BANACH PRESS

JOURNAL OF CONVEX ANALYSIS
Volume 34 (2027), No.1, 271–293
https://journalofconvexanalysis.com

New efficient accelerated and stochastic gradient descent algorithms based on locally Lipschitz gradient constants

Nguyen Phung Hai Chung[1], Pham Thi Hoai[1,*], Hoang Van Chung[1]

1. Hanoi University of Science and Technology, Hanoi, Vietnam

Abstract: This paper revisits the recently introduced stepsize in the gradient descent scheme called NGD, proposed by Hoai et al. [8]. We first investigate the NGD stepsize in conjunction with two well-known acceleration techniques: the Heavy Ball method and Nesterov's method. In the convex setting of unconstrained nonlinear optimization problems, we establish the ergodic convergence of the iterates generated by the accelerated versions of NGD at a sublinear rate. We introduce stochastic versions of the proposed accelerated algorithms and analyze their convergence in nonconvex settings. Although our proposed algorithms require global Lipschitz continuity of the gradient, we do not utilize the global Lipschitz constant during computations. Instead, we leverage information about local Lipschitz constants derived from previous iterations. Numerical experiments on various practical machine learning and deep learning problems demonstrate the efficiency of our proposed methods compared to the recent ones.

Keywords: Stochastic gradient descent, Nesterov's acceleration, Heavy Ball, adaptive stepsize, machine learning, deep learning.
2020 Mathematics Subject Classification: 90C25, 90C15, 65K05, 68T05.

1. Introduction

Unconstrained nonlinear optimization problems have received significant attention from researchers due to their wide range of applications in economics, data science, machine learning, deep learning, etc, see e.g. [19, 15] and the references therein. The problem is formulated as follows:

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. Let X^* denote the set of optimal solutions of Problem (1), and assume that X^* is nonempty. For any $x^* \in X^*$, we write $f_* = f(x^*)$ for the optimal value of Problem (1).

Leveraging the differentiability of the objective function f, first-order methods have been studied widely for solving Problem (1) in the literature, see [17, 13, 2, 1, 3]. Among these methods, gradient descent plays an important role due to its ease of implementation and efficient performance. Gradient descent was originally proposed by Cauchy [10] in 1847 and has since become one of the most fundamental optimization methods. Starting at some point $x_0 \in \mathbb{R}^n$, this method uses the idea of updating the variable $x_k \in \mathbb{R}^n$ at each iteration $k \in \mathbb{N}$ by the formula

$$x_{k+1} = x_k - \lambda_k \nabla f(x_k), \tag{GD}$$

where $\lambda_k > 0$ is the stepsize at iteration k. The standard condition on f to guarantee the convergence of the gradient scheme (GD) is the global Lipschitz continuity of the gradient ∇f over $\mathbb{R}^n$, i.e., there exists a constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|, \forall x, y \in \mathbb{R}^n.$$

When f is convex and λ_k is a constant in $\left(0, \frac{1}{L}\right]$ or λ_k is computed using backtracking line search procedures, the convergence rate of (GD) is $O\left(\frac{1}{k}\right)$ for $f(x_k) - f_*$, see e.g. [1, 3]. However, estimating

*Corresponding author.
E-mail addresses: N. P. H. Chung: hchung1997@gmail.com; P. T. Hoai*: hoai.phamthi@hust.edu.vn; H. V. Chung: chunghoangvan44@gmail.com
https://doi.org/10.68381/jca34020
Received date: May 3, 2025; Accepted date: September 2, 2026
© 2027 Published by Banach Press LLC. (*Dedicated to Professor Alexander Plakhov on the occasion of his 65th birthday*)

the global constant L is often impractical, and backtracking line search can be computationally expensive. Furthermore, both strategies can suffer from small stepsizes caused by a large L or frequent backtracking calculations. To address these drawbacks, adaptive stepsize selections used for (GD) have been proposed by Malitsky and Mischenko [11] and Hoai et al. [8] recently. Specifically, the AdGD method proposed by Malitsky and Mishchenko [11] updates the stepsize as follows:

$$\lambda_k = \min\left\{\sqrt{1+\theta_{k-1}}\lambda_{k-1}, \frac{\|x_k - x_{k-1}\|}{2\|\nabla f(x_k) - \nabla f(x_{k-1})\|}\right\}, \quad k \geq 1, \tag{2}$$

where $\theta_0 = +\infty$, $\theta_k = \lambda_k/\lambda_{k-1}$; and NGD provided by Hoai et al. [8] updates λ_k as follows: for $0 < \eta_1 < \eta_0 < 1/2$, and a convergence positive series $\sum_{k=0}^{+\infty} \varepsilon_k$,

$$\lambda_k = \begin{cases} \eta_1 \frac{\|x_k - x_{k-1}\|}{\|\nabla f(x_k) - \nabla f(x_{k-1})\|}, & \text{if } \|\nabla f(x_k) - \nabla f(x_{k-1})\| > \frac{\eta_0}{\lambda_{k-1}}\|x_k - x_{k-1}\| \\ (1+\varepsilon'_{k-1})\lambda_{k-1}, & \text{otherwise} \end{cases}, \tag{3}$$

where $\varepsilon'_k = \begin{cases} \varepsilon_k & \text{if } \frac{\lambda_{k-1}}{\lambda_{k-2}} \geq 1 \\ \varepsilon'_{k-1} = \min\left\{\varepsilon_{k-1}, \sqrt{1+\frac{\lambda_{k-1}}{\lambda_{k-2}}} - 1\right\}, & \text{otherwise} \end{cases}$.

Neither AdGD nor NGD requires a backtracking line search or knowledge of the global Lipschitz constant L; instead, they utilize local Lipschitz constant estimates to derive the stepsize. It is also worth noting that AdGD and NGD work without using the global Lipschitz condition on ∇f, they only require ∇f to be locally Lipschitz continuous.

In order to improve the speed of (GD) scheme, one can use accelerated techniques such as the two classical accelerated techniques introduced by Polyak [16, 17] and Nesterov [13]. In particular, the Heavy Ball technique given in [16, 17] takes

$$x_{k+1} = x_k - \lambda_k \nabla f\,(x_k) + \gamma(x_k - x_{k-1}), \tag{Heavy Ball method}$$

and Nesterov's acceleration in [13] determines

$$\begin{aligned} y_{k+1} &= x_k - \lambda_k \nabla f(x_k) \\ x_{k+1} &= y_{k+1} + \gamma(y_{k+1} - y_k), \end{aligned} \tag{Nesterov's method}$$

where the constant $\gamma > 0$ in the Heavy Ball method or Nesterov's method is called an accelerated parameter.

Contributions: Our contributions in this paper are as follows:

(i) First, we address an open problem raised by Hoai et al. [8] concerning whether it is possible to accelerate the NGD algorithm. To address this, two new algorithms, **NGDh** and **NGDn** which combine NGD with the Heavy Ball method and Nesterov's method respectively, are proposed in this paper.

From a **theoretical perspective**, our convergence results are comparable to those of other methods in the literature that employ the Heavy Ball method and Nesterov's method. In particular,

(1) In the case where f is convex, we prove the *ergodic convergence* of **NGDn** and **NGDh** with the rate $O(\frac{1}{K})$ from some fixed iteration. This convergence result is similar to that of the Heavy Ball method given in [6] which employs a fixed stepsize λ_k and requires knowledge of the global Lipschitz constant L of ∇f. Compared with the convergence rate $O(\frac{1}{k^2})$ of Nesterov's method in [13] (with $\lambda_k = \frac{1}{L}$ or λ_k computed by backtracking line search), ours is theoretically weaker. However, **in practice**, **NGDh** and **NGDn** use *adaptive stepsizes* λ_k, which are easy to implement without estimating the global Lipschitz constant L or incurring the high computational costs of backtracking procedures.

(2) It is worth noting that Malitsky and Mishchenko [11] proposed AdGD-accel which combines Nesterov's acceleration with the adaptive stepsize of AdGD; however, its convergence analysis has not yet been established.

(ii) Our **second contribution** is the study of stochastic versions of **NGDh** and **NGDn**. We proposed two new stochastic algorithms, named **SNGDh** and **SNGDn**. It is well known that, the stochastic approach is useful for Problem (1) in machine learning and deep learning (see e.g., [4] and the references therein) when working with big data. This is because it overcomes the high computational cost of the deterministic approach that requires the full computation of the gradient at each iteration.

From a theoretical point of view, we obtain typical convergence results for our stochastic algorithms SNGDn and SNGDh under standard conditions. In particular,

(1) Under the classical conditions including: 1. the uniform boundedness of the stochastic gradients (see e.g., [12, 18, 7]) and 2. the global Lipschitz continuity of the gradient of the *nonconvex* objective function f; we prove that the minimum expected squared norm of the gradient is bounded. This ensures that *the best iterates generated by our proposed stochastic algorithms are close to some stationary points of the objective function*.

(2) Compared with the related AdSGD algorithm [11] (the stochastic version of AdGD), our convergence results for SNGDn and SNGDh are obtained for nonconvex objective functions f, which are common in machine learning, whereas the convergence of AdSGD is established only for strongly convex functions.

(iii) From a practical standpoint, the adaptive stepsize based on the local Lipschitz constant, as used in our new algorithms NGDn, NGDh, SNGDh, and SNGDn, provides significantly better performance than that of recent state-of-the-art methods. This is demonstrated by numerical results for numerous benchmark problems in machine learning and deep learning in Section 5.

The rest of the paper is organized as follows. After recalling some fundamental results in Section 2 we propose the new accelerated algorithms in Section 3 with the convergence analysis. The stochastic versions of our new algorithms together with their convergences are presented in Section 4. Finally, the numerical results are reported in Section 5 with available codes in our repository https://github.com/hoaiphamthi/Accelerated-and-Stochastic-NGD. We end the paper with a brief conclusion in Section 6.

2. Preliminaries

Throughout this paper, our underlying space is $\mathbb{R}^n$ equipped with the standard Euclidean norm, denoted by $\|x\| = \sqrt{\langle x, x\rangle}$, for $x \in \mathbb{R}^n$.

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \quad \text{for all } x, y \in \mathbb{R}^n,\ \lambda \in [0,1].$$

Definition 2.2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *L-smooth* if it is differentiable and its gradient ∇f is L-Lipschitz continuous, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n.$$

The following lemma recalls some fundamental properties of L-smooth functions.

Lemma 2.3 ([2, 13]). *If f is L-smooth on $\mathbb{R}^n$, then*

$$f(y) \le f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2 \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Furthermore, if f is also convex, the following inequality holds:

$$f(x) - f(y) + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2 \le \langle x - y, \nabla f(x)\rangle \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Lemma 2.4. *Let $a \in [0,1)$, then $\sum_{q=1}^{+\infty} q a^q = \frac{a}{(1-a)^2}$. Consequently, for any integers $1 \le i \le Q$, we have*

$$\sum_{q=i}^{Q} q a^q \le \frac{a}{(1-a)^2}.$$

In this paper, we consider the optimization problem (1) under the following assumption

Assumption 2.5. The set of optimal solutions of Problem (1), denoted by X^*, is nonempty. We denote by f_* the optimal value of Problem (1), i.e., $f_* = f(x^*)$ for $x^* \in X^*$.

For specific settings in the convergence analysis, any further assumptions will be clarified explicitly at the beginning of the respective sections.

3. New accelerated gradient descent algorithms

In this section, we propose two accelerated algorithms that incorporate the Heavy Ball [16] and Nesterov [13] momentum techniques into the NGD adaptive stepsize framework [8]. Unlike standard methods that rely on a global Lipschitz constant, our approach adapts the stepsize based on local curvature information approximated by finite differences of gradients.

