

On Branching Points in the Gilbert-Steiner Problem

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Abstract: The Gilbert-Steiner problem is a generalization of the Steiner tree problem and specific optimal mass transportation, which allows the use of additional (branching) points in a transport plan. A specific feature of the problem is that the cost of transporting a mass m along a segment of length l is equal to $l \cdot m^p$ for a fixed $0 < p < 1$ and segments may end at points not belonging to the supports of given measures (branching points). The main result of this paper determines all pairs of (p, d) for which the Gilbert-Steiner problem in $\mathbb{R}^d$ admits only branching points of degree 3. Namely, it happens if and only if $d = 2$ or $p < 1/2$.

Keywords: Gilbert-Steiner problem, branched optimal transport.

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1. Introduction

One of the first models for branched transport was introduced by Gilbert [10]. The difference from the optimal transportation problem is that the extra geometric points may be of use; this explains the naming in honor of Steiner. Sometimes it is also referred to as *optimal branched transport*; a large part of book [2] is devoted to this problem. Let us proceed with the formal definition.

Definition 1.1. Let μ^+, μ^- be two finite measures on a metric space $(X, \rho(\cdot, \cdot))$ with finite supports such that total masses $\mu^+(X) = \mu^-(X)$ are equal. Let $V \subset X$ be a finite set containing the support of the signed measure $\mu^+ - \mu^-$, the elements of V are called *vertices*. Further, let E be a finite collection of unordered pairs $\{x, y\} \subset V$ which we call *edges*. So, (V, E) is a simple undirected finite graph. Assume that for every $\{x, y\} \in E$ two non-zero real numbers $m(x, y)$ and $m(y, x)$ are defined so that $m(x, y) + m(y, x) = 0$. This data set is called a (μ^+, μ^-) -flow if

$$\mu^+ - \mu^- = \sum_{\{x, y\} \in E} m(x, y) \cdot (\delta_y - \delta_x)$$

where δ_x denotes a delta-measure at x (note that the summand $m(x, y) \cdot (\delta_y - \delta_x)$ is well-defined in the sense that it does not depend on the order of x and y). $\square$

Let $c: [0, \infty) \rightarrow [0, \infty)$ be a *cost function*. The expression

$$\sum_{\{x, y\} \in E} c(|m(x, y)|) \cdot \rho(x, y)$$

is called the *Gilbert functional* of the (μ^+, μ^-) -flow. To avoid an overuse of modules we will use the notation $c(-x) := c(x)$.

Problem 1.2. (General finite Gilbert-Steiner problem) Let μ_+ and μ_- be measures in $\mathbb{R}^d$ with finite supports and $c(x)$ be a monotone, concave and non-negative function such that $c(0) = 0$. Find a (μ^+, μ^-) -flow which minimizes the Gilbert functional for μ^+, μ^- .

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A solution to Problem 1.2 is called a *minimal flow*. Note that concavity and non-negativity imply subadditivity, that is $c(m_1 + m_2) \leq c(m_1) + c(m_2)$. The greatest interest is attracted by the case of the cost function x^p .

Problem 1.3. (Gilbert-Steiner problem) Let μ_+ and μ_- be measures in $\mathbb{R}^d$ with finite supports and $c(x) = x^p$ for a fixed $0 < p < 1$. Find a (μ^+, μ^-) -flow which minimizes the Gilbert functional for μ^+, μ^- .

Vertices from $\text{supp}(\mu^+)$ and $\text{supp}(\mu^-)$ are called *sources* and *sinks*, respectively. A vertex from $V \setminus \text{supp}(\mu^+) \setminus \text{supp}(\mu^-)$ is called a *branching point*. Formally, we allow a branching point to have degree 2, but such points may be easily eliminated.

Local structure in the Gilbert-Steiner problem was discussed in [2], and the paper [12] deals with the planar case. A local configuration around a branching point b of degree 3 is well-understood thanks to Gilbert's original paper. Similarly to the finding of the Fermat-Torricelli point in the celebrated Steiner tree problem one can determine the angles around b in terms of masses (see Lemma 2.1).

