (UP). To verify the assumption (B1), observe that for all x in the neighbourhood of $\bar{x}$, $f_1(x, u_1)$ is continuous over $u_1 \in [\frac{\pi}{2}, 2\pi]$.

For Lipschitz continuity, one has

$$\begin{aligned} |f_1(x, u_1) - f_1(y, u_1)| &= |x_1 \sin(u_1) + x_2 \cos(u_1) - y_1 \sin(u_1) - y_2 \cos(u_1)| \\ &= |\sin(u_1)(x_1 - y_1) + \cos(u_1)(x_2 - y_2)| \\ &\leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \|x - y\|, \quad \forall x, y \in \mathbb{R}^2, \quad \forall u_1 \in \mathcal{U}_1. \end{aligned} \tag{18}$$

Similarly, under these conditions, assumption (B1) evidently holds for $f_2(x, u_2)$ and $g_t(x, v_t)$. Therefore, $\Omega := \{x \in \bigcap_{j=1}^2 C_j : -tx_1^2 - |x_2| \leq 0, t \in \mathcal{T}\} = C_1 \cap C_2$. By performing a direct computation, we obtain

$$\begin{aligned} \mathcal{N}(\bar{x}; \Omega) &= \mathcal{N}(\bar{x}; C_1 \cap C_2) = \{(\alpha, 0) : \alpha \geq 0\}, \\ \mathcal{N}(\bar{x}; C_1) &= \{(0, 0)\}, \quad \mathcal{N}(\bar{x}; C_2) = \{(\alpha, 0) : \alpha \geq 0\}, \\ G_t(x) &:= \max_{v_t \in \mathcal{V}_t} g_t(x, v_t) = -tx_1^2 - |x_2|, \quad t \in \mathcal{T}, \quad G_t(\bar{x}) = 0, \\ \mathcal{U}_1(\bar{x}) &= \left\{ \frac{\pi}{2}, 2\pi \right\}, \quad \mathcal{U}_2(\bar{x}) = \left\{ \frac{1}{2} \right\}, \quad \mathcal{V}_t(\bar{x}) = [-t, t], t \in \mathcal{T}. \end{aligned}$$

Hence, we obtain $\partial G_t(\bar{x}) = \{(0, -1), (0, 1)\}$, which ensures that

$$\bigcup_{\lambda \in \Delta(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \partial G_t(\bar{x}) \right] + \mathcal{N}(\bar{x}; C_1 \cap C_2) = \bigcup_{\lambda \in \mathbb{R}_+^{|\mathcal{T}|}} \left[\sum_{t > 0} \{(0, -\lambda_t), (0, \lambda_t)\} \right] + \{(\alpha, 0) : \alpha \geq 0\}.$$

To validate RLCQ, we need to show that

$$\mathcal{N}(\bar{x}; \Omega) \subset \bigcup_{\lambda \in \mathbb{R}_+^{|\mathcal{T}|}} \left[\sum_{t > 0} \{(0, -\lambda_t), (0, \lambda_t)\} \right] + \mathcal{N}(\bar{x}; C_1 \cap C_2).$$

Since $\mathcal{N}(\bar{x}; C_1 \cap C_2) = \mathcal{N}(\bar{x}; \Omega)$, therefore RLCQ can be written as follows:

$$\{(0, 0)\} \subset \bigcup_{\lambda \in \mathbb{R}_+^{|\mathcal{T}|}} \left[\sum_{t > 0} \{(0, -\lambda_t), (0, \lambda_t)\} \right],$$

which implies that there exists $\lambda \in \mathbb{R}_+^{|\mathcal{T}|}$, such that

$$(0, 0) \in \sum_{t \in \mathcal{T}} \{(0, -\lambda_t), (0, \lambda_t)\}.$$

By taking $\lambda = (1, 1, 0, \dots)$, we have the following

$$(0, 0) \in \{(0, -2), (0, 0), (0, 2)\}.$$

Consequently, RLCQ holds at $\bar{x} = (0, 0)$. Now for each $u_1 \in \mathcal{U}_1(\bar{x})$, $u_2 \in \mathcal{U}_2(\bar{x})$, and $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$, we have

$$\begin{aligned} \partial_x f_1(\bar{x}, u_1) &= \{(\sin u_1, \cos u_1)\} \implies co\{\partial_x f_1(\bar{x}, u_1) : u_1 \in \mathcal{U}_1(\bar{x})\} = \{(\sin u_1, \cos u_1)\}, \\ \partial_x f_2(\bar{x}, u_2) &= [-1, 1] \times \{0\} \implies co\{\partial_x f_2(\bar{x}, u_2) : u_2 \in \mathcal{U}_2(\bar{x})\} = [-1, 1] \times \{0\}, \\ \partial_x g_t(\bar{x}, v_t) &= \{0\} \times \{-1, 1\} \implies co\{\partial_x g_t(\bar{x}, v_t) : v_t \in \mathcal{V}_t(\bar{x})\} = \{0\} \times [-1, 1]. \end{aligned}$$

Therefore, assumption (B2) holds.

By taking $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2}$, $\lambda = (1, 0, 0, \dots) \in \mathbb{R}_+^{|\mathcal{T}|}$, we obtain the following:

$$\begin{aligned} (0, 0) &\in \mu_1\{(\sin u_1, \cos u_1)\} + \mu_2([-1, 1] \times \{0\}) \\ &+ \sum_{t \in \mathcal{T}} \lambda_t(\{0\} \times [-1, 1]) + \{(\alpha, 0) : \alpha \geq 0\}, \\ \lambda_t(-tx_1^2 - |x_2|) &= 0, \quad t \in \mathcal{T}. \end{aligned}$$

Accordingly, this confirms that the conditions specified in Theorem 3.7 are satisfied.

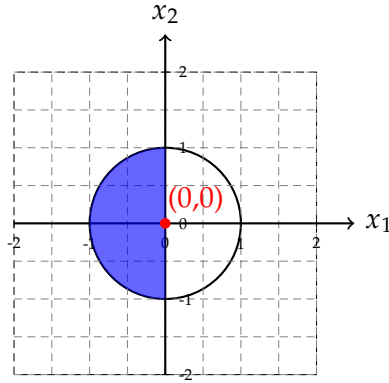


Figure 1. The blue shaded area represents the feasible region corresponding to Example 3.9.

To formulate sufficient conditions for robust weakly efficient solutions of the problem (UP), motivated by [12, 14, 16, 26, 57, 59], we discuss the following notion of generalized convexity. Recall here the notation, $F := (F_1, \dots, F_p)$, $G_{\mathcal{T}} := (G_t)_{t \in \mathcal{T}}$, $F_i(x) := \max_{u_i \in \mathcal{U}_i} f_i(x, u_i)$, $i \in \mathcal{P}$, and $G_t(x) := \max_{v_t \in \mathcal{V}_t} g_t(x, v_t)$, $t \in \mathcal{T}$.