Algorithm 1 NGD accelerated by the Heavy Ball method (NGDh)

1: **Initialization.** Select $\lambda_0 > 0$, $0 < \eta_1 < \eta_0$, $0 \le \gamma < 1$ and a positive real sequence $\{\varepsilon_k\}$ such that $\sum_{k=0}^{+\infty} \varepsilon_k < +\infty$. Choose $x_0 \in \mathbb{R}^n$, $x_1 = x_0 - \lambda_0 \nabla f(x_0)$.
2: **for** $k = 1, 2, \ldots$ **do**
3: **if** $\|\nabla f(x_k) - \nabla f(x_{k-1})\| > \frac{\eta_0}{\lambda_{k-1}} \|x_k - x_{k-1}\|$ **then**
4: $\lambda_k = \eta_1 \frac{\|x_k - x_{k-1}\|}{\|\nabla f(x_k) - \nabla f(x_{k-1})\|}$
5: **else**
6: $\lambda_k = (1 + \varepsilon_{k-1}) \lambda_{k-1}$
7: **end if**
8: $x_{k+1} = x_k - \lambda_k \nabla f(x_k) + \gamma(x_k - x_{k-1})$
9: **end for**

Algorithm 2 NGD accelerated by Nesterov's method (NGDn)

1: **Initialization.** Select $\lambda_0 > 0$, $0 < \eta_1 < \eta_0$, $0 \le \gamma < 1$ and a positive real sequence $\{\varepsilon_k\}$ such that $\sum_{k=0}^{+\infty} \varepsilon_k < +\infty$. Choose $x_0 \in \mathbb{R}^n$, $x_1 = x_0 - \lambda_0 \nabla f(x_0)$.
2: **for** $k = 1, 2, \ldots$ **do**
3: **if** $\|\nabla f(x_k) - \nabla f(x_{k-1})\| > \frac{\eta_0}{\lambda_{k-1}} \|x_k - x_{k-1}\|$ **then**
4: $\lambda_k = \eta_1 \frac{\|x_k - x_{k-1}\|}{\|\nabla f(x_k) - \nabla f(x_{k-1})\|}$
5: **else**
6: $\lambda_k = (1 + \varepsilon_{k-1}) \lambda_{k-1}$
7: **end if**
8: $y_{k+1} = x_k - \lambda_k \nabla f(x_k)$
9: $x_{k+1} = y_{k+1} + \gamma(y_{k+1} - y_k)$
10: **end for**

It is observed that our proposed algorithms, Algorithm 1 (NGDh) and Algorithm 2 (NGDn), dynamically update the stepsize λ_k. If the local curvature condition (Line 4) is violated, the stepsize is reduced inversely proportional to the local Lipschitz estimate. Otherwise, it is slightly increased by a factor $(1+\varepsilon_{k-1})$ to accelerate convergence.

Remark 3.1. Algorithm 1 is the heavy-ball version of NGD, whereas Algorithm 2 incorporates Nesterov's acceleration. When $\gamma = 0$ and $0 < \eta_1 < \eta_0 < 1/2$, both algorithms reduce to the NGD scheme proposed in [8]. Compared with Algorithm 2.1 in [8], the present algorithms allow the parameters η_0 and η_1 to be chosen in a wider range, namely $0 < \eta_1 < \eta_0$, and do not require the additional update of the sequence $\{\varepsilon_k\}$. These modifications simplify the implementation.

Next, we will investigate the convergence of **NGDh** and **NGDn** with additional assumptions on f below.

Assumption 3.2. The objective function f is L-smooth on $\mathbb{R}^n$.

Assumption 3.3. The objective function f is convex on $\mathbb{R}^n$.

Before establishing the convergence of Algorithms 1 and 2, it is essential to examine the behavior of the adaptive stepsize. The following lemma, which is analogous to Lemma 2.3 in [8], asserts that the sequence $\{\lambda_k\}$ is bounded and converges to a positive limit.

Lemma 3.4. *Let $\{\lambda_k\}$ be the sequence generated by Algorithm 1 or Algorithm 2. If f satisfies Assumptions 2.5 and 3.2, then:*

(i) For all $k \geq 0$, we have

$$\lambda_k \geq \min\left\{\lambda_0, \frac{\eta_1}{L}\right\}. \tag{4}$$

(ii) The sequence $\{\lambda_k\}$ converges to a limit $\overline{\lambda} \in (0, +\infty)$.

Proof. Note that both Algorithms 1 and 2 share the same stepsize update rule for λ_k (lines 2-6 of each algorithm) hence, we present the proof for Algorithm 1, as the arguments for Algorithm 2 are identical.

(i) Clearly, inequality (4) holds for $k = 0$. For $k \geq 1$, we consider two cases.

Case 1: If $\|\nabla f(x_k) - \nabla f(x_{k-1})\| > \frac{\eta_0}{\lambda_{k-1}}\|x_k - x_{k-1}\|$, then

$$\lambda_k = \frac{\eta_1\|x_k - x_{k-1}\|}{\|\nabla f(x_k) - \nabla f(x_{k-1})\|} \geq \frac{\eta_1}{L},$$

which follows from the L-smoothness of f.

Case 2: If $\|\nabla f(x_k) - \nabla f(x_{k-1})\| \leq \frac{\eta_0}{\lambda_{k-1}}\|x_k - x_{k-1}\|$, then

$$\lambda_k = (1 + \varepsilon_{k-1})\lambda_{k-1} \geq \lambda_{k-1}.$$

By induction, we conclude that $\lambda_k \geq \min\left\{\lambda_0, \frac{\eta_1}{L}\right\}$ for all $k \geq 0$.

(ii) From Algorithm 1, it follows that

$$a_k = \ln\left(\frac{\lambda_{k+1}}{\lambda_k}\right) \leq \ln(1 + \varepsilon_k) \leq \varepsilon_k, \quad \forall k \geq 0.$$

Decompose $a_k = a_k^+ - a_k^-$, where $a_k^+ = \max(0, a_k)$ and $a_k^- = -\min(0, a_k)$. Consequently, $a_k^+ \geq 0$ and $a_k^- \geq 0$ for all $k \geq 0$. Moreover, since $a_k^+ \leq \varepsilon_k$, the convergence of $\sum_{k=0}^{+\infty} \varepsilon_k$ implies the convergence of $\sum_{k=0}^{+\infty} a_k^+$.

Now, consider the partial sum:

$$\sum_{i=0}^{k} a_i = \ln(\lambda_{k+1}) - \ln(\lambda_0) = \sum_{i=0}^{k}(a_i^+ - a_i^-) = \sum_{i=0}^{k} a_i^+ - \sum_{i=0}^{k} a_i^-. \tag{5}$$

If $\lim_{k\to+\infty} \sum_{i=0}^{k} a_i^- = +\infty$, then $\lim_{k\to+\infty} \ln(\lambda_k) = -\infty$, which is equivalent to $\lim_{k\to+\infty} \lambda_k = 0$. This contradicts Lemma 3.4(i), which states that $\lambda_k \geq \min\left\{\lambda_0, \frac{\eta_1}{L}\right\} > 0$ for all $k \geq 0$. Therefore, the series $\sum_{k=0}^{+\infty} a_k^-$ is convergent. From (5), we conclude that $\lim_{k\to+\infty} \lambda_k = \overline{\lambda} < +\infty$.

□

Lemma 3.5. *Suppose that f satisfies Assumption 3.2. Let $\{\lambda_k\}$ be the sequence generated by Algorithm 1 (or Algorithm 2). Then, there exists an integer $\overline{k}$ such that*

$$\|\nabla f(x_k) - \nabla f(x_{k-1})\| \leq \frac{\eta_0}{\lambda_{k-1}}\|x_k - x_{k-1}\|, \quad \forall k \geq \overline{k}. \tag{6}$$

Consequently, $\lambda_{k-1} \leq \lambda_k \leq \overline{\lambda} = \lim_{j\to+\infty} \lambda_j$ for all $k \geq \overline{k}$.

Proof. Suppose, for the sake of contradiction, that there exists a subsequence $\{k_j\}$ with $k_j \to +\infty$ such that

$$\|\nabla f(x_{k_j}) - \nabla f(x_{k_j-1})\| > \frac{\eta_0}{\lambda_{k_j-1}}\|x_{k_j} - x_{k_j-1}\|.$$

In this case, the algorithm updates λ_{k_j} as:

$$\lambda_{k_j} = \eta_1 \frac{\|x_{k_j} - x_{k_j-1}\|}{\|\nabla f(x_{k_j}) - \nabla f(x_{k_j-1})\|}.$$

This implies

$$\frac{\eta_1\|x_{k_j} - x_{k_j-1}\|}{\lambda_{k_j}} = \|\nabla f(x_{k_j}) - \nabla f(x_{k_j-1})\| > \frac{\eta_0}{\lambda_{k_j-1}}\|x_{k_j} - x_{k_j-1}\|.$$

Therefore, we have $\frac{\lambda_{k_j}}{\lambda_{k_j-1}} < \frac{\eta_1}{\eta_0}$ for all k_j. On the other hand, from Lemma 3.4, we know that $\lim_{k_j\to+\infty} \lambda_{k_j} = \lim_{k_j\to+\infty} \lambda_{k_j-1} = \lim_{k\to+\infty} \lambda_k = \overline{\lambda}$. Taking the limit, we deduce that

$$\frac{\overline{\lambda}}{\overline{\lambda}} \leq \frac{\eta_1}{\eta_0} < 1,$$

which implies $1 < 1$. This is a contradiction, completing the proof. □

Lemma 3.5 establishes the relationship between the local Lipschitz approximation of ∇f and the stepsize for sufficiently large iterations. Furthermore, it shows that the sequence of stepsizes becomes monotonically increasing and converges to a finite limit after a certain iteration.

3.1. The convergence of Algorithm 1 (NGDh)

In this subsection, we establish the convergence rate of the NGDh algorithm. Specifically, rather than analyzing the standard sequence of iterates $\{x_k\}$, we evaluate the convergence behavior using a weighted average of these generated points. In optimization theory, this approach is formally referred to as **ergodic convergence**. To this end, we define the ergodic iterate at step K as

$$\overline{x}_K = \frac{\gamma x_K + (1-\gamma)\sum_{k=1}^{K} x_k}{\gamma + (1-\gamma)K}.$$

Under this framework, the ergodic convergence is established by proving that the function value gap $f(\overline{x}_K) - f_*$ vanishes at a sublinear rate of $O(1/K)$.

Theorem 3.6. *Consider Problem (1) under Assumptions 2.5, 3.2, and 3.3. Let $\overline{k}$ be the iteration index defined in Lemma 3.5. Suppose the parameters satisfy the condition:*

$$e^{\sum_{j=\overline{k}}^{+\infty} \varepsilon_{j-1}} \frac{\eta_1\|x_{\overline{k}} - x_{\overline{k}-1}\|}{\|\nabla f(x_{\overline{k}}) - \nabla f(x_{\overline{k}-1})\|} \leq \frac{1-\gamma}{L}. \tag{7}$$

Then, for any $K > \overline{k}$, the weighted average iterate $\overline{x}_K$ defined by

$$\overline{x}_K = \frac{\gamma x_K + (1-\gamma)\sum_{k=1}^{K} x_k}{\gamma + (1-\gamma)K}$$

satisfies the following convergence bound:

$$f(\overline{x}_K) - f_* \leq \frac{C}{\gamma + (1-\gamma)K} = O\left(\frac{1}{K}\right),$$

where C is a positive constant depending on the initialization and the function values at the first $\overline{k}$ steps.