Theorem 1.4. (Cherkashin-Petrov, 2025 [4]) Suppose that the cost function $c(\cdot)$ satisfies the decomposition

$$c(t) = \sqrt{\int \sin^2(tx) \, d\lambda(x)} = \frac{1}{2} \|e^{2itx} - 1\|_{L^2(\lambda)}, \quad (1)$$

where λ is a Borel measure on $\mathbb{R}$ with uncountable support such that

$$\int \min(x^2, 1) \, d\lambda(x) < \infty.$$

Then a solution to the planar general Gilbert-Steiner problem for $c(x)$ has no branching point of degree at least 4. In particular, this holds for $c(x) = x^p$ (via $d\lambda = \text{const} \cdot x^{-2p-1} dx$).

Note that condition (1) implies $c(0) = 0$, non-negativity and subadditivity of c ; on the other hand, it does not imply monotonicity.

It is known that an optimal flow for a strictly concave $c(\cdot)$ does not contain cycles (see Proposition 7.8 in [2]). Thus it can be represented as a collection of trees.

An *irrigation* (or *arborescence*) setup, which restricts μ^+ to be a Dirac measure was considered in Minkowski space by Volz, Brazil, Ras, Swanepoel and Thomas [18]. The arborescence naming is given because in this case an optimal flow corresponds to a tree; the case $p = 0$ corresponds to the Steiner tree problem. In a Euclidean space (of an arbitrarily dimension) they obtained that a degree of a branching is at most 3 provided that $c(\cdot)$ is a concave monotone function, the function $(c^2)'$ is convex and $c(0) > 0$. Note that for $p \in (1/2, 1)$ the convexity condition fails.

The main result of this paper is the following theorem.

Theorem 1.5. If $d = 2$ or $p < 1/2$ then degree of any branching point in a solution to the Gilbert-Steiner problem in $\mathbb{R}^d$ is equal to three. If $p \geq 1/2$, then there is a solution to the Gilbert-Steiner problem consisting of $d + 1$ segment adjacent to the same branching point.

Structure of the paper. Section 2 contains well-known lemmas which are used in our proofs. Section 3 adopts and simplifies methods from [18]; namely, we show (Theorems 3.2 and 3.4) that the condition $c(0) > 0$ can be replaced with strict convexity of $(c^2)'$ which is more natural in the context of optimal transportation and that the result can be generalized from irrigation to the general case. Section 4 provides an example of branching with degree $d + 1$ for $p \geq 1/2$ in d -dimensional Euclidean space. In Section 5 we study behavior of a solution in a neighborhood of a source (sink) point and finally Section 6 discusses applications and open problems.

2. Preliminaries

We will need the following lemmas.

Lemma 2.1. (Folklore) Let PQR be a triangle and w_1, w_2, w_3 be non-negative reals. For every point $X \in \mathbb{R}^2$ consider the value

$$L(X) := w_1 \cdot |PX| + w_2 \cdot |QX| + w_3 \cdot |RX|.$$

Then

- (i) a minimum of $L(X)$ is achieved at a unique point $X_{\min}$;
- (ii) if $X_{\min} = P$ then $w_1 \geq w_2 + w_3$ or there is a triangle Δ with sides w_1, w_2, w_3 and the angle $\angle P$ of ΔPQR is at least the exterior angle between w_2 and w_3 in Δ .
- (iii) if $X_{\min} \notin \{P, Q, R\}$ then triangle Δ exists and angles $\angle PX_{\min}Q, \angle PX_{\min}R, \angle QX_{\min}R$ are equal to the corresponding exterior angles of Δ .

Consider a Gilbert flow and two adjacent edges with masses m_1 and m_2 (they may be of different signs). Lemma 2.1 compares it with a blue tripod competitor with additional branching point, see Figure 1.

Another application of Lemma 2.1(iii) immediately describes the behavior of a minimal flow in a neighborhood of a triple branching. The angles around the point are equal to the exterior angles of a triangle with sides $c(m_1), c(m_2)$ and $c(m_3)$ (under the constraint $m_1 + m_2 + m_3 = 0$).