Definition 3.10. The function $(F, G_{\mathcal{T}})$ is said to be

- (a) *generalized convex* on $\bigcap_{j=1}^k C_j$ at the point $\bar{x} \in \bigcap_{j=1}^k C_j$ iff for any $x \in \bigcap_{j=1}^k C_j$, there exists $\eta \in (\sum_{j=1}^k \mathcal{N}(\bar{x}; C_j))^{\circ}$, such that

$$\begin{aligned} f_i(x, u_i) - f_i(\bar{x}, u_i) &\geq \langle \bar{x}_{f_i}^*, \eta \rangle, \quad \forall \bar{x}_{f_i}^* \in \partial_x f_i(\bar{x}, u_i), \quad \forall u_i \in \mathcal{U}_i(\bar{x}), \quad i \in \mathcal{P}, \\ g_t(x, v_t) - g_t(\bar{x}, v_t) &\geq \langle \bar{x}_{g_t}^*, \eta \rangle, \quad \forall \bar{x}_{g_t}^* \in \partial_x g_t(\bar{x}, v_t), \quad \forall v_t \in \mathcal{V}_t(\bar{x}), \quad t \in \mathcal{T}; \end{aligned} \quad (19)$$

- (b) *generalized pseudoquasiconvex* on $\bigcap_{j=1}^k C_j$ at the point $\bar{x} \in \bigcap_{j=1}^k C_j$ iff for any $x \in \bigcap_{j=1}^k C_j$, $\bar{x}_{f_i}^* \in \partial_x f_i(\bar{x}, u_i)$, $u_i \in \mathcal{U}_i(\bar{x})$, $i \in \mathcal{P}$, and $\bar{x}_{g_t}^* \in \partial_x g_t(\bar{x}, v_t)$, $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$, there exists $\eta \in (\sum_{j=1}^k \mathcal{N}(\bar{x}; C_j))^{\circ}$, such that

$$\begin{aligned} f_i(x, u_i) < f_i(\bar{x}, u_i) &\implies \langle \bar{x}_{f_i}^*, \eta \rangle < 0, \quad i \in \mathcal{P}, \\ g_t(x, v_t) \leq g_t(\bar{x}, v_t) &\implies \langle \bar{x}_{g_t}^*, \eta \rangle \leq 0, \quad t \in \mathcal{T}. \end{aligned} \quad (20)$$

Remark 3.11. (i) It is significant to observe that when $\mathcal{U}_i(\bar{x})$, $i \in \mathcal{P}$, $\mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$ are singleton sets, then Definition 3.10 (a) reduces to [16, Definition 3.3 (i)].

(ii) When $p = 1$ and $\mathcal{T}$ is finite, then our definition reduces to [26, Definition 3.3].

(iii) Since, for locally Lipschitz functions, limiting subdifferentials are examples of convexificator (see, e.g., [52, Remark 2.2]), therefore, the generalized pseudoquasiconvex is analogous to [12, Definition 2.9].

(iv) In terms of Clarke subdifferential, if $C_j = \mathbb{R}^n$, $j \in \mathcal{K}$ and $\mathcal{T}$ is finite, then Definition 3.10 (a) is analogous to [59, Definition 3.3]. $\square$

The following example illustrates that generalized pseudoquasiconvexity holds, but generalized convexity does not.

Example 3.12. Let $x \in \mathbb{R}$, $C_1 = [0, 1]$, $u_1 \in \mathcal{U}_1 = [0, 1]$, $t \in \mathcal{T} = [0, 1]$, and $v_t \in \mathcal{V}_t = [1 - t, 1 + t]$, $t \in \mathcal{T}$. Consider the functions

$$f_1(x) := x + u_1, \quad g_t(x, v_t) := -tx^2 + v_t.$$

Then, $F_1(x) = x + 1$, $G_t(x) = -tx^2 + 1 + t$, $\mathcal{U}_1(x) = \{1\}$, $\mathcal{V}_t(x) = \{1 + t\}$. Let us consider, $\bar{x} = 1 \in C_1$, $u_1 = 1$, and $v_t = 1 + t$. It is easy to see that, $\partial_x f_1(\bar{x}, u_1) = \{1\}$, $\partial_x g_t(\bar{x}, v_t) = \{-2t\}$, and $(\mathcal{N}(\bar{x}; C_1))^\circ = (-\infty, 0]$. In this setting, we can verify that the generalized pseudoquasiconvexity of (F_1, G_t) , $t \in \mathcal{T}$ is satisfied at $\bar{x} = 1$, but the generalized convexity is not. $\square$

The subsequent theorem establishes the sufficient conditions for the optimality of (RP) under the generalized convexity of $(F, G_{\mathcal{T}})$.

Theorem 3.13. (Robust sufficient optimality condition) *Let $\bar{x} \in \Omega$ and assume that $(F, G_{\mathcal{T}})$ is generalized convex at $\bar{x} \in \bigcap_{j=1}^k C_j$. Assume that there exist some $\mu = (\mu_1, \dots, \mu_p) \in \mathbb{R}_+^p$ with $\sum_{i=1}^p \mu_i = 1$, $\lambda \in \mathbb{R}_+^{|\mathcal{T}|}$, such that condition (7) holds. Then $\bar{x} \in \Omega_{UP}^{rwe}$.*

Proof. Suppose to the contrary that $\bar{x} \notin \Omega_{UP}^{rwe}$. Then, there exists $\hat{x} \in \Omega$ such that

$$F_i(\hat{x}) < F_i(\bar{x}), \quad \forall i \in \mathcal{P},$$

$$\text{i.e.} \quad \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \quad \forall i \in \mathcal{P}.$$

Given that $\bar{x}$ fulfills the robust KKT condition, one can identify $\mu = (\mu_1, \dots, \mu_p) \in \mathbb{R}_+^p$ satisfying $\sum_{i=1}^p \mu_i = 1$, $\lambda \in \mathbb{R}_+^{|\mathcal{T}|}$, $\bar{x}_{f_i}^* \in \text{co} \partial_x f_i(\bar{x}, u_i)$, $u_i \in \mathcal{U}_i(\bar{x})$, $i \in \mathcal{P}$, and $\bar{x}_{g_t}^* \in \text{co} \partial_x g_t(\bar{x}, v_t)$, $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$, such that

$$-\left(\sum_{i=1}^p \mu_i \bar{x}_{f_i}^* + \sum_{t \in \mathcal{T}} \lambda_t \bar{x}_{g_t}^* \right) \in \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j), \quad (21)$$

$$\lambda_t \max_{v_t \in \mathcal{V}_t} g_t(\bar{x}, v_t) = 0, \quad t \in \mathcal{T}.$$

Therefore, there exist $\mu_{im} \geq 0$, $\bar{x}_{f_{im}}^* \in \partial_x f_i(\bar{x}, u_{im})$, $u_{im} \in \mathcal{U}_i(\bar{x})$, $m = 1, \dots, \alpha$, $\alpha \in \mathbb{N}$, $i \in \mathcal{P}$, $\lambda_{tl} \geq 0$, $\bar{x}_{g_{tl}}^* \in \partial_x g_t(\bar{x}, v_{tl})$, $v_{tl} \in \mathcal{V}_t(\bar{x})$, $l = 1, \dots, \beta$, $\beta \in \mathbb{N}$, $t \in \mathcal{T}$, such that

$$\sum_{m=1}^{\alpha} \mu_{im} = 1, i \in \mathcal{P}, \quad \sum_{l=1}^{\beta} \lambda_{tl} = 1, t \in \mathcal{T}, \quad (22)$$

$$\sum_{m=1}^{\alpha} \mu_{im} \bar{x}_{f_{im}}^* = \bar{x}_{f_i}^*, i \in \mathcal{P}, \quad \sum_{l=1}^{\beta} \lambda_{tl} \bar{x}_{g_{tl}}^* = \bar{x}_{g_t}^*, t \in \mathcal{T}.$$