Proof. From the update rule of Algorithm 1, for each iteration $k \geq 1$, we have:

$$x_{k+1} - \gamma x_k = x_k - \gamma x_{k-1} - \lambda_k \nabla f(x_k). \tag{8}$$

Therefore, for any $x^* \in X^*$, we evaluate the expression:

$$\begin{aligned}
\|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2 &= \|x_k - \gamma x_{k-1} - (1-\gamma)x^*\|^2 + \lambda_k^2 \|\nabla f(x_k)\|^2 \\
&\quad - 2\langle x_k - \gamma x_{k-1} - (1-\gamma)x^*, \lambda_k \nabla f(x_k)\rangle \\
&= \|x_k - \gamma x_{k-1} - (1-\gamma)x^*\|^2 + \lambda_k^2 \|\nabla f(x_k)\|^2 \\
&\quad - 2\langle (1-\gamma)(x_k - x^*) + \gamma(x_k - x_{k-1}), \lambda_k \nabla f(x_k)\rangle \\
&= \|x_k - \gamma x_{k-1} - (1-\gamma)x^*\|^2 + \lambda_k^2 \|\nabla f(x_k)\|^2 \\
&\quad - 2(1-\gamma)\lambda_k \langle x_k - x^*, \nabla f(x_k)\rangle \\
&\quad - 2\gamma\lambda_k \langle x_k - x_{k-1}, \nabla f(x_k)\rangle.
\end{aligned} \tag{9}$$

Since f is L-smooth and convex, we apply inequality (2.3) from Lemma 2.3 to (9):

$$\begin{aligned}
\|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2 &\leq \|x_k - \gamma x_{k-1} - (1-\gamma)x^*\|^2 + \lambda_k^2 \|\nabla f(x_k)\|^2 \\
&\quad - 2(1-\gamma)\lambda_k \left(f(x_k) - f_* + \frac{1}{2L}\|\nabla f(x_k)\|^2 \right) \\
&\quad - 2\gamma\lambda_k \left(f(x_k) - f(x_{k-1}) + \frac{1}{2L}\|\nabla f(x_k) - \nabla f(x_{k-1})\|^2 \right) \\
&\leq \|x_k - \gamma x_{k-1} - (1-\gamma)x^*\|^2 - 2(1-\gamma)\lambda_k \left(f(x_k) - f_* \right) \\
&\quad - 2\gamma\lambda_k \left(f(x_k) - f(x_{k-1}) \right) + \left(\lambda_k^2 - \frac{(1-\gamma)\lambda_k}{L} \right) \|\nabla f(x_k)\|^2.
\end{aligned} \tag{10}$$

Dividing by $2\lambda_k$, we obtain:

$$\begin{aligned}
\frac{1}{2\lambda_k}\|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2 &\leq \frac{1}{2\lambda_k}\|x_k - \gamma x_{k-1} - (1-\gamma)x^*\|^2 - (1-\gamma)\left(f(x_k) - f_* \right) \\
&\quad - \gamma \left(f(x_k) - f(x_{k-1}) \right) + \frac{1}{2}\left(\lambda_k - \frac{1-\gamma}{L} \right) \|\nabla f(x_k)\|^2.
\end{aligned} \tag{11}$$

Summing (11) from $k = 1$ to K (where $K > \bar{k}$), we get:

$$\begin{aligned}
\sum_{k=1}^{K} \left((1-\gamma)(f(x_k) - f_*) + \gamma(f(x_k) - f(x_{k-1})) \right) &\leq \frac{1}{2\lambda_1} \|x_1 - \gamma x_0 - (1-\gamma)x^*\|^2 \\
&\quad + \sum_{k=1}^{K-1} \left(\frac{1}{2\lambda_{k+1}} - \frac{1}{2\lambda_k} \right) \|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2 \\
&\quad - \frac{1}{2\lambda_K} \|x_{K+1} - \gamma x_K - (1-\gamma)x^*\|^2 \\
&\quad + \frac{1}{2} \sum_{k=1}^{K} \left(\lambda_k - \frac{1-\gamma}{L} \right) \|\nabla f(x_k)\|^2.
\end{aligned} \tag{12}$$

Since $\bar{k}$ is the smallest integer satisfying inequality (6) in Lemma 3.5, the following properties hold:

- First, $\lambda_k \leq \lambda_{k+1}$ for all $k \geq \bar{k}$, which yields:

$$\sum_{k=\bar{k}}^{K-1} \left(\frac{1}{2\lambda_{k+1}} - \frac{1}{2\lambda_k} \right) \|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2 \leq 0. \tag{13}$$

- Second, since $\lambda_{\overline{k}-1} = \frac{\eta_1\|x_{\overline{k}} - x_{\overline{k}-1}\|}{\|\nabla f(x_{\overline{k}}) - \nabla f(x_{\overline{k}-1})\|}$, it follows from (7) that for all $k \geq \overline{k}$:

$$\lambda_k = (1+\varepsilon_{k-1})\lambda_{k-1} = \ldots = \prod_{j=\overline{k}}^{k}(1+\varepsilon_{j-1})\lambda_{\overline{k}-1} \leq e^{\sum\limits_{j=\overline{k}}^{k}\varepsilon_{j-1}}\lambda_{\overline{k}-1}$$
$$\leq e^{\sum\limits_{j=\overline{k}}^{+\infty}\varepsilon_{j-1}}\frac{\eta_1\|x_{\overline{k}} - x_{\overline{k}-1}\|}{\|\nabla f(x_{\overline{k}}) - \nabla f(x_{\overline{k}-1})\|} \leq \frac{1-\gamma}{L}. \tag{14}$$

Substituting (13) and (14) into (12), we obtain:

$$(1-\gamma)\sum_{k=1}^{K}(f(x_k) - f_*) + \gamma(f(x_K) - f(x_0)) \leq \sum_{k=1}^{\overline{k}-1}\left(\frac{1}{2\lambda_{k+1}} - \frac{1}{2\lambda_k}\right)\|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2$$
$$+ \frac{1}{2\lambda_1}\|x_1 - \gamma x_0 - (1-\gamma)x^*\|^2 + \frac{1}{2}\sum_{k=1}^{\overline{k}-1}\left(\lambda_k - \frac{1-\gamma}{L}\right)\|\nabla f(x_k)\|^2. \tag{15}$$

Let us define the constant C as:

$$C = \gamma(f(x_0) - f_*) + \sum_{k=1}^{\overline{k}-1}\left(\frac{1}{2\lambda_{k+1}} - \frac{1}{2\lambda_k}\right)\|x_{k+1} - \gamma x_k - (1-\gamma)x^*\|^2$$
$$+ \frac{1}{2\lambda_1}\|x_1 - \gamma x_0 - (1-\gamma)x^*\|^2 + \frac{1}{2}\sum_{k=1}^{\overline{k}-1}\left(\lambda_k - \frac{1-\gamma}{L}\right)\|\nabla f(x_k)\|^2.$$

Then, inequality (15) can be rewritten as:

$$\gamma(f(x_K) - f_*) + (1-\gamma)\sum_{k=1}^{K}\left(f(x_k) - f_*\right) \leq C. \tag{16}$$

By the convexity of f, we have:

$$f\left(\frac{\gamma x_K + (1-\gamma)\sum\limits_{k=1}^{K}x_k}{\gamma + (1-\gamma)K}\right) \leq \frac{\gamma f(x_K) + (1-\gamma)\sum\limits_{k=1}^{K}f(x_k)}{\gamma + (1-\gamma)K}. \tag{17}$$

Combining (16) and (17), we conclude that:

$$f(\overline{x}_K) - f_* \leq \frac{C}{\gamma + (1-\gamma)K} = O\left(\frac{1}{K}\right), \quad \forall K > \overline{k}, \tag{18}$$

where

$$\overline{x}_K = \frac{\gamma x_K + (1-\gamma)\sum\limits_{k=1}^{K}x_k}{\gamma + (1-\gamma)K}.$$

□

Remark 3.7. Following the classical scheme in [17], λ_k is chosen as a constant (i.e., $\lambda_k = \lambda \in (0, \frac{2(1+\gamma)}{L})$ for all $k = 0, 1, ...$) and the corresponding *local convergence* result of the Heavy Ball method was obtained for an objective function f satisfying the following conditions: 1. μ-strong convexity, 2. twice differentiable, 3. having a global Lipschitz gradient. Recently, in [6], the authors provided the *global convergence* of the Heavy Ball method for convex f and fixed stepsize λ_k. They obtained the ergodic convergence that is, the convergence evaluated at a weighted average of the generated iterates with the sublinear rate $O(\frac{1}{k})$. Hence, the convergence result established in Theorem 3.6 is comparable to those in the literature and has the advantage of eliminating the need to know the global Lipschitz constant L when computing λ_k.

3.2. The convergence of Algorithm 2 (NGDn)

We now establish the ergodic convergence of the Nesterov-accelerated version (NGDn). Similar to the previous section, the convergence analysis relies on the eventual stability of the stepsize.

Theorem 3.8. *Consider Problem (1) under Assumptions 2.5, 3.2, and 3.3. Let $\bar{k}$ be the smallest integer satisfying the condition in Lemma 3.5. Suppose the parameters satisfy:*

$$e^{\sum_{j=\bar{k}}^{+\infty}\varepsilon_{j-1}}\frac{\eta_1\|x_{\bar{k}}-x_{\bar{k}-1}\|}{\|\nabla f(x_{\bar{k}})-\nabla f(x_{\bar{k}-1})\|}\leq \min\left\{\frac{1-\gamma}{L}+\gamma\lambda_{min},\frac{1}{L}\right\}. \tag{19}$$

Then, for any $K>\bar{k}$, the sequence $\{x_k\}$ generated by Algorithm 2 satisfies:

$$f(\overline{x}_K)-f_*\leq\frac{D}{K(1-\gamma)+\gamma}=O\left(\frac{1}{K}\right),$$

where D is a positive constant and the ergodic iterate is defined as

$$\overline{x}_K=\frac{\gamma x_K+(1-\gamma)\sum_{k=1}^{K}x_k}{\gamma+(1-\gamma)K}.$$

Proof. From the update rules of Algorithm 2, we have for $k\geq 0$ (with $y_1=y_0$):

$$\begin{aligned}y_{k+1}&=x_k-\lambda_k\nabla f(x_k),\\ x_{k+1}&=y_{k+1}+\gamma(y_{k+1}-y_k).\end{aligned} \tag{20}$$

Let us define $a_0=\frac{\gamma}{1-\gamma}\lambda_0\nabla f(x_0)$ and

$$a_{k+1}=\frac{\gamma}{1-\gamma}\left(x_{k+1}-x_k+\lambda_k\nabla f(x_k)\right),\quad \text{for } k\geq 0. \tag{21}$$

It is straightforward to verify that

$$x_{k+1}+a_{k+1}=x_{k+1}+\frac{\gamma}{1-\gamma}\left(x_{k+1}-x_k+\lambda_k\nabla f(x_k)\right)=\frac{x_{k+1}}{1-\gamma}+\frac{\gamma}{1-\gamma}(\lambda_k\nabla f(x_k)-x_k).$$

From (20), we can write $x_{k+1}=(1+\gamma)\left(x_k+\lambda_k\nabla f(x_k)\right)-\gamma\left(x_{k-1}-\lambda_{k-1}\nabla f(x_{k-1})\right)$. Hence, for all $k\geq 0$:

$$\begin{aligned}x_{k+1}+a_{k+1}&=\frac{1+\gamma}{1-\gamma}\left(x_k-\lambda_k\nabla f(x_k)\right)-\frac{\gamma}{1-\gamma}\left(x_{k-1}-\lambda_{k-1}\nabla f(x_{k-1})\right)+\frac{\gamma}{1-\gamma}\left(\lambda_k\nabla f(x_k)-x_k\right)\\ &=x_k+\frac{\gamma}{1-\gamma}x_k-\frac{\lambda_k\nabla f(x_k)}{1-\gamma}-\frac{\gamma}{1-\gamma}\left(x_{k-1}-\lambda_{k-1}\nabla f(x_{k-1})\right)\\ &=x_k+\frac{\gamma}{1-\gamma}\left(x_k-x_{k-1}+\lambda_{k-1}\nabla f(x_{k-1})\right)-\frac{\lambda_k\nabla f(x_k)}{1-\gamma}\\ &=x_k+a_k-\frac{\lambda_k\nabla f(x_k)}{1-\gamma}.\end{aligned}$$

Squaring the norm of both sides yields:

$$\begin{aligned}\|x_{k+1}+a_{k+1}-x^*\|^2&=\left\|x_k+a_k-\frac{\lambda_k\nabla f(x_k)}{1-\gamma}-x^*\right\|^2\\ &=\|x_k+a_k-x^*\|^2+\frac{\lambda_k^2}{(1-\gamma)^2}\|\nabla f(x_k)\|^2-\frac{2\lambda_k}{1-\gamma}\langle x_k+a_k-x^*,\nabla f(x_k)\rangle\\ &=\|x_k+a_k-x^*\|^2+\frac{\lambda_k^2}{(1-\gamma)^2}\|\nabla f(x_k)\|^2-\frac{2\lambda_k}{1-\gamma}\langle x_k-x^*,\nabla f(x_k)\rangle\\ &\quad-\frac{2\gamma\lambda_k}{(1-\gamma)^2}\langle x_k-x_{k-1},\nabla f(x_k)\rangle-\frac{2\gamma\lambda_k\lambda_{k-1}}{(1-\gamma)^2}\langle\nabla f(x_{k-1}),\nabla f(x_k)\rangle.\end{aligned}$$