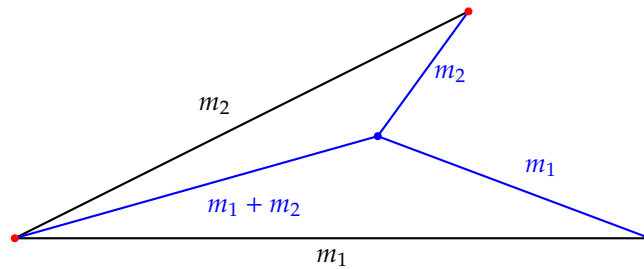


Figure 1. Basic optimality test: comparison of two adjacent edges and a tripod

Definition 2.2. We say that a non-increasing n -tuple $x_1, \dots, x_n$ majorizes a non-increasing n -tuple $y_1, \dots, y_n$ if the inequalities

$$x_1 + \dots + x_i \geq y_1 + \dots + y_i \quad \text{hold for all } i \in \{1, \dots, n-1\},$$

and

$$x_1 + \dots + x_n = y_1 + \dots + y_n.$$

Lemma 2.3. (Karamata's inequality, [11]) Let $I \subset \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be a convex function. If $x_1, \dots, x_n, y_1, \dots, y_n \in I$ are such that $(x_1, \dots, x_n)$ majorizes $(y_1, \dots, y_n)$, then

$$f(x_1) + \dots + f(x_n) \geq f(y_1) + \dots + f(y_n).$$

If f is strictly convex, then the equality holds if and only if we have $x_i = y_i$ for all $i \in \{1, \dots, n\}$.

3. Upper bounds on the degree of a branching point

Hereafter we consider a Euclidean space $\mathbb{R}^d$, $d \geq 2$. Recall that a cost function $c(x)$ is always concave, increasing and $c(0) = 0$.

Definition 3.1. We say that the set of unit vectors $e_1, \dots, e_k \in \mathbb{R}^d$ with masses $m_1, \dots, m_k$ respectively is a *satisfactory configuration* if $m_1 + \dots + m_k = 0$ and the inequality

$$\langle e_i, e_j \rangle \leq \frac{c^2(m_i + m_j) - c^2(m_i) - c^2(m_j)}{2c(m_i)c(m_j)} \quad (2)$$

holds for every $i \neq j$. □

Obviously, if A is a branching point¹ of an optimal flow then the unit vectors emanating along the flow edges adjacent to A with the corresponding masses produce a satisfactory configuration. Otherwise, by Lemma 2.1 a tripod is shorter than a pair of edges for which the inequality (2) fails. Thus a local transformation produces a better competitor, contradiction.

Theorem 3.2. Let c be a cost function such that $c^2(x)$ has a strictly convex derivative. Then there is no satisfactory configuration with unique negative mass and $k > 3$. If there is a unique negative mass and $k = 3$ then e_1, e_2 and e_3 are coplanar.

¹ If A is a source, then the condition $m_1 + \dots + m_k = 0$ may be omitted.

Proof. Consider a satisfactory configuration with unique negative mass, say, m_1 . Clearly

$$\begin{aligned} 0 &\leq \left\langle \sum_{i=1}^k c(m_i)e_i, \sum_{i=1}^k c(m_i)e_i \right\rangle = \sum_{i=1}^k c^2(m_i) + 2 \sum_{1 \leq i < j \leq k} c(m_i)c(m_j)\langle e_i, e_j \rangle \\ &\leq \sum_{i=1}^k c^2(m_i) + \sum_{1 \leq i < j \leq k} (c^2(m_i + m_j) - c^2(m_i) - c^2(m_j)) \end{aligned}$$

by the property of satisfaction. Substitute $m_1 = -m_2 - \dots - m_k$ and use $c(-x) = c(x)$ to get

$$\begin{aligned} 0 &\leq c^2\left(\sum_{i=2}^k m_i\right) + \sum_{i=2}^k c^2(m_i) \\ &\quad + \sum_{2 \leq i \leq k} \left(c^2\left(\left(\sum_{j=2}^k m_j\right) - m_i\right) - c^2(m_i) - c^2\left(\sum_{j=2}^k m_j\right) \right) \\ &\quad + \sum_{2 \leq i < j \leq k} (c^2(m_i + m_j) - c^2(m_i) - c^2(m_j)) =: f(m_2, \dots, m_k) \\ &= \sum_{2 \leq i \leq k} c^2\left(\left(\sum_{j=2}^k m_j\right) - m_i\right) + \sum_{2 \leq i < j \leq k} c^2(m_i + m_j) \\ &\quad - (k-2) \sum_{i=2}^k c^2(m_i) - (k-2)c^2\left(\sum_{i=2}^k m_i\right). \end{aligned}$$