From equation (21), it follows that

$$-\left(\sum_{i=1}^p \mu_i \sum_{m=1}^{\alpha} \mu_{im} \bar{x}_{f_{im}}^* + \sum_{t \in \mathcal{T}} \lambda_t \sum_{l=1}^{\beta} \lambda_{tl} \bar{x}_{g_{tl}}^* \right) \in \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j). \quad (23)$$

For any $\eta \in (\sum_{j=1}^k \mathcal{N}(\bar{x}; C_j))^\circ$, implies that

$$\sum_{i=1}^p \mu_i \sum_{m=1}^{\alpha} \mu_{im} \langle \bar{x}_{f_{im}}^*, \eta \rangle + \sum_{t \in \mathcal{T}} \lambda_t \sum_{l=1}^{\beta} \lambda_{tl} \langle \bar{x}_{g_{tl}}^*, \eta \rangle \geq 0. \quad (24)$$

By generalized convexity of $(F, G_{\mathcal{T}})$ at $\bar{x} \in \bigcap_{j=1}^k C_j$, for the above $\hat{x}$ one can find $\eta \in (\sum_{j=1}^k \mathcal{N}(\bar{x}; C_j))^\circ$ such that

$$\begin{aligned} & \sum_{i=1}^p \mu_i \left(\sum_{m=1}^{\alpha} \mu_{im} (f_i(\hat{x}, u_{im}) - f_i(\bar{x}, u_{im})) \right) \\ & + \sum_{t \in \mathcal{T}} \lambda_t \left(\sum_{l=1}^{\beta} \lambda_{tl} (g_t(\hat{x}, v_{tl}) - g_t(\bar{x}, v_{tl})) \right) \geq 0. \end{aligned} \quad (25)$$

Due to the fact that $\lambda_{tl} g_t(\bar{x}, v_{tl}) = 0$, $v_{tl} \in \mathcal{V}_t(\bar{x})$, $l = 1, \dots, \beta$, $t \in \mathcal{T}$ and from the feasibility of $\hat{x}$, i.e., $\lambda_{tl} g_t(\hat{x}, v_{tl}) \leq 0$, $l = 1, \dots, \beta$, $t \in \mathcal{T}$, we get

$$\sum_{i=1}^p \mu_i \left(\sum_{m=1}^{\alpha} \mu_{im} (f_i(\hat{x}, u_{im}) - f_i(\bar{x}, u_{im})) \right) \geq 0. \quad (26)$$

It is important to observe that, based on the definition of the max function $F_i(x)$, $i \in \mathcal{P}$, we obtain $f_i(\hat{x}, u_{im}) \leq F_i(\hat{x})$, $f_i(\bar{x}, u_{im}) = F_i(\bar{x})$, $m = 1, \dots, \alpha$, $i \in \mathcal{P}$. It follows that

$$F_i(\hat{x}) - F_i(\bar{x}) \geq 0. \quad (27)$$

This yields a contradiction, which establishes the result and completes the proof. $\square$

The subsequent theorem establishes the sufficient conditions for the optimality of the (RP) under the generalized pseudoquasiconvexity of $(F, G_{\mathcal{T}})$.

Theorem 3.14. Let $\bar{x} \in \Omega$ and assume that $(F, G_{\mathcal{T}})$ is generalized pseudoquasiconvex at $\bar{x} \in \bigcap_{j=1}^k C_j$. Assume that there exist $\mu = (\mu_1, \dots, \mu_p) \in \mathbb{R}_+^p$ with $\sum_{i=1}^p \mu_i = 1$, $\lambda \in \mathbb{R}_+^{|\mathcal{T}|}$, such that condition (7) holds. Then $\bar{x} \in \Omega_{UP}^{rwe}$.

Proof. Suppose to contrary that $\bar{x} \notin \Omega_{UP}^{rwe}$. Then, there exists $\hat{x} \in \Omega$ such that

$$\begin{aligned} & F_i(\hat{x}) < F_i(\bar{x}), \quad \forall i \in \mathcal{P}, \\ \text{i.e.} \quad & \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \quad \forall i \in \mathcal{P}, \end{aligned}$$

which implies that

$$\max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < f_i(\bar{x}, u_i), \quad \forall u_i \in \mathcal{U}_i(\bar{x}), \quad \forall i \in \mathcal{P},$$

which gives

$$f_i(\hat{x}, u_i) < f_i(\bar{x}, u_i), \quad \forall u_i \in \mathcal{U}_i(\bar{x}), \quad \forall i \in \mathcal{P}.$$

Subsequently, from the feasibility of $\hat{x}$, one has

$$G_t(\hat{x}) \leq 0 = G_t(\bar{x}), \quad \forall t \in \mathcal{T}(\bar{x}),$$

which gives

$$g_t(\hat{x}, v_t) \leq g_t(\bar{x}, v_t), \quad \forall v_t \in \mathcal{V}_t, \quad \forall t \in \mathcal{T}(\bar{x}).$$

By the generalized pseudoquasiconvexity of $(F, G_{\mathcal{T}})$ at $\bar{x} \in \bigcap_{j=1}^k C_j$, there exist $\eta \in (\sum_{j=1}^k \mathcal{N}(\bar{x}; C_j))^\circ$, $\bar{x}_{f_i}^* \in \partial_x f_i(\bar{x}, u_i)$, $u_i \in \mathcal{U}_i(\bar{x})$, $i \in \mathcal{P}$, and $\bar{x}_{g_t}^* \in \partial_x g_t(\bar{x}, v_t)$, $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}(\bar{x})$, it follows that

$$\langle \bar{x}_{f_i}^*, \eta \rangle < 0, \quad i \in \mathcal{P}, \quad \langle \bar{x}_{g_t}^*, \eta \rangle \leq 0, \quad t \in \mathcal{T}(\bar{x}). \quad (28)$$

Hence, for any $\bar{x}_{f_i}^* \in \text{cod}_x f_i(\zeta, u_i)$, $u_i \in \mathcal{U}_i(\bar{x})$, $i \in \mathcal{P}$, $\bar{x}_{g_t}^* \in \text{cod}_x g_t(\bar{x}, v_t)$, $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}(\bar{x})$, and $d \in \sum_{j=1}^k \mathcal{N}(\bar{x}; C_j)$, one has

$$\left\langle \sum_{i=1}^p \mu_i \bar{x}_{f_i}^* + \sum_{t \in \mathcal{T}(\bar{x})} \lambda_t \bar{x}_{g_t}^* + d, \eta \right\rangle < 0, \quad (29)$$

which is a contradiction to condition (7) and hence the proof. $\square$

The example below demonstrates the importance of the generalized convexity assumption.

Example 3.15. Consider the following (UP) under data uncertainty:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & (f_1(x, u_1), f_2(x, u_2)) \\ \text{s.t.} \quad & x \in C_1, \quad g_t(x, v_t) \leq 0, \quad t \in \mathcal{T}, \end{aligned}$$

where $f_1(x, u_1) := u_1 x_1 + x_2^3$, $f_2(x, u_2) := x_1^3 - u_2 x_2$, $C_1 := \{x \in \mathbb{R}^2 : x_2 \geq -|x_1|\}$, $g_t(x, v_t) := (v_t - 2t)(x_1^2 + x_2^2)$, $u_1 \in [0, 1]$, $u_2 \in [-1, 0]$, $v_t \in \mathcal{V}_t$, $\mathcal{V}_t := [-t, t]$, $t \in \mathcal{T} := (0, +\infty)$, and $x \in \mathbb{R}^2$.