By the L-smoothness and convexity of f, and applying Lemma 2.3 (specifically inequality (2.3)), we obtain the following inequality:

$$\begin{aligned}\|x_{k+1}+a_{k+1}-x^*\|^2 \le \|x_k+a_k-x^*\|^2 &- \frac{2\lambda_k}{1-\gamma}(f(x_k)-f_*) - \frac{\lambda_k}{(1-\gamma)L}\|\nabla f(x_k)\|^2 \\ &+ \frac{\lambda_k^2\|\nabla f(x_k)\|^2}{(1-\gamma)^2} - \frac{2\gamma\lambda_k}{(1-\gamma)^2}(f(x_k)-f(x_{k-1})) \\ &- \frac{\gamma\lambda_k}{L(1-\gamma)^2}\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2 - \frac{2\gamma\lambda_k\lambda_{k-1}}{(1-\gamma)^2}\langle\nabla f(x_{k-1}),\nabla f(x_k)\rangle. \end{aligned} \tag{22}$$

Using the identity $-2\langle a,b\rangle = \|a-b\|^2 - \|a\|^2 - \|b\|^2$, inequality (22) can be rewritten as:

$$\begin{aligned}\|x_{k+1}+a_{k+1}-x^*\|^2 \le \|x_k+a_k-x^*\|^2 &- \frac{2\lambda_k}{1-\gamma}(f(x_k)-f(x^*)) - \frac{2\gamma\lambda_k}{(1-\gamma)^2}(f(x_k)-f(x_{k-1})) \\ &+ \left(\frac{\lambda_k^2}{(1-\gamma)^2} - \frac{\lambda_k}{(1-\gamma)L} - \frac{\gamma\lambda_k\lambda_{k-1}}{(1-\gamma)^2}\right)\|\nabla f(x_k)\|^2 \\ &+ \left(\frac{\gamma\lambda_k\lambda_{k-1}}{(1-\gamma)^2} - \frac{\gamma\lambda_k}{L(1-\gamma)^2}\right)\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2 \\ &- \frac{\gamma\lambda_k\lambda_{k-1}}{(1-\gamma)^2}\|\nabla f(x_{k-1})\|^2. \end{aligned} \tag{23}$$

Consequently, for all $k \ge 1$, multiplying by $\frac{(1-\gamma)^2}{2\lambda_k}$, we get:

$$\begin{aligned}\frac{(1-\gamma)^2}{2\lambda_k}\|x_{k+1}+a_{k+1}-x^*\|^2 \le \frac{(1-\gamma)^2}{2\lambda_k}\|x_k+a_k-x^*\|^2 &- (1-\gamma)(f(x_k)-f(x^*)) \\ &- \gamma(f(x_k)-f(x_{k-1})) - \frac{\gamma\lambda_{k-1}}{2}\|\nabla f(x_{k-1})\|^2 \\ &+ \underbrace{\frac{1}{2}\left(\lambda_k - \frac{1-\gamma}{L} - \gamma\lambda_{k-1}\right)\|\nabla f(x_k)\|^2}_{A_k} \\ &+ \underbrace{\frac{1}{2}\left(\gamma\lambda_{k-1} - \frac{\gamma}{L}\right)\|\nabla f(x_k)-\nabla f(x_{k-1})\|^2}_{B_k}. \end{aligned} \tag{24}$$

Recall that since $\bar{k}$ is the smallest integer satisfying inequality (6) in Lemma 3.5, we have $\lambda_{\bar{k}-1} = \frac{\eta_1\|x_{\bar{k}}-x_{\bar{k}-1}\|}{\|\nabla f(x_{\bar{k}})-\nabla f(x_{\bar{k}-1})\|}$. It follows that for all $k \ge \bar{k}$:

$$\begin{aligned}\lambda_k = (1+\varepsilon_{k-1})\lambda_{k-1} = \ldots = \prod_{j=\bar{k}}^{k}(1+\varepsilon_{j-1})\lambda_{\bar{k}-1} &\le e^{\sum_{j=\bar{k}}^{k}\varepsilon_{j-1}}\lambda_{\bar{k}-1} \\ &\le e^{\sum_{j=\bar{k}}^{+\infty}\varepsilon_{j-1}}\frac{\eta_1\|x_{\bar{k}}-x_{\bar{k}-1}\|}{\|\nabla f(x_{\bar{k}})-\nabla f(x_{\bar{k}-1})\|} \overset{(19)}{\le} \min\left\{\frac{1-\gamma}{L}+\gamma\lambda_{\min}, \frac{1}{L}\right\}. \end{aligned} \tag{25}$$

From (25), we derive that $A_k \le 0$ and $B_k \le 0$ for all $k \ge \bar{k}$. Now, taking $K > \bar{k}$ and summing (24) over $k = 1,\ldots,K$, we obtain:

$$\begin{aligned}\sum_{k=1}^{K}\frac{(1-\gamma)^2}{2\lambda_k}\|x_{k+1}+a_{k+1}-x^*\|^2 \le \sum_{k=1}^{K}\frac{(1-\gamma)^2}{2\lambda_k}\|x_k+a_k-x^*\|^2 &- (1-\gamma)\sum_{k=1}^{K}\big(f(x_k)-f_*\big) \\ &- \gamma\sum_{k=1}^{K}\big(f(x_k)-f(x_{k-1})\big) + \sum_{k=1}^{\bar{k}-1}(A_k+B_k). \end{aligned}$$

Rearranging the terms yields:

$$\begin{aligned}(1-\gamma)\sum_{k=1}^{K}\left(f(x_k)-f_*\right) \leq & -\sum_{k=1}^{K-1}\frac{(1-\gamma)^2}{2\lambda_k}\|x_{k+1}+a_{k+1}-x^*\|^2-\frac{(1-\gamma)^2}{2\lambda_K}\|x_{K+1}+a_{K+1}-x^*\|^2\\ &+\sum_{k=2}^{K}\frac{(1-\gamma)^2}{2\lambda_k}\|x_k+a_k-x^*\|^2+\frac{(1-\gamma)^2}{2\lambda_1}\|x_1+a_1-x^*\|^2\\ &-\gamma\left(f(x_K)-f(x_0)\right)+\sum_{k=1}^{\overline{k}-1}(A_k+B_k)\\ \leq & -\sum_{k=1}^{K-1}\frac{(1-\gamma)^2}{2\lambda_k}\|x_{k+1}+a_{k+1}-x^*\|^2+\sum_{k=1}^{K-1}\frac{(1-\gamma)^2}{2\lambda_{k+1}}\|x_{k+1}+a_{k+1}-x^*\|^2\\ &+\frac{(1-\gamma)^2}{2\lambda_1}\|x_1+a_1-x^*\|^2-\gamma\left(f(x_K)-f(x_0)\right)+\sum_{k=1}^{\overline{k}-1}(A_k+B_k)\\ \leq & (1-\gamma)^2\sum_{k=1}^{K-1}\left(\frac{1}{2\lambda_{k+1}}-\frac{1}{2\lambda_k}\right)\|x_{k+1}+a_{k+1}-x^*\|^2\\ &+\frac{(1-\gamma)^2}{2\lambda_1}\|x_1+a_1-x^*\|^2-\gamma\left(f(x_K)-f(x_0)\right)+\sum_{k=1}^{\overline{k}-1}(A_k+B_k). \end{aligned} \tag{26}$$

Recall from Lemma 3.5 that with the chosen $\overline{k}$, we have $\lambda_{k+1} > \lambda_k$ for all $k \geq \overline{k}$. Therefore:

$$(1-\gamma)^2\sum_{k=\overline{k}}^{K-1}\left(\frac{1}{2\lambda_{k+1}}-\frac{1}{2\lambda_k}\right)\|x_{k+1}+a_{k+1}-x^*\|^2 \leq 0.$$

Combining this with (26), we have:

$$(1-\gamma)\sum_{k=1}^{K}\left(f(x_k)-f_*\right)+\gamma(f(x_K)-f_*) \leq D,$$

where D is defined as:

$$\begin{aligned}D = \gamma\left(f(x_0)-f_*\right)+(1-\gamma)^2\sum_{k=1}^{\overline{k}-1}\left(\frac{1}{2\lambda_{k+1}}-\frac{1}{2\lambda_k}\right)\|x_{k+1}+a_{k+1}-x^*\|^2\\ +\frac{(1-\gamma)^2}{2\lambda_1}\|x_1+a_1-x^*\|^2+\sum_{k=1}^{\overline{k}-1}(A_k+B_k).\end{aligned}$$

By the convexity of f, we conclude that

$$f(\overline{x}_K)-f_* \leq \frac{D}{\gamma+(1-\gamma)K} = O\left(\frac{1}{K}\right), \quad \forall K > \overline{k}. \tag{27}$$

where

$$\overline{x}_K = \frac{\gamma x_K+(1-\gamma)\sum_{k=1}^{K}x_k}{\gamma+(1-\gamma)K}.$$

□

Remark 3.9. We highlight the following items regarding Algorithm 2 (**NGDn**):

(i) The classical Nesterov accelerated gradient method has strong convergence rate $O\left(\frac{1}{k^2}\right)$ under the same assumptions but typically requires a fixed stepsize $\lambda_k = 1/L$ or a backtracking line search

procedure (see Section 2.2, Theorem 2.2.2 [13]). While the former requires knowledge of the global constant L, which is often unavailable, the latter increases computational cost per iteration. Although the ergodic convergence rate of Algorithm 2 is $O(1/K)$ from some fixed iteration, it overcomes the limitations of classical stepsize used in classical Nesterov gradient descent method by dynamically adapting the stepsize using local Lipschitz constant of f. Its advantages will be demonstrated more clearly by numerical experiments for benchmark examples in machine learning in Section 5.

(ii) It is worth noting that the authors of [11] also proposed AdGD-accel which combines AdGD with Nesterov's method technique. However, the convergence of AdGD-accel has not yet been provided.

4. Stochastic versions of NGDh and NGDn

In this section, we extend our proposed methods to the stochastic setting, which is essential for large-scale machine learning and deep learning applications. We propose Algorithm 3 (SNGDh) and Algorithm 4 (SNGDn) for solving Problem (1) where the objective function is potentially nonconvex and has a finite-sum structure

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{1}{d}\sum_{i=1}^{d} f_i(x), \quad d \in \mathbb{N}. \tag{28}$$

This form of f is prevalent in machine learning and deep learning problems, where $x \in \mathbb{R}^n$ corresponds to the model parameters, d denotes the total number of training samples, $f_i(x)$, $i = 1, ..., d$, represents the loss on the training point i and the aim is to minimize the average loss $f(x)$ across training points.

A classical example of this structure is the least-squares linear regression problem, where each component is defined as $f_i(x) = \frac{1}{2}(a_i^\top x - b_i)^2$, with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ representing the input features and target responses, respectively. Another foundational example is binary logistic regression, where $f_i(x) = \log(1 + \exp(-b_i a_i^\top x))$ with labels $b_i \in \{-1, 1\}$. In modern applications, this finite-sum formulation extends naturally to training deep neural networks, where $f_i(x)$ denotes the loss incurred by the network (parameterized by weights x) on the i-th data point or mini-batch.

The critical challenge in large-scale learning is that the number of training examples d can be extremely large, making the evaluation of the full deterministic gradient $\nabla f(x) = \frac{1}{d}\sum_{i=1}^{d} \nabla f_i(x)$ computationally prohibitive at each iteration. To address this challenge, the stochastic gradient descent (SGD) approach is commonly used:

$$x_{k+1} = x_k - \lambda_k \nabla f_{\xi_k}(x_k),$$

where $\lambda_k > 0$ is the stepsize, and $\nabla f_{\xi_k}(x_k)$ is an unbiased stochastic estimator of the full gradient, satisfying

$$\mathbb{E}\left[\nabla f_{\xi_k}(x_k)\right] = \mathbb{E}\left[\nabla f_{\xi_k}(x_k)\middle| x_k\right] = \nabla f(x_k).$$

In practice, this estimator is computed using a mini-batch $\xi_k \subseteq \{1, \ldots, d\}$. Crucially, these mini-batches are drawn i.i.d. from the uniform distribution over the dataset at each iteration, yielding $\nabla f_{\xi_k}(x_k) = \frac{1}{|\xi_k|}\sum_{i\in\xi_k} \nabla f_i(x_k)$ to ensure the unbiasedness.