Let $g(x)$ denote $(c^2(x))'$. Then

$$\begin{aligned} \frac{df}{dm_2} &= \sum_{3 \leq i \leq k} g\left(\left(\sum_{j=2}^k m_j\right) - m_i\right) \\ &\quad + \sum_{3 \leq i \leq k} g(m_2 + m_i) - (k-2)g(m_2) - (k-2)g\left(\sum_{i=2}^k m_i\right). \end{aligned}$$

Note that the $(2k-4)$ -tuple $x_1, \dots, x_{k-2} = \sum_{i=2}^k m_i, x_{k-3}, \dots, x_{2k-4} = m_2$ majorizes any $(2k-4)$ -tuple with the same sum and entries in the interval $\left[m_2, \sum_{i=2}^k m_i\right]$, in particular the ordered non-increasing $(2k-4)$ -tuple with entries

$$\left(\sum_{j=2}^k m_j\right) - m_i, \quad m_2 + m_i, \quad \text{where } 3 \leq i \leq k.$$

Since g is strictly convex on $[0, +\infty)$ by Lemma 2.3 we have $\frac{df}{dm_2} \leq 0$. Thus we have $0 \leq f(0, \dots, 0) = 0$, so we have equalities in all inequalities including the application of Lemma 2.3. In particular the second $(2k-4)$ -tuple contains $\sum_{i=2}^k m_i$ which is possible only for some $m_2 + m_i$, which means $k \leq 3$.

In the case $k = 3$ the equalities

$$\langle e_i, e_j \rangle = \frac{c^2(m_i + m_j) - c^2(m_i) - c^2(m_j)}{2c(m_i)c(m_j)}$$

imply that angles between e_1, e_2 and e_3 are equal to exterior angle of the triangle with sides $c(m_1), c(m_2)$ and $c(m_3)$, so their sum is 2π and they are coplanar. $\square$

Remark 3.3. A convexity condition on $(c^2)'$ can be replaced with a milder condition

$$\int_0^a \int_0^b \int_0^c (c^2(x+y+z))''' dx dy dz > 0$$

for every positive a, b , and c .

Theorem 3.4. Let $c(x)$ be a differentiable function such that $(c^2)'$ is strictly convex. Then there is no satisfactory configuration with $k > 3$.

Proof. Suppose that there is a satisfactory configuration consisting of vectors $e_1, e_2, \dots, e_k$ with positive masses $m_1, \dots, m_k$, respectively and vectors $e_{k+1}, \dots, e_{k+l}$ with negative masses $-n_1, \dots, -n_l$, respectively. Without loss of generality let

$$m_1 = \max(m_1, \dots, m_k, |n_1|, \dots, |n_l|).$$

Erase the vectors $e_2, \dots, e_k$ with the corresponding masses and replace m_1 with $M := m_1 + \dots + m_k$. We claim that vectors $e_1, e_{k+1}, \dots, e_{k+l}$ and masses $M, -n_1, \dots, -n_l$ form a satisfactory configuration.

Indeed, the bounds (2) on $\langle e_i, e_j \rangle$ for $i, j > k$ do not change so let us analyze the bounds (2) on $\langle e_1, e_i \rangle$ for $i > k$. Take the derivative

$$\begin{aligned} & \frac{d}{dm} \frac{c^2(m-n) - c^2(m) - c^2(n)}{2c(m)c(n)} \\ &= \frac{2c'(m-n)c(m-n)c(m) - c'(m)(c^2(m) + c^2(m-n) - c^2(n))}{2c^2(m)c(n)}. \end{aligned}$$

Since $c(x)$ is concave and monotone $c'(m-n) > c'(m) > 0$ and

$$2c(m-n)c(m) > c^2(m) + c^2(m-n) - c^2(n)$$

because

$$\frac{c^2(m) + c^2(m-n) - c^2(n)}{2c(m-n)c(m)}$$

is a cosine of (acute) angle in the triangle with sides $c(m), c(m-n), c(n)$ which exists due to subadditivity of $c(x)$. So the derivative is positive and thus every condition on the angle between e_1 and $e_i, i > k$ becomes milder.