The corresponding robust counterpart (RP) is given by:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \left(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2) \right) \\ \text{s.t.} \quad & x \in \{x \in \mathbb{R}^2 : x_2 \geq -|x_1|\}, \\ & g_t(x, v_t) := (v_t - 2t)(x_1^2 + x_2^2) \leq 0, \quad \forall v_t \in \mathcal{V}_t, t \in \mathcal{T}, \end{aligned}$$

where
$$F_1(x) := \max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) := \begin{cases} x_1 + x_2^3, & x_1 > 0 \\ x_2^3, & x_1 \leq 0 \end{cases}$$

and
$$F_2(x) := \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2) := \begin{cases} x_1^3 + x_2, & x_1 > 0 \\ x_1^3, & x_1 \leq 0 \end{cases}.$$

It is easy to see that assumptions (B1) and (B2) are automatically satisfied. Therefore, $\Omega := \{x \in C_1 : -t(x_1^2 + x_2^2) \leq 0, t \in \mathcal{T}\} = C_1$. By performing a direct computation at $\bar{x} = (0, 0)$, we derive

$$\mathcal{N}(\bar{x}; \Omega) = \mathcal{N}(\bar{x}; C_1) := \{(\alpha, \alpha) : \alpha \leq 0\} \cup \{(\alpha, -\alpha) : \alpha \geq 0\},$$

$$G_t(x) := \max_{v_t \in \mathcal{V}_t} g_t(x, v_t) = -t(x_1^2 + x_2^2), \quad t \in \mathcal{T}, \quad G_t(\bar{x}) = 0,$$

$$\mathcal{U}_1(\bar{x}) = [0, 1], \quad \mathcal{U}_2(\bar{x}) = [-1, 0], \quad \mathcal{V}_t(\bar{x}) = [-t, t], \quad t \in \mathcal{T}.$$

Hence, we obtain $\nabla G_t(\bar{x}) = \{(0, 0)\}$, which ensures that

$$\begin{aligned} & \bigcup_{\lambda \in \Delta(\bar{x})} \left[\sum_{t \in \mathcal{T}} \lambda_t \nabla G_t(\bar{x}) \right] + \mathcal{N}(\bar{x}; C_1) \\ &= \bigcup_{\lambda \in \mathbb{R}_+^{|\mathcal{T}|}} \left[\sum_{t \geq 0} \{ (0, 0) \} \right] + \{(\alpha, \alpha) : \alpha \leq 0\} \cup \{(\alpha, -\alpha) : \alpha \geq 0\}. \end{aligned}$$

Consequently, RLCQ holds at $\bar{x} = (0, 0)$. Now for each $u_1 \in \mathcal{U}_1(\bar{x})$, $u_2 \in \mathcal{U}_2(\bar{x})$, and $v_t \in \mathcal{V}_t(\bar{x})$, $t \in \mathcal{T}$, we have

$$\nabla_x f_1(\bar{x}, u_1) = \{(u_1, 0)\}, \quad \nabla_x f_2(\bar{x}, u_2) = \{(0, u_2)\}, \quad \nabla_x g_t(\bar{x}, v_t) = (0, 0).$$

Therefore, assumption (B2) holds. By taking, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2}$, $\lambda_t = (1, 0, 0, \dots) \in \mathbb{R}_+^{|\mathcal{T}|}$, we have the following

$$\begin{aligned} & (0, 0) \in \mu_1 \{(u_1, 0)\} + \mu_2 \{(0, u_2)\} \\ & + \sum_{t \in \mathcal{T}} \lambda_t (0, 0) + \{(\alpha, \alpha) : \alpha \leq 0\} \cup \{(\alpha, -\alpha) : \alpha \geq 0\}, \\ & \lambda_t \max_{v_t \in \mathcal{V}_t} g_t(\bar{x}, v_t) = 0, \quad t \in \mathcal{T}. \end{aligned}$$

Hence, from Theorem 3.7, $\bar{x}$ may be an optimal solution for the (UP). However, for any $x^* = (-2, 0) \in \Omega = C_1$, we observe that

$$F(x^*) - F(\bar{x}) = (-2, -8) - (0, 0) \in -\text{int}\mathbb{R}_+^2.$$

This implies that $\bar{x} = (0, 0)$ is not a robust weakly efficient solution of the (UP). Consequently, Theorem 3.13 fails to hold in this case, which highlights the necessity of the generalized convexity at x^* .

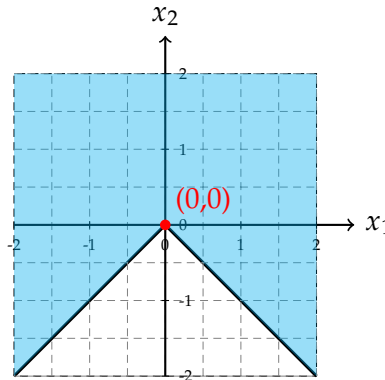


Figure 2. The cyan shaded area represents the nonconvex feasible region corresponding to Example 3.15.

4. Robust duality

The notion of duality is of extreme importance in optimization theory [10, 25, 51]. A mixed-type dual model, incorporating both the Wolfe type [64] and Mond-Weir type [44] dual formulations as special cases, was introduced by Xu [65]. Many researchers have extensively studied the mixed-type dual model in recent years [4, 21, 32, 42, 53].

In the framework of robust optimization, Boş et al. [11] established robust duality for the family of parametric convex optimization problems. Moreover, Wang et al. [63] have developed a robust duality theory for uncertain nonconvex optimization problems. Building on this, Chen et al. [12] developed a mixed-type robust dual model for uncertain multiobjective optimization problems by employing convexifiers. Sun et al. [58, 60] developed mixed-type robust dual models for uncertain single-objective optimization problems using the Clarke subdifferential. These developments were extended in [57] to uncertain semi-infinite single-objective optimization problems. Subsequently, generalization of these developments to the more complex setting of uncertain semi-infinite multiobjective optimization problems, within the framework of the Clarke subdifferential, is given in [56, 59].

Since the robust primal (RP) and the mixed-type robust dual (OD) problems attain equal optimal objective values under appropriate convexity assumptions and suitable constraint qualifications. Because of this, instead of solving the primal problem directly, we can solve its dual problem, which is often easier to handle (see e.g., [14, 30, 55]). Motivated by this, the mixed-type robust dual model (OD) is introduced to extend duality theory to a broader class of problems by combining features of both Wolfe type and Mond-Weir type dual formulations. Within this framework, the Lagrangian function $\mathcal{L}$ serves as a bridge between the uncertain primal and dual problems by incorporating the constraints into the objective function.