To define Algorithm 3 (SNGDh) and Algorithm 4 (SNGDn) we need the following assumption.

Assumption 4.1. (Bounded Second Moment). There exists a constant $\sigma > 0$ such that the unbiased stochastic estimator $\nabla f_{\xi_k}(x_k)$ satisfies:

$$\mathbb{E}\left[\|\nabla f_{\xi_k}(x_k)\|^2\right] = \mathbb{E}\left[\|\nabla f_{\xi_k}(x_k)\|^2 | x_k\right] \le \sigma^2, \quad \forall k \ge 0.$$

Algorithm 3 Stochastic version of NGDh (SNGDh)

Initialization. Select $\lambda_0 > 0$, $0 < \eta_1 < \eta_0$, $0 \leq \gamma < 1$ and a real positive sequence $\{\varepsilon_k\}$, $x_0 \in \mathbb{R}^n$, $\lambda_{\max} > \lambda_0$, $\lambda_{\max}$ is a sufficiently large number, and ξ_0 is the initial sample.
$v_1 = \nabla f_{\xi_0}(x_0)$
$x_1 = x_0 - \lambda_0 v_1$
for $k = 1, 2, \ldots$ **do**
  Sampling ξ_k
  if $\left\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\right\| > \frac{\eta_0}{\lambda_{k-1}} \|x_k - x_{k-1}\|$ **then**
    $\lambda_k = \eta_1 \frac{\|x_k - x_{k-1}\|}{\left\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\right\|}$
  else
    $\lambda_k = \min\{(1 + \varepsilon_{k-1})\lambda_{k-1}, \lambda_{\max}\}$
  end if
  $v_{k+1} = \gamma v_k + \nabla f_{\xi_k}(x_k)$
  $x_{k+1} = x_k - \lambda_k v_{k+1}$
end for

Algorithm 4 Stochastic version of NGDn (SNGDn)

Initialization. Select $\lambda_0 > 0$, $0 < \eta_1 < \eta_0$, $0 \leq \gamma < 1$ and a real positive sequence $\{\varepsilon_k\}$, $x_0 \in \mathbb{R}^n$, $\lambda_{\max} > \lambda_0$, $\lambda_{\max}$ is a sufficiently large number, and ξ_0 is the initial sample.
$v_1 = \nabla f_{\xi_0}(x_0)$
$x_1 = x_0 - \lambda_0 v_1$
for $k = 1, 2, \ldots$ **do**
  Sampling ξ_k
  if $\left\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\right\| > \frac{\eta_0}{\lambda_{k-1}} \|x_k - x_{k-1}\|$ **then**
    $\lambda_k = \eta_1 \frac{\|x_k - x_{k-1}\|}{\left\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\right\|}$
  else
    $\lambda_k = \min\{(1 + \varepsilon_{k-1})\lambda_{k-1}, \lambda_{\max}\}$
  end if
  $v_{k+1} = \gamma v_k + \nabla f_{\xi_k}(x_k)$
  $x_{k+1} = x_k - \lambda_k \left(\gamma v_{k+1} + \nabla f_{\xi_k}(x_k)\right)$
end for

Remark 4.2. It is worth noting that when $\gamma = 0$ Algorithms 3 and 4 reduce to Algorithms 1 and 2, respectively, with slight differences.

Specifically, we relax the condition imposed on the sequence $\{\varepsilon_k\}$ in Algorithms 3 and 4 by removing the requirement that $\sum_{k=0}^{+\infty} \varepsilon_k < +\infty$ which is required in Algorithms 1 and 2. Moreover, we introduce a new parameter $\lambda_{\max} > 0$ to provide an upper bound for the stepsize.

The bounded second moment assumption (Assumption 4.1) is very common, but it fails when the objective function f is strongly convex (as shown by Lam et al.[14]). Therefore, in this paper we will investigate the convergence of SNGDh and SNGDn without using the strongly convex assumption imposed on f. We only use the $L-$ smoothness property as follows.

Assumption 4.3. (Smoothness) Each function $f_i : \mathbb{R}^n \to \mathbb{R}, i \in \{1, \ldots, d\}$, is L-smooth, i.e., there exists a constant $L > 0$ such that

$$\left\|\nabla f_i(x) - \nabla f_i(y)\right\| \leq L \left\|x - y\right\|, \forall x, y \in \mathbb{R}^n.$$

Assumption 4.3 implies that f is also L-smooth.

Remark 4.4. It should be emphasized that there are two major distinctions between our proposed SNGD schemes and the related AdSGD algorithm given in [11]:

- **Acceleration:** SNGDh and SNGDn incorporate an explicit momentum term $\gamma \geq 0$. When $\gamma = 0$, they revert to a stochastic gradient descent without acceleration, whereas AdSGD is inherently a one-step method.

- **Nonconvexity:** The convergence analysis of AdSGD typically relies on the strong convexity of the objective function. In contrast, we establish convergence results for SNGDh and SNGDn in the nonconvex setting, which is more relevant for deep neural networks.

Next, in order to prove the convergence of our stochastic algorithms, we should prepare auxiliary results regarding the boundedness of the sequence of stepsizes generated by Algorithms 3 and 4 in the following lemma.

Lemma 4.5. *Suppose that Problem (1) satisfies Assumptions 2.5, 4.1 and 4.3. Let $\{\lambda_k\}$ be the sequence generated by Algorithm 3 (or Algorithm 4, respectively). Then,*

$$\min\left\{\lambda_0, \frac{\eta_1}{L}\right\} = \lambda_{\min} \leq \lambda_k \leq \lambda_{\max}, \quad \forall k \geq 0. \tag{29}$$

Proof. It is evident that $\lambda_k \geq \min\left\{\lambda_0, \frac{\eta_1}{L}\right\}$ holds for $k = 0$. For $k \geq 1$, if $\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\| > \frac{\eta_0}{\lambda_{k-1}}\|x_k - x_{k-1}\|$, then

$$\lambda_k = \frac{\eta_1\|x_k - x_{k-1}\|}{\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\|} \geq \frac{\eta_1}{L},$$

which follows from the L-smoothness of f_{ξ_k}. In the remaining case, where $\|\nabla f_{\xi_k}(x_k) - \nabla f_{\xi_k}(x_{k-1})\| \leq \frac{\eta_0}{\lambda_{k-1}}\|x_k - x_{k-1}\|$, we have

$$\lambda_k = (1 + \varepsilon_{k-1})\lambda_{k-1} \geq \lambda_{k-1}.$$

By induction, we conclude that $\lambda_k \geq \min\left\{\lambda_0, \frac{\eta_1}{L}\right\}$ for all $k \geq 0$.

Similarly, the upper bound of $\{\lambda_k\}$ is proved by induction. For $k = 0$, $\lambda_0 \leq \lambda_{\max}$ clearly holds. Suppose that $\lambda_i \leq \lambda_{\max}$ for all $i \leq k$. For the next step, if $\|\nabla f_{\xi_{k+1}}(x_{k+1}) - \nabla f_{\xi_{k+1}}(x_k)\| > \frac{\eta_0}{\lambda_k}\|x_{k+1} - x_k\|$, then $\frac{\eta_0\|x_{k+1}-x_k\|}{\|\nabla f_{\xi_{k+1}}(x_{k+1})-\nabla f_{\xi_{k+1}}(x_k)\|} < \lambda_k$. Consequently, since $\eta_1 < \eta_0$:

$$\lambda_{k+1} = \frac{\eta_1\|x_{k+1} - x_k\|}{\|\nabla f_{\xi_{k+1}}(x_{k+1}) - \nabla f_{\xi_{k+1}}(x_k)\|} < \frac{\eta_0\|x_{k+1} - x_k\|}{\|\nabla f_{\xi_{k+1}}(x_{k+1}) - \nabla f_{\xi_{k+1}}(x_k)\|} < \lambda_k \leq \lambda_{\max}. \tag{30}$$

In the other case, where $\|\nabla f_{\xi_{k+1}}(x_{k+1}) - \nabla f_{\xi_{k+1}}(x_k)\| \leq \frac{\eta_0}{\lambda_k}\|x_{k+1} - x_k\|$, we have:

$$\lambda_{k+1} = \min\{(1 + \varepsilon_{k-1})\,\lambda_{k-1}, \lambda_{\max}\} \leq \lambda_{\max}. \tag{31}$$

Both (30) and (31) imply $\lambda_{k+1} \leq \lambda_{\max}$. The induction proof is complete. □

The next lemma is useful for the convergence analysis of SNGDh and SNGDn.

Lemma 4.6. *Suppose that Problem (1) satisfies Assumptions 4.1 and 4.3. Let $\{x_k\}$ and $\{v_k\}$ be defined by Algorithm 3 (or Algorithm 4, respectively). Then we have:*

(i)

$$v_{k+1} = \sum_{i=0}^{k} \gamma^i \nabla f_{\xi_{k-i}}(x_{k-i}). \tag{32}$$

(ii)

$$\mathbb{E}\left[\|v_{k+1}\|^2\right] \leq \frac{\sigma^2}{(1-\gamma)^2}, \quad \forall k \geq 0. \tag{33}$$

(iii)

$$\mathbb{E}\left[\langle \nabla f(x_k), \nabla f_{\xi_k}(x_k)\rangle\right] = \mathbb{E}\left[\|\nabla f(x_k)\|^2\right], \quad \forall k \geq 1. \tag{34}$$

Proof. (i) This follows immediately from the update rule of v_{k+1}.

(ii) Using the expansion of the squared norm and the independence assumption:

$$\begin{aligned}
\mathbb{E}\left[\|v_{k+1}\|^2\right] &= \mathbb{E}\left[\left\|\sum_{i=0}^{k}\gamma^i\nabla f_{\xi_{k-i}}(x_{k-i})\right\|^2\right] \\
&= \mathbb{E}\left[\left\langle\sum_{i=0}^{k}\gamma^i\nabla f_{\xi_{k-i}}(x_{k-i}),\sum_{j=0}^{k}\gamma^j\nabla f_{\xi_{k-j}}(x_{k-j})\right\rangle\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{k}\sum_{j=0}^{k}\gamma^i\gamma^j\langle\nabla f_{\xi_{k-i}}(x_{k-i}),\nabla f_{\xi_{k-j}}(x_{k-j})\rangle\right] \\
&\le \mathbb{E}\left[\sum_{i=0}^{k}\sum_{j=0}^{k}\gamma^i\gamma^j\left(\frac{\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2}{2}+\frac{\|\nabla f_{\xi_{k-j}}(x_{k-j})\|^2}{2}\right)\right] \\
&\le \sum_{i=0}^{k}\sum_{j=0}^{k}\gamma^i\gamma^j\mathbb{E}\left[\frac{\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2}{2}+\frac{\|\nabla f_{\xi_{k-j}}(x_{k-j})\|^2}{2}\right] \\
&\le \sum_{i=0}^{k}\sum_{j=0}^{k}\gamma^i\gamma^j\sigma^2 \le \frac{\sigma^2}{(1-\gamma)^2}.
\end{aligned}$$

(iii) Using the law of iterated expectations and the unbiasedness property:

$$\begin{aligned}
\mathbb{E}\left[\langle\nabla f(x_k),\nabla f_{\xi_k}(x_k)\rangle\right] &= \mathbb{E}\left[\mathbb{E}\left[\langle\nabla f(x_k),\nabla f_{\xi_k}(x_k)\rangle \mid x_k\right]\right] \\
&= \mathbb{E}\left[\langle\nabla f(x_k),\mathbb{E}[\nabla f_{\xi_k}(x_k)\mid x_k]\rangle\right] \\
&= \mathbb{E}\left[\|\nabla f(x_k)\|^2\right], \quad \forall k\ge 1. \qquad (35)
\end{aligned}$$

□

4.1. Convergence analysis of Algorithm 3 (SNGDh)

In this section, we study the convergence of Algorithm 3. We first establish a descent-type lemma, which plays a crucial role in deriving the final convergence result.