Theorem 3.2 says that $l = 2$ (if $l = 1$, then n_1 has maximal absolute value) and if e_1, e_{k+1}, e_{k+2} are coplanar with equalities in the inequality of satisfaction. If $k > 2$, then $M > m_1$ and these equalities break. $\square$

Corollary 3.5. The Gilbert-Steiner problem for an arbitrary d and $p < 1/2$ admits only triple branching.

Proof. Recall that by Lemma 2.1 a branching point of degree k produces a satisfactory configuration which consists of k vectors. By Theorem 3.4 this is impossible for $p < 1/2$ since $(c^2)' = 2px^{2p-1}$ is strictly convex. $\square$

Proposition 3.6. For $p = 1/2$ the maximal degree of a branching point in a d -dimensional solution to the Gilbert-Steiner problem is at most $d + 1$.

Proof. Let the vectors parallel to edges emanating from the branching point be $e_1, \dots, e_k$. The satisfactory inequality (2) in the case $p = 1/2$ says that the angles between masses of the same sign are at least $\pi/2$ and the angles between masses of different signs are strictly greater than $\pi/2$.

We use induction in d . The base $d = 2, k \leq 3$ is obvious, since at least one angle should be obtuse. For the step choose any vector $v \in \{e_1, \dots, e_k\}$ and project everything onto the hyperplane $v^\perp$. Note that the scalar product of projections of any vectors from $\{e_1, \dots, e_k\} \setminus v$ not increases. Also no two vectors can share the projection. If zero is not represented as a projection we are immediately done. In the remaining case if $v, -v$ have masses of the same sign then there is no vector with mass of the opposite sign; otherwise there are no more vectors at all. $\square$

4. Examples

The following theorem was proved in [18] for a general Minkowski space (the authors do not use the condition $c(0) > 0$ through the proof).

Theorem 4.1. (Volz-Brazil-Ras-Swanepoel-Thomas, 2013 [18]) Consider a single source at q and sinks $p_1, \dots, p_n \in \mathbb{R}^d$. Let the mass demand at p_i be m_i for $i = 1, \dots, n$. Then a star flow with a unique branching point at the origin o is optimal if and only if

$$\sum_{i=1}^n c(m_i)e_i + c\left(\sum_{i=1}^n m_i\right)e = \bar{0} \quad (3)$$

and

$$\left\langle \sum_{i \in I} c(m_i) e_i, \sum_{i \in I} c(m_i) e_i \right\rangle \leq c^2 \left(\sum_{i \in I} m_i \right) \quad \text{for all } I \subseteq \{1, \dots, n\}, \quad (4)$$

where e_i is the unit vector codirected with op_i and e is the unit vector codirected with oq .

Note that Theorem 4.1 in particular shows that local optimality of an irrigation star implies global optimality (this fact can be also obtained for a general star by rescaling argument).

Example 4.2. Let $e_1, \dots, e_d$ be a standard basis of $\mathbb{R}^d$ and $c(x) = \sqrt{x}$. Consider any positive masses $m_1, \dots, m_d$ and define e by

$$\sum_{i=1}^n \sqrt{m_i} \cdot e_i + \sqrt{\sum_{i=1}^n m_i} \cdot e = \bar{0}.$$

Clearly, e has unit length, so (3) holds. Also for every $I \subseteq \{1, \dots, d\}$ one has an equality in (4). Thus for $p = 1/2$ there is an example of $(d+1)$ -branching. $\square$

This example can be extended to the case $p \geq 1/2$ as follows:

Example 4.3. Let $e_1, \dots, e_d$ be unit vectors in $\mathbb{R}^d$ and $c(x) = x^p$ such that $\langle e_i, e_j \rangle = a$, where

$$\sqrt{d(1 + (d-1)a)} = d^p \quad \text{so that} \quad a = \frac{d^{2p-1} - 1}{d-1}.$$

Consider masses $m_1 = \dots = m_d = 1$ and define e by

$$\sum_{i=1}^n e_i + d^p \cdot e = \bar{0}.$$

Clearly, e has unit length, so (3) holds. Inequality (4) is equivalent to

$$|I| + |I| \cdot (|I| - 1)a \leq |I|^{2p}$$

and

$$\frac{d^{2p-1} - 1}{d-1} \leq \frac{|I|^{2p-1} - 1}{|I| - 1}. \quad (5)$$

Derive the right-hand side putting $x = |I|$:

$$\frac{d}{dx} \frac{x^{2p-1} - 1}{x-1} = \frac{1 - (2p-1)x^{2p-2} + (2p-2)x^{2p-1}}{(x-1)^2}.$$