In this section, we introduce a mixed-type robust dual problem corresponding to the uncertain optimization problem (UP). Under the assumption of generalized convexity, we establish both robust weak and strong duality relationships between primal and dual formulations. The mixed-type optimistic dual optimization problem is expressed as a deterministic maximization problem, which can be formulated as follows:

$$\begin{aligned} \text{(OD)} \quad & \max \mathcal{L}(y, u, \lambda, v) := \left(f_1(y, u_1) + \sum_{t \in \mathcal{T}} \lambda_t g_t(y, v_t), \dots, f_p(y, u_p) + \sum_{t \in \mathcal{T}} \lambda_t g_t(y, v_t) \right) \\ & \text{s.t. } (y, \mu, u, \lambda, \beta, v) \in \Omega_{OD}, \end{aligned}$$

where the feasible set Ω_{OD} is defined by

$$\begin{aligned} \Omega_{OD} := & \left\{ (y, \mu, u, \lambda, \beta, v) \in \bigcap_{j=1}^k C_j \times \mathbb{R}_+^p \times \mathcal{U} \times \mathbb{R}_+^{2|\mathcal{T}|} \times \mathcal{V} : \right. \\ & 0 \in \sum_{i=1}^p \mu_i \text{co} \left\{ \partial_x f_i(y, u_i) : u_i \in \mathcal{U}_i(y) \right\} + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) \text{co} \left\{ \partial_x g_t(y, v_t) : v_t \in \mathcal{V}_t(y) \right\} \\ & \left. + \sum_{j=1}^k \mathcal{N}(y; C_j), \quad \sum_{i=1}^p \mu_i = 1, \quad \beta_t g_t(y, v_t) \geq 0, \quad \forall v_t \in \mathcal{V}_t(y), t \in \mathcal{T} \right\}. \end{aligned}$$

Remark 4.1. (i) If $\lambda_t = 0, t \in \mathcal{T}$, (OD) reduces to the following uncertain Mond-Weir type dual problem:

$$\begin{aligned} \max \quad & \mathcal{L}(y, u) := (f_1(y, u_1), \dots, f_p(y, u_p)) \\ \text{s.t.} \quad & 0 \in \sum_{i=1}^p \mu_i \text{co} \left\{ \partial_x f_i(y, u_i) : u_i \in \mathcal{U}_i(y) \right\} + \sum_{t \in \mathcal{T}} \beta_t \text{co} \left\{ \partial_x g_t(y, v_t) : v_t \in \mathcal{V}_t(y) \right\} \\ & + \sum_{j=1}^k \mathcal{N}(y; C_j), \quad \sum_{i=1}^p \mu_i = 1, \\ & \beta_t g_t(y, v_t) \geq 0, \quad \forall v_t \in \mathcal{V}_t(y), t \in \mathcal{T}; \end{aligned}$$

(ii) If $\beta_t = 0, t \in \mathcal{T}$, (OD) reduces to the following uncertain Wolfe type dual problem:

$$\begin{aligned} \max \quad & \mathcal{L}(y, u, \lambda, v) := (f_1(y, u_1) + \sum_{t \in \mathcal{T}} \lambda_t g_t(y, v_t), \dots, f_p(y, u_p) + \sum_{t \in \mathcal{T}} \lambda_t g_t(y, v_t)) \\ \text{s.t.} \quad & 0 \in \sum_{i=1}^p \mu_i \text{co} \left\{ \partial_x f_i(y, u_i) : u_i \in \mathcal{U}_i(y) \right\} + \sum_{t \in \mathcal{T}} \beta_t \text{co} \left\{ \partial_x g_t(y, v_t) : v_t \in \mathcal{V}_t(y) \right\} \\ & + \sum_{j=1}^k \mathcal{N}(y; C_j), \quad \sum_{i=1}^p \mu_i = 1. \end{aligned}$$

Building upon the concept of a robust weakly efficient solution in (UP), we now introduce a corresponding robust weakly efficient solution for (OD), which will serve as a foundation for the subsequent discussion.

Definition 4.2. A point $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{v}) \in \Omega_{OD}$ is a *robust weakly efficient solution* of the (OD) iff $\mathcal{L}(y, u, \lambda, v) \not\prec \mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}, \bar{v})$ for all $y \in \Omega_{OD}$.

To give a robust weak duality theorem, we need the following concept of generalized pseudoconvexity.

Definition 4.3. A function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *generalized pseudoconvex* on $\bigcap_{j=1}^k C_j$ at $\bar{y} \in \bigcap_{j=1}^k C_j$ iff for any $y \in \bigcap_{j=1}^k C_j$, there exists $\eta \in (\sum_{j=1}^k \mathcal{N}(\bar{y}; C_j))^\circ$, such that

$$h(y) < h(\bar{y}) \implies \langle y_h^*, \eta \rangle < 0, \quad \forall y_h^* \in \partial h(\bar{y}).$$

Remark 4.4. Observe that if h is pseudoconvex then h is generalized pseudoconvex on $\bigcap_{j=1}^k C_j$ at $\bar{y} \in \bigcap_{j=1}^k C_j$ with $\eta := y - \bar{y}$ for each $y \in \bigcap_{j=1}^k C_j$. Moreover, as the next example shows, the converse is not true in general. $\square$

Example 4.5. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$h(y) = \begin{cases} -\frac{y_2}{2}, & y_2 \geq 0, \\ y_1 - y_2, & y_2 < 0. \end{cases}$$

Consider the set

$$C_1 = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 2\},$$

and the point $\bar{y} = (0, 0) \in C_1$. Since $\bar{y}$ is an interior point of C_1 , therefore the normal cone satisfies $\mathcal{N}(\bar{y}; C_1) = \{(0, 0)\}$, and hence $(\mathcal{N}(\bar{y}; C_1))^\circ = \mathbb{R}$. The limiting subdifferential of h at $\bar{y}$ is given by

$$\partial h(\bar{y}) = \left\{ \left(0, -\frac{1}{2}\right), (0, -1) \right\}.$$

It is easy to observe that h is generalized pseudoconvex on C_1 at $\bar{y} \in C_1$. Note that for any y with $y_2 \geq 0$, $h(y) < h(0) = 0$. Take $y = (1, 1) \in C_1$. Then $h(y) < h(0)$, but $\langle (1, -1), y \rangle \not\leq 0$. Hence h is not pseudoconvex on C_1 at $\bar{y} \in C_1$. $\square$

Define a function $\mathcal{H} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathcal{H}(y) = \sum_{i=1}^p \mu_i F_i(y) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) G_t(y).$$

Now, we present the robust weak duality theorem to ensure that, even in the worst-case scenario, the robust dual objective functions provide a bound on the robust primal objective functions.

Theorem 4.6. (Robust Weak Duality) *Let $x \in \Omega$, and $(y, \mu, u, \lambda, \beta, v) \in \Omega_{OD}$ along with $u := (u_1, \dots, u_i)$, $u_i \in \mathcal{U}_i(y)$, $i \in \mathcal{P}$, $v := (v_1, \dots, v_t)$, $v_t \in \mathcal{V}_t(y)$, $t \in \mathcal{T}$. If one of the following holds:*

- (i) *If the function $(F, G_{\mathcal{T}})$ is generalized convex at y .*
- (ii) *Under assumption (B1) and (B2), if function $\mathcal{H}$ is generalized pseudoconvex at y .*

Then $F(x) \not\leq \mathcal{L}(y, u, \lambda, v)$.