Lemma 4.7. *Under Assumptions 2.5, 4.1 and 4.3, let $\{x_k\},\{v_k\},\{\lambda_k\}$ be the sequences defined by Algorithm 3. Then,*

$$\mathbb{E}\left[\langle\nabla f(x_k),v_{k+1}\rangle\right] \ge \sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] - \frac{\lambda_{\max}L\gamma\sigma^2}{(1-\gamma)^3}. \qquad (36)$$

Proof. Consider the expansion:

$$\begin{aligned}
\langle\nabla f(x_k),v_{k+1}\rangle &= \sum_{i=0}^{k}\gamma^i\langle\nabla f(x_k),\nabla f_{\xi_{k-i}}(x_{k-i})\rangle \\
&= \sum_{i=0}^{k}\gamma^i\langle\nabla f(x_{k-i}),\nabla f_{\xi_{k-i}}(x_{k-i})\rangle + \sum_{i=1}^{k}\gamma^i\underbrace{\langle\nabla f(x_k)-\nabla f(x_{k-i}),\nabla f_{\xi_{k-i}}(x_{k-i})\rangle}_{H_i}. \qquad (37)
\end{aligned}$$

By Assumption 4.3, f is L-smooth. Thus, we have:

$$\begin{aligned}
\|\nabla f(x_k)-\nabla f(x_{k-i})\|^2 \le L^2\|x_k-x_{k-i}\|^2 &\le L^2\left\|\sum_{l=1}^{i}(x_{k-l+1}-x_{k-l})\right\|^2 \\
&\le L^2\left\|\sum_{l=1}^{i}\lambda_{k-l}v_{k-l+1}\right\|^2 \le L^2\left\|\sum_{l=1}^{i}\lambda_{\max}v_{k-l+1}\right\|^2 \\
&\le \lambda_{\max}^2L^2i\sum_{l=1}^{i}\|v_{k-l+1}\|^2, \quad \forall i=1,\dots,k. \qquad (38)
\end{aligned}$$

On the other hand, applying Cauchy-Schwarz and Young's inequality:

$$H_i \geq -\frac{1}{2}\left(\frac{1-\gamma}{i\lambda_{\max}L}\|\nabla f(x_k)-\nabla f(x_{k-i})\|^2+\frac{i\lambda_{\max}L}{1-\gamma}\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right)$$
$$\overset{(38)}{\geq} -\frac{\lambda_{\max}L}{2}\left((1-\gamma)\sum_{l=1}^{i}\|v_{k-l+1}\|^2+\frac{i}{1-\gamma}\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right). \tag{39}$$

Therefore, substituting back into (37), we obtain:

$$\langle\nabla f(x_k),v_{k+1}\rangle \geq \sum_{i=0}^{k}\gamma^i\langle\nabla f(x_{k-i}),\nabla f_{\xi_{k-i}}(x_{k-i})\rangle$$
$$-\lambda_{\max}L\sum_{i=1}^{k}\frac{\gamma^i}{2}\left((1-\gamma)\sum_{l=1}^{i}\|v_{k-l+1}\|^2+\frac{i}{1-\gamma}\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right). \tag{40}$$

Taking the expectation of both sides of (40), we derive that:

$$\mathbb{E}\left[\langle\nabla f(x_k),v_{k+1}\rangle\right] \geq \sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\langle\nabla f(x_{k-i}),\nabla f_{\xi_{k-i}}(x_{k-i})\rangle\right]$$
$$-\lambda_{\max}L\sum_{i=1}^{k}\frac{\gamma^i}{2}\left((1-\gamma)\sum_{l=1}^{i}\mathbb{E}\left[\|v_{k-l+1}\|^2\right]+\frac{i}{1-\gamma}\mathbb{E}\left[\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right]\right). \tag{41}$$

Using Lemmas 2.4 and 4.6, combined with Assumption 4.1, we deduce:

$$\mathbb{E}\left[\langle\nabla f(x_k),v_{k+1}\rangle\right] \geq \sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right]$$
$$-\lambda_{\max}L\sum_{i=1}^{k}\frac{\gamma^i}{2}\left((1-\gamma)\left(\sum_{l=1}^{i}\frac{\sigma^2}{(1-\gamma)^2}\right)+\frac{i}{1-\gamma}\sigma^2\right)$$
$$\geq \sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right]-\frac{\lambda_{\max}\gamma L\sigma^2}{(1-\gamma)^3}. \tag{42}$$

□

The following theorem establishes an upper bound for the minimum expected value of $|\nabla f(x_k)|^2$ over the index k, implying that the best iterate in the sequence x_k lies within a neighborhood of a stationary point of f.

Theorem 4.8. *Consider Problem (1) under Assumptions 4.1, 4.3, and 2.5. Then, the sequence $\{x_k\}$ generated by Algorithm 3 satisfies:*

$$\min_{k=0,\ldots,K-1}\mathbb{E}\left[\|\nabla f(x_k)\|^2\right] \leq \frac{(1-\gamma)\left(f(x_0)-f_*\right)}{\lambda_{\min}(K+\gamma^2-\gamma)}+\frac{K}{(K+\gamma^2-\gamma)}\left(\frac{\lambda_{\max}L\gamma\sigma^2}{(1-\gamma)^2}+\frac{L\lambda_{\max}^2\sigma^2}{2\lambda_{\min}(1-\gamma)}\right). \tag{43}$$

Proof. By Assumption 4.3, f is L-smooth. Therefore, from Lemma 2.3 (inequality (2.3)), we have:

$$f(x_{k+1}) \leq f(x_k)+\langle x_{k+1}-x_k,\nabla f(x_k)\rangle+\frac{L}{2}\|x_{k+1}-x_k\|^2$$
$$\leq f(x_k)-\lambda_{k+1}\langle v_{k+1},\nabla f(x_k)\rangle+\frac{L}{2}\lambda_{k+1}^2\|v_{k+1}\|^2. \tag{44}$$

Recall from Lemma 4.6 that $\lambda_{\min}\leq\lambda_k\leq\lambda_{\max}<\infty$ for all $k\geq 0$. By taking expectations of both sides of (44), we have:

$$\mathbb{E}\left[f(x_{k+1})\right] \leq \mathbb{E}\left[f(x_k)\right]-\lambda_{\min}\mathbb{E}\left[\langle v_{k+1},\nabla f(x_k)\rangle\right]+\frac{L}{2}\lambda_{\max}^2\mathbb{E}\left[\|v_{k+1}\|^2\right]$$
$$\overset{(36),(33)}{\leq} \mathbb{E}\left[f(x_k)\right]-\lambda_{\min}\left(\sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right]-\frac{\lambda_{\max}L\gamma\sigma^2}{(1-\gamma)^3}\right)+\frac{L}{2}\lambda_{\max}^2\frac{\sigma^2}{(1-\gamma)^2}$$
$$\leq \mathbb{E}\left[f(x_k)\right]-\lambda_{\min}\left(\sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right]\right)+\frac{\lambda_{\min}\lambda_{\max}L\gamma\sigma^2}{(1-\gamma)^3}+\frac{L\lambda_{\max}^2\sigma^2}{2(1-\gamma)^2}. \tag{45}$$

Summing (45) over $k \in \{0, \dots, K-1\}$ and rearranging terms yields:

$$\lambda_{\min} \sum_{k=0}^{K-1} \sum_{i=0}^{k} \gamma^i \mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] \leq f(x_0) - \mathbb{E}\left[f(x_K)\right] + K\left(\frac{\lambda_{\min}\lambda_{\max} L\gamma\sigma^2}{(1-\gamma)^3} + \frac{L\lambda_{\max}^2\sigma^2}{2(1-\gamma)^2}\right). \tag{46}$$

On the other hand, changing the order of summation:

$$\begin{aligned}\sum_{k=0}^{K-1} \sum_{i=0}^{k} \gamma^i \mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] &= \sum_{k=0}^{K-1} \sum_{j=0}^{k} \gamma^{k-j} \mathbb{E}\left[\|\nabla f(x_j)\|^2\right] \\ &= \sum_{j=0}^{K-1} \mathbb{E}\left[\|\nabla f(x_j)\|^2\right] \sum_{k=j}^{K-1} \gamma^{k-j} \\ &= \frac{1}{1-\gamma} \sum_{j=0}^{K-1} \mathbb{E}\left[\|\nabla f(x_j)\|^2\right](1-\gamma^{K-j}).\end{aligned} \tag{47}$$

Observing that

$$\sum_{j=0}^{K-1}(1-\gamma^{K-j}) = K - \gamma\frac{1-\gamma^K}{1-\gamma} \geq K - \frac{\gamma}{1-\gamma}, \tag{48}$$

and combining (46), (47), with (48), we derive:

$$\lambda_{\min}\left(\frac{K}{1-\gamma} - \gamma\right) \min_{k=0,\dots,K-1} \mathbb{E}\left[\|\nabla f(x_k)\|^2\right] \leq f(x_0) - f_* + K\left(\frac{\lambda_{\min}\lambda_{\max} L\gamma\sigma^2}{(1-\gamma)^3} + \frac{L\lambda_{\max}^2\sigma^2}{2(1-\gamma)^2}\right).$$

This is equivalent to:

$$\min_{k=0,\dots,K-1} \mathbb{E}\left[\|\nabla f(x_k)\|^2\right] \leq \frac{(1-\gamma)\left(f(x_0) - f_*\right)}{\lambda_{\min}(K + \gamma^2 - \gamma)} + \frac{K}{(K+\gamma^2-\gamma)}\left(\frac{\lambda_{\max} L\gamma\sigma^2}{(1-\gamma)^2} + \frac{L\lambda_{\max}^2\sigma^2}{2\lambda_{\min}(1-\gamma)}\right).$$

□

4.2. Convergence analysis of Algorithm 4 (SNGDn)

Similar to the previous section, we first prove a descent inequality in the following lemma.

Lemma 4.9. *Under Assumptions 4.1, 4.3, and 2.5, let $\{x_k\}, \{v_k\}, \{\lambda_k\}$ be defined by Algorithm 4. Then,*

$$\mathbb{E}\left[\langle \nabla f(x_k), v_{k+1}\rangle\right] \geq \sum_{i=0}^{k} \gamma^i \mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] - \frac{\lambda_{\max}\gamma L\sigma^2(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3}. \tag{49}$$

Proof. Consider the expansion:

$$\begin{aligned}\langle \nabla f(x_k), v_{k+1}\rangle &= \sum_{i=0}^{k} \gamma^i \langle \nabla f(x_k), \nabla f_{\xi_{k-i}}(x_{k-i})\rangle \\ &= \sum_{i=0}^{k} \gamma^i \langle \nabla f(x_{k-i}), \nabla f_{\xi_{k-i}}(x_{k-i})\rangle + \sum_{i=1}^{k} \gamma^i \underbrace{\langle \nabla f(x_k) - \nabla f(x_{k-i}), \nabla f_{\xi_{k-i}}(x_{k-i})\rangle}_{N_i}.\end{aligned} \tag{50}$$

By Assumption 4.3, f is L-smooth. Thus:

$$\begin{aligned}\|\nabla f(x_k) - \nabla f(x_{k-i})\|^2 &\le L^2\|x_k - x_{k-i}\|^2 \le L^2\left\|\sum_{l=1}^{i}(x_{k-l+1} - x_{k-l})\right\|^2 \\ &\le L^2\left\|\sum_{l=1}^{i}\lambda_{k-l}\left(\gamma v_{k-l+1} + \nabla f_{\xi_{k-l}}(x_{k-l})\right)\right\|^2 \\ &\le \lambda_{\max}^2 L^2\left\|\sum_{l=1}^{i}\left(\gamma v_{k-l+1} + \nabla f_{\xi_{k-l}}(x_{k-l})\right)\right\|^2 \\ &\le 2\lambda_{\max}^2 L^2\left(\left\|\gamma\sum_{l=1}^{i} v_{k-l+1}\right\|^2 + \left\|\sum_{l=1}^{i}\nabla f_{\xi_{k-l}}(x_{k-l})\right\|^2\right) \\ &\le 2i\lambda_{\max}^2 L^2\left(\gamma^2\sum_{l=1}^{i}\|v_{k-l+1}\|^2 + \sum_{l=1}^{i}\|\nabla f_{\xi_{k-l}}(x_{k-l})\|^2\right), \quad \forall i = 1,\dots,k. \end{aligned} \tag{51}$$