The denominator is always positive, so

$$\frac{d}{dx} (1 - (2p-1)x^{2p-2} + (2p-2)x^{2p-1}) = (2p-2)(2p-1)x^{2p-3}(x-1) < 0$$

means that

$$1 - (2p-1)x^{2p-2} + (2p-2)x^{2p-1} < 1 - (2p-1) + (2p-2) = 0,$$

so $\frac{x^{2p-1}-1}{x-1}$ decreases on $(1, +\infty)$ which implies (5).

Thus for $p \geq 1/2$ there is an example of $(d+1)$ -branching.

5. Degree of a source

The following theorem is a version of Theorem 4.1 which bounds the degree of a source.

Theorem 5.1. Suppose that sinks $p_1, \dots, p_n \in \mathbb{R}^d$, and a single source in the origin o . Let the flow demand at p_i be m_i for $i = 1, \dots, n$. Then a star flow is optimal if and only if

$$\left\langle \sum_{i \in I} c(m_i) e_i, \sum_{i \in I} c(m_i) e_i \right\rangle \leq c^2 \left(\sum_{i \in I} m_i \right) \quad \text{for all } I \subseteq \{1, \dots, n\}, \quad (6)$$

where e_i is a unit vector codirected with op_i .

Proof. ($\Rightarrow$) Assume without loss of generality that $I \neq \emptyset$. Consider a competitor in which some mass goes from the source point o via vector te , for some $t \in \mathbb{R}_+$ and a unit vector e . In this new network, for each index $i \notin I$, the source p_i remains connected to o with flow demand m_i and for indices $i \in I$, the source p_i is now connected to te with flow demand m_i , and te is connected to o with flow demand $\sum_{i \in I} m_i$. Minimality of the original network then implies that, for all unit vectors e , the function

$$\psi_e(t) = \sum_{i \in I} c(m_i) \cdot (\|p_i - te\| - \|p_i\|) + c\left(\sum_{i \in I} m_i\right)|t|$$

attains its minimum at $t = 0$. Although ψ_e may not be differentiable at 0, we can still take the right derivative to conclude that

$$0 \leq \lim_{t \rightarrow 0^+} \frac{\psi_e(t)}{t} = \lim_{t \rightarrow 0^+} \left(\sum_{i \in I} c(m_i) \frac{\|p_i - te\| - \|p_i\|}{t} + c\left(\sum_{i \in I} m_i\right) \right).$$

This inequality yields

$$\left\langle \sum_{i \in I} c(m_i) e_i, e \right\rangle \leq c\left(\sum_{i \in I} m_i\right).$$

Substituting e to be parallel $\sum_{i \in I} c(m_i) e_i$, we obtain the desired inequality.

($\Leftarrow$) Now assume that a flow satisfies (6). Let T be any competitor for the given data. For each $i = 1, \dots, n$, let P_i denote the unique path in T from p_i to o , say

$$P_i = x_1^{(i)} x_2^{(i)} \dots x_{k_i}^{(i)}, \quad x_1^{(i)} = p_i, \quad x_{k_i}^{(i)} = o,$$

with the edges $x_j^{(i)} x_{j+1}^{(i)}$ distinct for $j = 1, \dots, k_i - 1$. For each edge e of T , define the index set $S_e = \{i : e \text{ lies on } P_i\}$. Then the flow on edge e is $\sum_{i \in S_e} m_i$, and its cost is

$$c\left(\sum_{i \in S_e} m_i\right) \|x - y\|,$$

where e connects vertices x and y . Hence, the total cost of T is

$$c(T) := \sum_{e=xy \text{ edge of } T} c\left(\sum_{i \in S_e} m_i\right) \|x - y\|.$$