Proof. Since $(y, u, \lambda, v) \in \Omega_{OD}$ there exist $\mu \in \mathbb{R}_+^p$ with $\sum_{i=1}^p \mu_i = 1$,

$$\lambda, \beta \in \mathbb{R}_+^{|\mathcal{T}|}, y_{f_i}^* \in \text{co}\{\partial_x f_i(y, u_i) : u_i \in \mathcal{U}_i(y)\}, y_{g_t}^* \in \text{co}\{\partial_x g_t(y, v_t) : v_t \in \mathcal{V}_t(y)\} \quad (30)$$

such that

$$\begin{aligned} -\left(\sum_{i=1}^p \mu_i y_{f_i}^* + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) y_{g_t}^* \right) &\in \sum_{j=1}^k \mathcal{N}(y; C_j), \\ \beta_t g_t(y, v_t) &\geq 0, \quad \forall v_t \in \mathcal{V}_t(y), t \in \mathcal{T}. \end{aligned} \quad (31)$$

Assume to the contrary that $F(x) < \mathcal{L}(y, u, \lambda, v)$.

Then, we get

$$F_i(x) < f_i(y, u_i) + \sum_{t \in \mathcal{T}} \lambda_t g_t(y, v_t), \quad \forall i \in \mathcal{P}.$$

This implies

$$\sum_{i=1}^p \mu_i F_i(x) < \sum_{i=1}^p \mu_i f_i(y, u_i) + \sum_{t \in \mathcal{T}} \lambda_t g_t(y, v_t).$$

Since we have by definition $f_i(y, u_i) \leq \max_{u_i \in \mathcal{U}_i} f_i(y, u_i) = F_i(y)$, $i \in \mathcal{P}$, and $g_t(y, v_t) \leq \max_{v_t \in \mathcal{V}_t} g_t(y, v_t) = G_t(y)$, $t \in \mathcal{T}$, it follows that

$$\sum_{i=1}^p \mu_i F_i(x) < \sum_{i=1}^p \mu_i F_i(y) + \sum_{t \in \mathcal{T}} \lambda_t G_t(y). \quad (32)$$

From (30), it follows that there exist $\mu_{im} \geq 0$, $y_{f_{im}}^* \in \partial_x f_i(y, u_{im})$, $u_{im} \in \mathcal{U}_i(y)$, $m = 1, \dots, \alpha$, $\alpha \in \mathbb{N}$, $i \in \mathcal{P}$, $\lambda_{tl} \geq 0$, $\beta_{tl} \geq 0$, $y_{g_{tl}}^* \in \partial_x g_t(y, v_{tl})$, $v_{tl} \in \mathcal{V}_t(y)$, $l = 1, \dots, \delta$, $\delta \in \mathbb{N}$, $t \in \mathcal{T}$, such that

$$\begin{aligned}
\sum_{m=1}^{\alpha} \mu_{im} &= 1, i \in \mathcal{P}, \quad \sum_{l=1}^{\delta} (\lambda_{tl} + \beta_{tl}) = 1, t \in \mathcal{T}, \\
\sum_{m=1}^{\alpha} \mu_{im} y_{f_{im}}^* &= y_{f_i}^*, i \in \mathcal{P}, \quad \sum_{l=1}^{\delta} (\lambda_{tl} + \beta_{tl}) y_{g_{tl}}^* = y_{g_t}^*, t \in \mathcal{T}.
\end{aligned} \tag{33}$$

(i) Under generalized convexity at $y \in \bigcap_{j=1}^k C_j$, for any $x \in \bigcap_{j=1}^k C_j$, there exist $\eta \in (\sum_{j=1}^k \mathcal{N}(y; C_j))^{\circ}$, $u_{f_i}^* \in \partial_x f_i(y, u_i)$, $u_i \in \mathcal{U}_i(y)$, $i \in \mathcal{P}$, and $v_{g_t}^* \in \partial_x g_t(y, v_t)$, $v_t \in \mathcal{V}_t(y)$, $t \in \mathcal{T}$, we get

$$\begin{aligned}
f_i(x, u_i) - f_i(y, u_i) &\geq \langle u_{f_i}^*, \eta \rangle, \quad i \in \mathcal{P}, \\
g_t(x, v_t) - g_t(y, v_t) &\geq \langle v_{g_t}^*, \eta \rangle, \quad t \in \mathcal{T}.
\end{aligned} \tag{34}$$

Consequently, from (33) we get

$$\begin{aligned}
\langle y_{f_i}^*, \eta \rangle &= \sum_{m=1}^{\alpha} \mu_{im} \langle y_{f_{im}}^*, \eta \rangle \leq \sum_{m=1}^{\alpha} \mu_{im} (f_i(x, u_{im}) - f_i(y, u_{im})), \quad i \in \mathcal{P}, \\
\langle y_{g_t}^*, \eta \rangle &= \sum_{l=1}^{\delta} \lambda_{tl} \langle y_{g_{tl}}^*, \eta \rangle \leq \sum_{l=1}^{\delta} (\lambda_{tl} + \beta_{tl}) (g_t(x, v_{tl}) - g_t(y, v_{tl})), \quad t \in \mathcal{T}.
\end{aligned} \tag{35}$$

Following the definition of max function, we arrive at $f_i(x, u_{im}) \leq F_i(x)$, $f_i(y, u_{im}) \leq F_i(y)$, $m = 1, \dots, \alpha$, $i \in \mathcal{P}$, $g_t(x, v_{tl}) \leq G_t(x)$, $g_t(y, v_{tl}) \leq G_t(y)$, $l = 1, \dots, \delta$, $t \in \mathcal{T}$, it follows that

$$\begin{aligned}
\langle y_{f_i}^*, \eta \rangle &\leq \sum_{m=1}^{\alpha} \mu_{im} (F_i(x) - F_i(y)), \\
\langle y_{g_t}^*, \eta \rangle &\leq \sum_{l=1}^{\delta} (\lambda_{tl} + \beta_{tl}) (G_t(x) - G_t(y)).
\end{aligned} \tag{36}$$

Using the previous inequalities, we derive:

$$\begin{aligned}
0 &\leq \sum_{i=1}^p \mu_i \langle y_{f_i}^*, \eta \rangle + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) \langle y_{g_t}^*, \eta \rangle \\
&\leq \sum_{i=1}^p \mu_i (F_i(x) - F_i(y)) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) (G_t(x) - G_t(y)),
\end{aligned} \tag{37}$$

where the first inequality follows from equation (31) together with the definition of the polar cone. Rearranging, we obtain

$$\sum_{i=1}^p \mu_i F_i(x) \geq \sum_{i=1}^p \mu_i F_i(y) - \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) G_t(x) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) G_t(y). \tag{38}$$

Since x is a feasible point for the primal problem and y is a feasible point for the dual problem, respectively, therefore we have

$$G_t(x) \leq 0, \quad \beta_t g_t(y, v_t) \geq 0, \quad \forall v_t \in \mathcal{V}_t(y), t \in \mathcal{T}.$$

Thus, it follows that

$$\sum_{i=1}^p \mu_i F_i(x) \geq \sum_{i=1}^p \mu_i F_i(y) + \sum_{t \in \mathcal{T}} \lambda_t G_t(y).$$

This contradicts our initial assumption. Hence, the claim is proved.