Applying Cauchy-Schwarz and Young's inequality for N_i:

$$\begin{aligned} N_i &\ge -\frac{1}{2}\left(\frac{1-\gamma}{i\lambda_{\max}L}\|\nabla f(x_k) - \nabla f(x_{k-i})\|^2 + \frac{i\lambda_{\max}L}{1-\gamma}\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right) \\ &\overset{(51)}{\ge} -\frac{\lambda_{\max}L}{2}\left(2\gamma^2(1-\gamma)\sum_{l=1}^{i}\|v_{k-l+1}\|^2 + 2(1-\gamma)\sum_{l=1}^{i}\|\nabla f_{\xi_{k-l}}(x_{k-l})\|^2 + \frac{i}{1-\gamma}\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right). \end{aligned} \tag{52}$$

Substituting back into (50):

$$\begin{aligned}\langle\nabla f(x_k), v_{k+1}\rangle &\ge \sum_{i=0}^{k}\gamma^i\langle\nabla f(x_{k-i}), \nabla f_{\xi_{k-i}}(x_{k-i})\rangle \\ &\quad - \lambda_{\max}L\sum_{i=1}^{k}\frac{\gamma^i}{2}\left(2\gamma^2(1-\gamma)\sum_{l=1}^{i}\|v_{k-l+1}\|^2 + 2(1-\gamma)\sum_{l=1}^{i}\|f_{\xi_{k-l}}(x_{k-l})\|^2\right. \\ &\quad \left. + \frac{i}{1-\gamma}\|\nabla f_{\xi_{k-i}}(x_{k-i})\|^2\right). \end{aligned} \tag{53}$$

Taking expectations and applying Lemmas 2.4, 4.6, and Assumption 4.1:

$$\begin{aligned}\mathbb{E}\left[\langle\nabla f(x_k), v_{k+1}\rangle\right] &\ge \sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] \\ &\quad - \lambda_{\max}L\sum_{i=1}^{k}\frac{\gamma^i i}{2}\left(2\gamma^2(1-\gamma)\frac{\sigma^2}{(1-\gamma)^2} + 2(1-\gamma)\sigma^2 + \frac{1}{1-\gamma}\sigma^2\right) \\ &\ge \sum_{i=0}^{k}\gamma^i\mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] - \frac{\lambda_{\max}\gamma L\sigma^2(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3}. \end{aligned} \tag{54}$$

□

The final result of this study establishes the convergence of Algorithm 4, demonstrating that the best iterate approaches a neighborhood of some stationary point of the objective function f.

Theorem 4.10. *Consider Problem (1) under Assumptions 4.1, 4.3, and 2.5. Then, the sequence $\{x_k\}$ generated by Algorithm 4 satisfies:*

$$\begin{aligned}\min_{k=0,\dots,K-1}\mathbb{E}\left[\|\nabla f(x_k)\|^2\right] &\le \frac{(1-\gamma)\left(f(x_0) - f_*\right)}{\lambda_{\min}(K + (K-\gamma)(1-\gamma))} + \\ &\quad + \frac{KL\lambda_{\max}\sigma^2}{K + (K-\gamma)(1-\gamma)}\left(\frac{\gamma(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3} + \frac{\lambda_{\max}(\gamma^2 - \gamma + 1)}{\lambda_{\min}(1-\gamma)^2}\right). \end{aligned} \tag{55}$$

Proof. By Assumption 4.3, f is L-smooth. From Lemma 2.3 (inequality (2.3)):

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \left\langle x_{k+1} - x_k, \nabla f(x_k) \right\rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2 \\ &\leq f(x_k) - \lambda_{k+1} \left\langle \gamma v_{k+1} + \nabla f_{\xi_k}(x_k), \nabla f(x_k) \right\rangle + \frac{L}{2}\lambda_{k+1}^2 \|\gamma v_{k+1} + \nabla f_{\xi_k}(x_k)\|^2. \end{aligned} \tag{56}$$

Using Lemma 4.6 ($\lambda_{\min} \leq \lambda_k \leq \lambda_{\max}$), we take expectations:

$$\begin{aligned} \mathbb{E}\left[f(x_{k+1})\right] \leq \mathbb{E}\left[f(x_k)\right] &- \lambda_{\min}\left(\gamma \mathbb{E}\left[\langle v_{k+1}, \nabla f(x_k)\rangle\right] + \mathbb{E}\left[\langle \nabla f_{\xi_k}(x_k), \nabla f(x_k)\rangle\right]\right) \\ &+ L\lambda_{\max}^2 \left(\gamma \mathbb{E}\left[\|v_{k+1}\|^2\right] + \mathbb{E}\left[\|\nabla f_{\xi_k}(x_k)\|^2\right]\right). \end{aligned} \tag{57}$$

Combining with (49) and (33), we evaluate:

$$\begin{aligned} \mathbb{E}\left[f(x_{k+1})\right] &\leq \mathbb{E}\left[f(x_k)\right] - \lambda_{\min}\left(\sum_{i=0}^{k} \gamma^i \mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] - \frac{\lambda_{\max}\gamma L\sigma^2(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3} + \mathbb{E}\left[\|\nabla f(x_k)\|^2\right]\right) \\ &\quad + L\lambda_{\max}^2\left(\frac{\gamma\sigma^2}{(1-\gamma)^2} + \sigma^2\right) \\ &\leq \mathbb{E}\left[f(x_k)\right] - \lambda_{\min}\left(\sum_{i=0}^{k} \gamma^i \mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] + \mathbb{E}\left[\|\nabla f(x_k)\|^2\right]\right) \\ &\quad + \frac{\lambda_{\min}\lambda_{\max}L\gamma\sigma^2(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3} + \frac{L\lambda_{\max}^2\sigma^2(\gamma^2 - \gamma + 1)}{(1-\gamma)^2}. \end{aligned} \tag{58}$$

Summing (58) over $k \in \{0, \ldots, K-1\}$ and reformulating:

$$\begin{aligned} &\lambda_{\min}\sum_{k=0}^{K-1}\left(\sum_{i=0}^{k} \gamma^i \mathbb{E}\left[\|\nabla f(x_{k-i})\|^2\right] + \mathbb{E}\left[\|\nabla f(x_k)\|^2\right]\right) \\ &\quad \leq f(x_0) - \mathbb{E}\left[f(x_K)\right] + K\left(\frac{\lambda_{\min}\lambda_{\max}L\gamma\sigma^2(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3} + \frac{L\lambda_{\max}^2\sigma^2(\gamma^2 - \gamma + 1)}{(1-\gamma)^2}\right). \end{aligned} \tag{59}$$

Using the identity from (47) and (48), we substitute back into (59) to derive:

$$\begin{aligned} &\lambda_{\min}\left(\frac{K}{1-\gamma} - \gamma + K\right)\min_{k=0,\ldots,K-1} \mathbb{E}\left[\|\nabla f(x_k)\|^2\right] \\ &\quad \leq f(x_0) - f_* + K\left(\frac{\lambda_{\min}\lambda_{\max}L\gamma\sigma^2(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3} + \frac{L\lambda_{\max}^2\sigma^2(\gamma^2 - \gamma + 1)}{(1-\gamma)^2}\right). \end{aligned}$$

This is equivalent to the stated result:

$$\begin{aligned} \min_{k=0,\ldots,K-1} \mathbb{E}\left[\|\nabla f(x_k)\|^2\right] \leq &\frac{(1-\gamma)\left(f(x_0) - f_*\right)}{\lambda_{\min}(K + (K-\gamma)(1-\gamma))} + \\ &+ \frac{KL\lambda_{\max}\sigma^2}{K + (K-\gamma)(1-\gamma)}\left(\frac{\gamma(4\gamma^2 - 4\gamma + 3)}{2(1-\gamma)^3} + \frac{\lambda_{\max}(\gamma^2 - \gamma + 1)}{\lambda_{\min}(1-\gamma)^2}\right). \end{aligned}$$

□

5. Numerical experiments

In this section, we evaluate the performance of the proposed algorithms on logistic regression and deep neural network training. All algorithms are implemented in Python. Our source code is available at https://github.com/hoaiphamthi/Accelerated-and-Stochastic-NGD.

5.1. Logistic Regression

In this subsection, experiments are conducted on the logistic regression problem, which minimizes the following objective function:

$$f(x) = \frac{1}{d}\sum_{i=1}^{d} \log(1 + \exp(-b_i a_i^T x)) + \frac{\ell_2}{2}\|x\|^2,$$

where $(a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$, $i = 1, \ldots, d$, denote the observations, and $\ell_2 > 0$ is the regularization parameter. Clearly, f is μ-strongly convex with $\mu = \ell_2$. We compare our proposed algorithms, **NGDh** and **NGDn**, with related algorithms including **GD** [10], **Nesterov-accel**, **Heavy-ball-accel**, **AdGD**, and **AdGD-accel** [11]. We utilize popular benchmark datasets [1] such as `a9a`, `cod-rna`, `ijcnn1`, `mushroom`, `phishing`, `skin_nonskin` and `w8a`. In the experiments, the parameters are configured as follows:

- **GD:** $\lambda_k = \frac{1}{L}$ for all $k \geq 0$, where L is the smoothness constant of f.
- **Nesterov-accel:** Nesterov's accelerated gradient descent configured with a constant stepsize $\lambda = \frac{1}{L}$ and a fixed momentum coefficient $\gamma = \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$.
- **Heavy-ball-accel:** Heavy Ball Gradient descent algorithm is configured with a fixed stepsize $\lambda = \frac{4}{(\sqrt{L}+\sqrt{\mu})^2}$ and a momentum coefficient $\gamma = \left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2$.
- **AdGD** and **AdGD-accel:** Default parameters are used as provided in the public source code[2].
- **NGD:** $\lambda_0 = 10^{-3}$, $\eta_0 = 0.2$, $\eta_1 = 0.15$, and $\varepsilon_{k-1} = \dfrac{2(\log k)^{4.5}}{k^{1.1}}$ for $k \geq 1$.
- **NGDn** and **NGDh:** $\lambda_0 = 0.01$, $\eta_0 = 0.2$, $\eta_1 = 0.19$, $\gamma = 0.9$ and $\varepsilon_{k-1} = \dfrac{3}{k^{1.1}}$ for $k \geq 1$.

As illustrated in Figure 1, both NGDh and NGDn demonstrate superior performance across most tested datasets. The results also show that on certain datasets, with a fixed stepsize, Heavy-ball-accel and Nesterov-accel algorithms exhibit severe oscillations and instability.