On the other hand, the cost of the star network is

$$\sum_{i=1}^n c(m_i) \langle e_i, p_i \rangle = \sum_{i=1}^n c(m_i) \sum_{j=1}^{k_i-1} \langle e_i, x_j^{(i)} - x_{j+1}^{(i)} \rangle = \sum_{e=xy \text{ edge of } T} \left\langle \sum_{i \in S_e} c(m_i) e_i, x - y \right\rangle$$

Then, by applying inequality (6) we obtain

$$\left\langle \sum_{i \in S_e} c(m_i) e_i, x - y \right\rangle \leq c\left(\sum_{i \in S_e} m_i\right) \|x - y\|.$$

Summing over all edges e shows that the cost of the star is not greater than $C(T)$. Thus, the star network is a minimal Gilbert flow. $\square$

Similarly for Examples 4.2 and 4.3, Theorem 5.1 provides a star with the source of degree $2d$ with edges $\pm e_i$ for $p \geq 1/2$.

6. Applications and further work

Universal irrigation flow. Having determined the maximal possible degrees of a source and a branching point one may try to construct a *universal* solution to the Gilbert-Steiner problem, id est a solution which contains every possible combinatorial structure with the same p and d as a substructure. Such results for Steiner trees were obtained in [5, 13, 14] and used in [1].

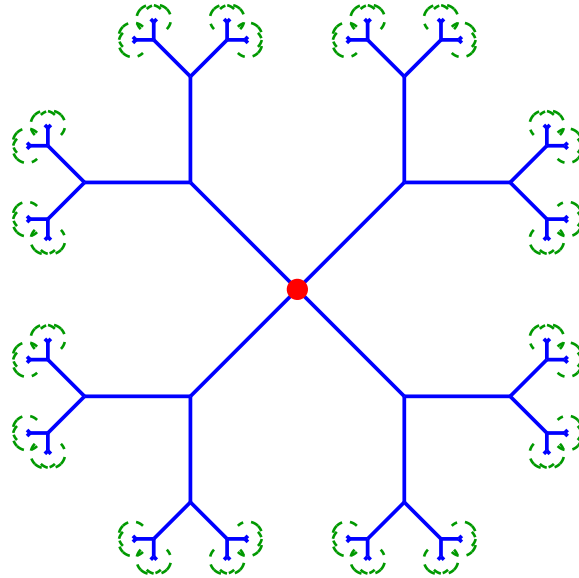


Figure 2. An example of a universal irrigation flow

At this point it seems to be possible only for a certain range of p , d and only for irrigation setup. In particular, for $p = 1/2$ and $d = 2$ the following tree is universal.

Classification of triple branching. It would be interesting to describe all cost functions for which the degree of *planar* branching is exactly 3. One of the motivations is the existence of *exact* (but very slow) algorithm in this case. No algorithm is currently known in $\mathbb{R}^d$ for $d > 2$ (see Problem 15.12 in [2]). One of the reasons is that the coordinates of the branching points, even for the Steiner tree problem for a general simplex in three-dimensional space, are no longer expressed in radicals [17].

Picture in high dimensions. Maximum possible degree of a source is heavily related with spherical codes and kissing number [3]. The kissing number of a metric space is defined as the greatest number of non-overlapping unit spheres that can be arranged in that space such that they each touch a common unit sphere. The exact answer for a Euclidean space is known only in dimensions 1, 2, 3, 4, 8, 24. Note that the presence of masses and conditions for $|I| > 2$ in Theorem 5.1 make the situation even more intricate since the energy (Gilbert functional) of a configuration has extremely complicated behavior. Relations with other types of energies require additional investigation.

It seems that the results of this paper rhymes with tight Rankin bounds for large non-intersecting spherical caps (see Theorem 1 in [16]).

Recall that in the case of branching point the conditions depend also on signs of masses. A major question for the maximum possible degree of a branching point is whether there exists an upper bound which depends only on d . A positive resolution seems inaccessible to current methods.

Open questions are listed in Section 15 of the book [2]. Open question 1 was solved in the recent series of papers [6, 7, 8, 9]. Namely, Colombo, Rosa, Marchese, Pegon and Prouff showed (under mild technical assumptions) that a weak-star limit of solutions to the Gilbert-Steiner problem for Borel probabilistic measures is a solution for a weak-star limit of the given data. This makes it possible to apply numerical methods in the general case of the problem (see [15]). Note that any upper bounds on the degree of a branching should speed up this approach.

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