(ii) Since $x \in \Omega$ and $(y, \mu, u, \lambda, \beta, v) \in \Omega_{OD}$, we have

$$g_t(x, v_t) \leq 0, \quad \beta_t g_t(y, v_t) \geq 0, \quad \forall v_t \in \mathcal{V}_t, \quad t \in \mathcal{T}.$$

By definition of the maximum functions $G_t, t \in \mathcal{T}$, it follows that

$$g_t(x, v_t) \leq \max_{v_t \in \mathcal{V}_t} g_t(x, v_t) = G_t(x), \quad g_t(y, v_t) \leq \max_{v_t \in \mathcal{V}_t} g_t(y, v_t) = G_t(y), \quad t \in \mathcal{T}.$$

Consequently, we get

$$G_t(x) \leq 0, \quad \beta_t G_t(y) \geq 0, \quad t \in \mathcal{T}. \quad (39)$$

Since $\lambda, \beta \in \mathbb{R}_+^{|\mathcal{T}|}$, therefore using equation (39) in (32), we obtain

$$\sum_{i=1}^p \mu_i F_i(x) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) G_t(x) < \sum_{i=1}^p \mu_i F_i(y) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) G_t(y). \quad (40)$$

From the definition of $\mathcal{H}$, it follows from (40) that $\mathcal{H}(x) < \mathcal{H}(y)$. By invoking generalized pseudoconvexity of $\mathcal{H}$ at y , we obtain

$$\langle \xi, \eta \rangle < 0, \quad \forall \xi \in \partial \mathcal{H}(\bar{y}).$$

Under assumption (B1), $F_i, i \in \mathcal{P}$, and $G_t, t \in \mathcal{T}$, are locally Lipschitz. Hence, from Lemma 2.1, one has

$$\partial \mathcal{H}(\bar{y}) = \partial \left(\sum_{i=1}^p \mu_i F_i(y) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) G_t(y) \right) \subset \sum_{i=1}^p \mu_i \partial F_i(y) + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) \partial G_t(y).$$

Furthermore, under assumptions (B1) and (B2), and using equations (16)–(17), we deduce

$$\partial \mathcal{H}(\bar{y}) \subset \sum_{i=1}^p \mu_i \text{co} \left\{ \partial_x f_i(y, u_i) : u_i \in \mathcal{U}_i(y) \right\} + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) \text{co} \left\{ \partial_x g_t(y, v_t) : v_t \in \mathcal{V}_t(y) \right\}.$$

Therefore, there exist

$$y_{f_i}^* \in \text{co} \left\{ \partial_x f_i(y, u_i) : u_i \in \mathcal{U}_i(y) \right\} \quad \text{and} \quad y_{g_t}^* \in \text{co} \left\{ \partial_x g_t(y, v_t) : v_t \in \mathcal{V}_t(y) \right\}$$

$$\text{such that} \quad \left\langle \sum_{i=1}^p \mu_i y_{f_i}^* + \sum_{t \in \mathcal{T}} (\lambda_t + \beta_t) y_{g_t}^*, \eta \right\rangle < 0.$$

which contradicts (31). Hence, the claim is proved. $\square$

To emphasize the role of generalized convexity of $(F, G_{\mathcal{T}})$ in the analysis, consider the following example.

Example 4.7. Consider the following primal problem (UP) under data uncertainty:

$$\begin{aligned} \min \quad & (f_1(x, u_1), f_2(x, u_2)) := (x^5 + u_1, x^3 + u_2) \\ \text{s.t.} \quad & x \in C_1 \cap C_2, \quad g_t(x, v_t) := (v_t - 2t)|x| \leq 0, \quad t \in \mathcal{T}, \end{aligned}$$

for $u_1 \in [0, 1]$, $u_2 \in [2, 3]$, $v_t \in \mathcal{V}_t := [-t, t]$, $t \in \mathcal{T} := (0, +\infty)$, $C_1 = [-1, 1]$, $C_2 = [-2, 0]$ and $x \in \mathbb{R}$. The feasible set for the primal problem is given by $\Omega := [-1, 0]$. Let $y \in \mathbb{R}$. We get $\mathcal{U}_1(y) = \{1\}$, $\mathcal{U}_2(y) = \{3\}$, $\mathcal{V}_t(y) = \{t\}$. At this point, for $\lambda, \beta \in \mathbb{R}_+^{|\mathcal{T}|}$, we introduce the dual problem (OD) for further analysis.

$$\begin{aligned} \min \quad & \mathcal{L}(y, u, \lambda, v) := (y^5 + u_1 + \sum_{t \geq 0} \lambda_t (v_t - 2t)|y|, y^3 + u_2 + \sum_{t \geq 0} \lambda_t (v_t - 2t)|y|) \\ \text{s.t.} \quad & 0 \in \mu_1 \partial_x f_1(y, 1) + \mu_2 \partial_x f_2(y, 3) + \sum_{t \geq 0} (\lambda_t + \beta_t) \partial_x g_t(y, t) + \mathcal{N}(y; C_1) + \mathcal{N}(y; C_2), \\ & \beta_t g_t(y, v_t) := \beta_t (v_t - 2t)|y| \geq 0, \quad \forall v_t \in \mathcal{V}_t(y) = \{t\}, t \in \mathcal{T}. \end{aligned}$$

By choosing $\bar{x} = -1 \in \Omega$, $\bar{y} = 0$, $\bar{u}_1 = 1$, $\bar{u}_2 = 3$, $\bar{\lambda}_t = \bar{\beta}_t = (1, 0, \dots)$, $\bar{v}_t = t$, it holds that $(\bar{y}, \bar{u}_1, \bar{u}_2, \bar{\lambda}_t, \bar{\beta}_t, \bar{v}_t) \in \Omega_{OD}$ and that

$$F(\bar{x}) = (0, 2) < (1, 3) = \mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}, \bar{v}),$$

which shows that Theorem 4.6 fails since $(F, G_{\mathcal{T}})$ is not generalized convex at $\bar{y}$. $\square$

We establish a robust strong duality relationship between the primal and dual problems as follows.

Theorem 4.8. (Robust Strong Duality) *Let $\bar{y} \in \Omega_{UP}^{lrwe}$. Assume that the conditions (B1) and (B2) hold at $\bar{y}$. Suppose that RLCQ is satisfied at $\bar{y}$, and equation $w_1 + \dots + w_k = 0$, where $w_j \in \mathcal{N}(\bar{x}; C_j)$, $j \in \mathcal{K}$, admits only a trivial solution $w_j = 0$, $j \in \mathcal{K}$. Then there exists $(\bar{\mu}, \bar{u}, \bar{\lambda}, \bar{v}) \in \mathbb{R}_+^p \setminus \{0\} \times \mathcal{U} \times \mathbb{R}_+^{|\mathcal{T}|} \times \mathcal{V}$ such that $(\bar{y}, \bar{\mu}, \bar{u}, \bar{\lambda}, 0, \bar{v}), (\bar{y}, \bar{\mu}, \bar{u}, 0, \bar{\lambda}, \bar{v}) \in \Omega_{DOUM}$ and $F(\bar{y}) = \mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}, \bar{v})$. Furthermore, if $(F, G_{\mathcal{T}})$ is generalized convex at $y \in \bigcap_{j=1}^k C_j$, then $(\bar{y}, \bar{\mu}, \bar{u}, \bar{\lambda}, 0, \bar{v})$ and $(\bar{y}, \bar{\mu}, \bar{u}, 0, \bar{\lambda}, \bar{v})$ are robust weakly efficient solutions of (OD).*