5.2. Deep Neural Network Training

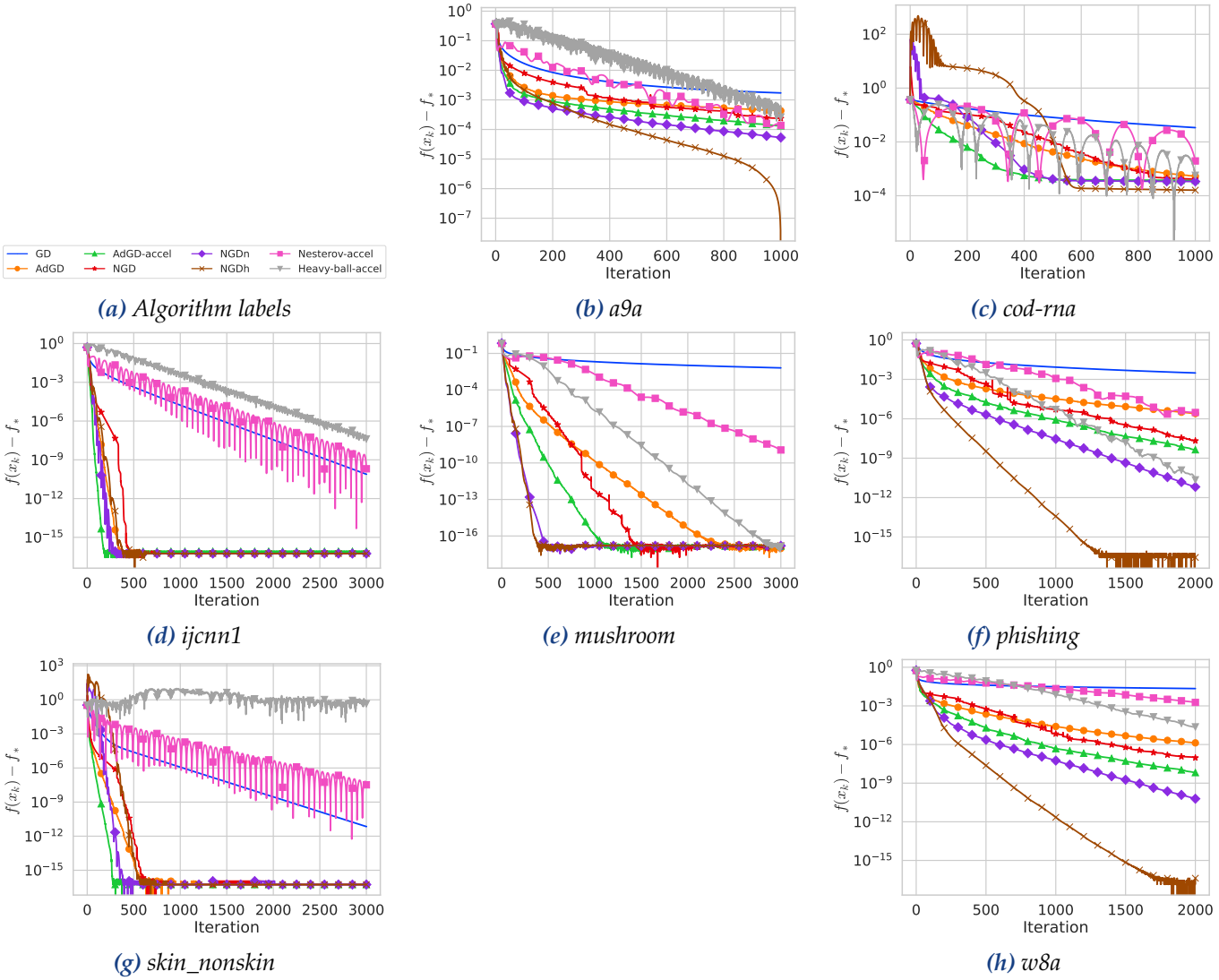
The subsequent experiments evaluate the performance of the stochastic algorithms, **SNGDh** and **SNGDn**, on the benchmark datasets **MNIST** [5], **FashionMNIST** [20], and **CIFAR-10** [9]. The compared algorithms include **SGD**, **SGD-Heavy Ball Accel**, **SGD-Nesterov Accel**, **AdSGD** [11], **Adam**, **SNGDn** and **SNGDh**. The hyperparameters are set as follows:

- **SGD**, **SGD-Heavy Ball Accel** and **SGD-Nesterov Accel**: Default parameters are used, with $\lambda_k = 0.01$, for all $k \geq 0$. The momentum parameters for SGD-Heavy Ball Accel and SGD-Nesterov Accel are both set to $\gamma = 0.9$.
- **AdSGD:** Default parameters are used as in the public source code of [11].
- **SNGDn** and **SNGDh:** $\lambda_0 = 10^{-5}$, $\eta_0 = 0.2$, $\eta_1 = 0.15$, $\varepsilon_{k-1} = \dfrac{1}{k^{0.9}}$, $k \geq 1$, and $\gamma = 0.9$. The parameter $\lambda_{\max}$ is initially set to a sufficiently large positive number (e.g., $\lambda_{\max} = 10$). However, empirical observations revealed that the effective stepsizes for **SNGDh** and **SNGDn** remained very small (less than 1, see Figure 3). Consequently, we omitted this parameter in our public Python implementation.

The numerical results are illustrated in Figures 2 and 3. As shown in Figure 2, **SNGDn** and **SNGDh** exhibit the best performance across all datasets. Particularly on FashionMNIST and CIFAR-10, **SNGDh** yields the lowest training losses.

Observing the stepsize in Figure 3, it is evident that our algorithms (**SNGDh** and **SNGDn**) exhibit adaptive behavior throughout the training phase. They maintain large stepsizes during the initial epochs and gradually decay upon converging to a stationary point. Consequently, our algorithms provide the best performances among the compared methods regarding objective function values (training losses).

[1] All datasets are available at https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/
[2] https://github.com/ymalitsky/adaptive_gd

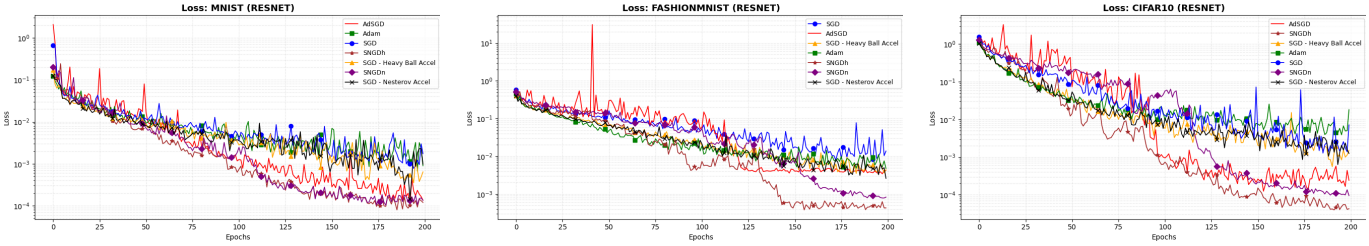


(a) Algorithm labels *(b)* a9a *(c)* cod-rna

(d) ijcnn1 *(e)* mushroom *(f)* phishing

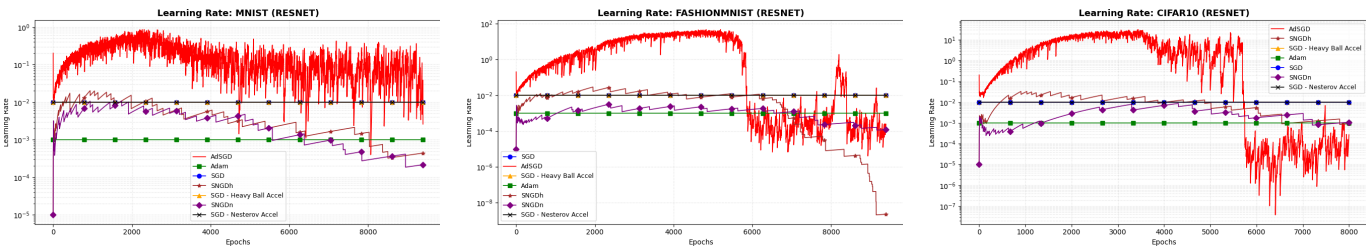
(g) skin_nonskin *(h)* w8a

Figure 1. *Numerical results on objective values for the logistic regression problem*



(a) MNIST *(b)* FashionMNIST *(c)* CIFAR-10

Figure 2. *Numerical results for the Deep Neural Network training loss*



(a) MNIST *(b)* FashionMNIST *(c)* CIFAR-10

Figure 3. *Numerical results on stepsize for the Deep Neural Network*

6. Conclusions

In this paper, we have developed efficient accelerated and stochastic variants of the NGD algorithm originally proposed in [8]. By incorporating Polyak's Heavy Ball and Nesterov's momentum techniques, we address the limitations of fixed stepsize and backtracking line search strategies. From some finite iteration, we establish the ergodic convergence of the proposed deterministic algorithms with a sublinear rate of $O(1/K)$ for convex objective functions, relying solely on local Lipschitz estimates rather than a global constant. Furthermore, we extend these methods to the stochastic setting to handle large-scale optimization problems. We provide a rigorous convergence analysis for the stochastic algorithms (SNGDh and SNGDn) within a nonconvex framework, proving that the expected gradient norm remains bounded under standard variance assumptions. Numerical experiments on logistic regression and deep neural network training (using the MNIST, FashionMNIST, and CIFAR10 datasets) verify that our proposed algorithms achieve superior performance and faster speed compared to existing methods.

Acknowledgments: The authors would like to thank the editors and the anonymous referees very much for their valuable comments, which significantly improved the quality of this paper. The author (Pham Thi Hoai) would like to express her gratitude to Professor Nguyen Dong Yen for providing valuable materials and constructive comments that greatly enhanced this work.

References

[1] A. Beck: *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB*, Society for Industrial and Applied Mathematics (2014).

[2] A. Beck: *First order methods in optimization*, Society for Industrial and Applied Mathematics, USA (2017).

[3] D. P. Bertsekas: *Nonlinear programming*, Athena Scientific, 3rd ed. (2016).

[4] L. Bottou, F. E. Curtis, J. Nocedal: *Optimization methods for large-scale machine learning*, SIAM Review 60 (2018) 223–311.

[5] L. Deng: *The mnist database of handwritten digit images for machine learning research*, IEEE Signal Processing Magazine 29(6) (2012) 141–142.

[6] E. Ghadimi, H. R. Feyzmahdavian, M. Johansson: *Global convergence of the Heavy-ball method for convex optimization*, 2015 European Control Conference (ECC) (2015) 310–315.

[7] E. Hazan, S. Kale: *Beyond the Regret Minimization Barrier: Optimal Algorithms for Stochastic Strongly-Convex Optimization*, Journal of Machine Learning Research 15(71) (2014) 2489–2512.

[8] P. T. Hoai, N. T. Vinh, N. P. H. Chung: *A novel stepsize for gradient descent method*, Operations Research Letters 53 (2024) 107072.

[9] A. Krizhevsky, G. Hinton: *Learning multiple layers of features from tiny images*, Master's thesis, Department of Computer Science, University of Toronto (2009).

[10] C. Lemaréchal: *Cauchy and the gradient method*, Doc. Math. Extra Vol. Optimization Stories (2012) 251–254.

[11] Y. Malitsky, K. Mishchenko: *Adaptive Gradient Descent without Descent*, Proceedings of the 37th International Conference on Machine Learning 119, PMLR (2020) 6702–6712.

[12] A. S. Nemirovski, A. B. Juditsky, G. Lan, A. Shapiro: *Robust Stochastic Approximation Approach to Stochastic Programming*, SIAM Journal on Optimization 19(4) (2009) 1574–1609.

[13] Y. Nesterov: *Lectures on Convex Optimization*, Springer Publishing Company, Incorporated, 2nd ed. (2018).

[14] L. Nguyen, P. H. Nguyen, M. van Dijk, P. Richtarik, K. Scheinberg, M. Takac: *SGD and Hogwild! Convergence Without the Bounded Gradients Assumption*, Proceedings of the 35th International Conference on Machine Learning 80, PMLR (2018) 3750–3758.

[15] J. Nocedal, S. J. Wright: *Numerical Optimization*, Springer Series in Operations Research and Financial Engineering (2006).

[16] B. T. Polyak: *Some methods of speeding up the convergence of iteration methods*, USSR Computational Mathematics and Mathematical Physics 4(5) (1964) 1–17.

[17] B. T. Polyak: *Introduction to optimization*, New York, Optimization Software (1987).

[18] B. Recht, C. Re, S. Wright, F. Niu: *Hogwild!: A Lock-Free Approach to Parallelizing Stochastic Gradient Descent*, Advances in Neural Information Processing Systems 24, Curran Associates, Inc. (2011).

[19] C. Suh: *Convex Optimization for Machine Learning*, Boston-Delft: Now Publishers (2022).

[20] H. Xiao, K. Rasul, R. Vollgraf: *Fashion-MNIST: a Novel Image Dataset for Benchmarking Machine Learning Algorithms*, arXiv:1708.07747 [cs.LG] (2017).