Proof. Since $\bar{y} \in \Omega_{UP}^{lrwe}$, it follows that $\bar{y} \in \bigcap_{j=1}^k C_j$. From Theorem 3.7, there exist $\bar{\mu} \in \mathbb{R}_+$ with $\sum_{i=1}^p \bar{\mu}_i = 1$, $\bar{\lambda} \in \mathbb{R}_+^{|\mathcal{T}|}$, such that

$$\begin{aligned} 0 \in & \sum_{i=1}^p \bar{\mu}_i \text{co} \left\{ \partial_x f_i(\bar{y}, \bar{u}_i) : \bar{u}_i \in \mathcal{U}_i(\bar{y}) \right\} \\ & + \sum_{t \in \mathcal{T}} \bar{\lambda}_t \text{co} \left\{ \partial_x g_t(\bar{y}, \bar{v}_t) : \bar{v}_t \in \mathcal{V}_t(\bar{y}) \right\} + \sum_{j=1}^k \mathcal{N}(\bar{y}; C_j), \\ & \bar{\lambda}_t \max_{\bar{v}_t \in \mathcal{V}_t} g_t(\bar{y}, \bar{v}_t) = 0, \quad t \in \mathcal{T}. \end{aligned} \quad (41)$$

Consequently, $(\bar{y}, \bar{\mu}, \bar{u}, \bar{\lambda}, 0, \bar{v})$ and $(\bar{y}, \bar{\mu}, \bar{u}, 0, \bar{\lambda}, \bar{v})$ are feasible solutions to (OD). From the conditions $\bar{\lambda}_t g_t(\bar{y}, \bar{v}_t) = 0$, $t \in \mathcal{T}$ and $f_i(\bar{y}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} f_i(\bar{y}, u_i)$, $i \in \mathcal{P}$, it follows that:

$$(f_1(\bar{y}, \bar{u}_1), \dots, f_p(\bar{y}, \bar{u}_p)) = \mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}, \bar{v}). \quad (42)$$

Since $(F, G_{\mathcal{T}})$ is generalized convex at $y \in \bigcap_{j=1}^k C_j$, therefore it follows from Theorem 4.6 that for any feasible solution $(y, \mu, u, \lambda, \beta, v)$ of (OD):

$$(f_1(\bar{y}, \bar{u}_1), \dots, f_p(\bar{y}, \bar{u}_p)) \not\leq \mathcal{L}(y, u, \lambda, v). \quad (43)$$

Thus, from equation (42), we obtain:

$$\mathcal{L}(\bar{y}, \bar{u}, \bar{\lambda}, \bar{v}) \not\leq \mathcal{L}(y, u, \lambda, v).$$

Thus, $(\bar{y}, \bar{\mu}, \bar{u}, \bar{\lambda}, 0, \bar{v})$ and $(\bar{y}, \bar{\mu}, \bar{u}, 0, \bar{\lambda}, \bar{v})$ are robust weakly efficient solutions of (OD). This completes the proof. $\square$

The following example demonstrates that the condition RLCQ is essential in Theorem 4.8.

Example 4.9. Consider the following primal problem (UP) under data uncertainty:

$$\begin{aligned} \min \quad & (f_1(x, u_1), f_2(x, u_2)) := (x + u_1, 3x + u_2) \\ \text{s.t.} \quad & x \in C_1 := \mathbb{R}, \quad g_t(x, v_t) := (v_t - 2t)x^2 \leq 0, \quad t \in \mathcal{T}, \end{aligned}$$

for $u_1 \in [0, 1]$, $u_2 \in [-2, -1]$, $v_t \in \mathcal{V}_t := [-t, t]$, $t \in \mathcal{T} := (0, +\infty)$, and $x \in \mathbb{R}$.

The feasible set of the primal problem is given by $\Omega = \{0\}$. Thus, $\bar{y} = 0 \in \Omega_{UP}^{lrwe}$. Now observe that $G_t(x) = -tx^2$, $\mathcal{U}_1(\bar{y}) = \{1\}$, $\mathcal{U}_2(\bar{y}) = \{-1\}$, $\mathcal{V}_t(\bar{y}) = [-t, t]$, $\partial_x f_1(\bar{y}, u_1) = \{1\}$, $\partial_x f_2(\bar{y}, u_2) = \{3\}$, $\partial_x g_t(\bar{y}, v_t) = \{0\}$, $v_t \in \mathcal{V}_t(\bar{y})$.

$$\bigcup_{\bar{\lambda} \in \Delta(\bar{y})} \left[\sum_{t \in \mathcal{T}} \bar{\lambda}_t \partial G_t(\bar{y}) \right] + \mathcal{N}(\bar{y}; C_1) = \{0\}.$$

However, $\mathcal{N}(\bar{y}; \Omega) = \mathbb{R}$. Hence, RLCQ does not hold at $\bar{y}$. Now, consider the dual problem (OD), we cannot find out $\bar{\mu} := (\bar{\mu}_1, \bar{\mu}_2) \in \mathbb{R}_+^p \setminus \{0\}$ and $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2) \in \mathbb{R}_+^{|\mathcal{T}|}$, such that $(\bar{y}, \bar{\mu}, \bar{u}, \bar{\lambda}, 0, \bar{v}), (\bar{y}, \bar{\mu}, \bar{u}, 0, \bar{\lambda}, \bar{v}) \in \Omega_{OD}$. It means that the Theorem 4.8 fails to hold even though $f_1(\cdot, u_1)$, $f_2(\cdot, u_2)$, and $g_t(\cdot, v_t)$, $t \in \mathcal{T}$, are convex and smooth functions. It is due to the fact that RLCQ does not hold.

5. Conclusion

This paper extends the work of Hung et al. [26] by generalizing the results from the scalar objective functions with finite inequality constraints to multiobjective optimization problems with an infinite number of constraints. By using the robust optimization framework and advanced variational analysis techniques, we introduced the robust limiting constraint qualification (RLCQ) to derive robust KKT necessary optimality conditions. Furthermore, we proposed the concepts of generalized convexity and generalized pseudoquasi-convexity to establish sufficient optimality conditions. Additionally, we developed a mixed-type robust dual model for the primal problem and examined the concepts of robust weak and strong duality.

This research opens several promising avenues for future exploration in optimization. This study focuses on the (UP), where uncertainties are confined to the objective and inequality constraint functions. There is significant potential to extend this work by incorporating uncertain parameters into other types of optimization problems, such as mathematical programs with vanishing constraints, mathematical programs with equilibrium constraints, and mathematical programs with switching constraints, and by exploring different uncertainty sets, such as interval-based, norm-based, or ellipsoidal uncertainty. Moreover, the current analysis is limited to limiting subdifferentials. Future work could generalize these results to more complex functions using alternative nonsmooth calculus tools such as convexificators [31, 20] and quasidifferentials [5]. Additionally, this framework could be used to extend the results for robust fractional semi-infinite multiobjective optimization problems in [56], for ε -quasi-weakly efficient solutions [49], for interval-valued approximate solutions [37], and for approximate solutions involving convex symmetric cone optimization [3].

Acknowledgments: We are thankful to the anonymous referees for the in-depth reading of the paper and for providing useful suggestions which helped to modify the paper in its revised form. We are also thankful to the editors for careful handling of the paper and for their continuous support.

Funding: This research of the first author is funded by the University Grants Commission, India via NTA reference No. 230510257519.

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