BANACH PRESS

JOURNAL OF CONVEX ANALYSIS
Volume 34 (2027), No.1, 294–308
https://journalofconvexanalysis.com

Asymptotics of Nonlocal Energies with a Bilateral Constraint

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Abstract: We consider a family of nonlocal functionals F_ε depending on a small parameter $\varepsilon > 0$ and impose a generic Dirichlet constraint on a periodic array of small balls of radius r_δ centered on a δ-periodic lattice. We investigate the asymptotic behaviour of this type of constrained functionals when the order parameter ε tends to zero. Such analysis is deeply affected by the interplay between the three scales ε, δ_ε and r_{δ_ε}. Assuming that $\varepsilon = o(r_{\delta_\varepsilon})$ we perform a complete analysis when the datum assigned on the perforation coincides with a function in $C^1(\mathbb{R}^d;\mathbb{R}^m)$. More precisely, according to the scaling among the periodicity and the radius of the 'perforation', we prove the Γ-convergence of F_ε, with respect to the strong L^p topology, to some local functionals defined on Sobolev spaces as $\varepsilon \to 0$, accounting for a possible penalization term of capacitary type.

Keywords: Convolution functionals, nonlocal energies, homogenization, periodic perforated domains.
2020 Mathematics Subject Classification: 49J45, 49J55, 74Q05, 35B27, 35B40, 45E10.

1. Introduction

In the past two decades there has been a growing interest in the variational analysis of nonlocal functionals of convolution type, motivated by applications ranging from peridynamics and phase transitions to image processing and collective dynamics (see e.g. [1, 6, 7, 12, 19, 20, 21, 22, 26]). A central aspect of these studies concerns the rigorous passage from nonlocal to local models as the interaction range tends to zero. This transition, first identified by Bourgain, Brezis and Mironescu [7] for Gagliardo seminorms, was later formalized in the framework of Γ-convergence by Ponce [21]. There, it was shown that, with fixed Ω a bounded regular set in R^d, families of energies of the form

$$E_\varepsilon(u) = \int_\Omega \int_\Omega f_\varepsilon(x, y, u(y) - u(x))\, dx\, dy$$

Γ-converge, as $\varepsilon \to 0$, to local integral functionals of Sobolev type, provided that the kernels f_ε satisfy some suitable growth conditions of order $p \geq 1$ and concentrate on the diagonal.

In this paper we will examine some problems related to such nonlocal energies, for $1 < p < d$, in particular when constraints are imposed at a fine scale. In the local case a large literature has been devoted to the effect of Dirichlet or pinning constraints imposed on a fine distribution of small inclusions, in the spirit of the classical homogenization in perforated domains. The seminal paper by Cioranescu and Murat [13] showed that, for local energies of the form

$$\int_\Omega f(x, \nabla u)\, dx,$$

with $f(x, \xi)$ satisfying growth conditions of order $p > 1$ in the variable ξ, and subject to homogeneous Dirichlet conditions on a δ-periodic array of small holes of radius r_δ, when $p < d$, a *critical scaling* $r_\delta \sim \delta^{d/(d-p)}$ leads to the appearance of an additional *capacitary term* in the limit functional. This term reflects the averaged energetic contribution of the pinning constraint. Subsequent works extended these results to

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https://doi.org/10.68381/jca34021
Received date: October 31, 2025; Accepted date: April 11, 2026
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vectorial problems, nonperiodic geometries, discrete-to-continuum setting, and nonlinear or degenerate growth conditions (see, among others, [5, 9, 24, 25]).

Actually, inhomogeneous Dirichlet conditions can be also considered in the same setting. In such cases, the limit energy acquires a penalization depending on the deviation of u from the prescribed datum ψ, yielding expressions of the type

$$\int_\Omega f_{\hom}(\nabla u)\,dx + C\int_\Omega |u-\psi|^p\,dx,$$

with C related to the p-capacity of the reference inclusion. These problems, which we may call *bilateral* or *inhomogeneous perforation problems*, constitute the local analogue of the setting studied in this paper.

In the nonlocal framework, the analysis of Dirichlet-type constraints on perforated domains has been initiated in [3], where the authors considered convolution-type functionals

$$F_{\varepsilon,\delta}(u) = \int_{\mathbb{R}^d}\int_{\Omega_\varepsilon(\xi)} f\left(\xi, \frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right)dx\,d\xi,$$

with $u = 0$ on a periodic array of balls of radius r_δ centered on a δ-periodic lattice. The asymptotic behavior of $F_{\varepsilon,\delta}$ depends crucially on the interplay among the three scales ε, δ, and r_δ. When $\varepsilon = o(r_\delta)$ and $r_\delta = O(\delta^{d/(d-p)})$, the Γ-limit coincides with a local homogenized functional augmented by a local capacitary term; when $\varepsilon = O(r_\delta)$, a *nonlocal capacitary density* appears, while slower decay of ε yields trivial or unconstrained limits. A fundamental tool in that analysis is a nonlocal Gagliardo-Nirenberg-Sobolev inequality (Theorem 5.1 in [3]), ensuring L^{p^*} control of suitable interpolations in terms of the nonlocal energy.

The present paper extends that analysis to the bilateral (inhomogeneous) Dirichlet setting, where the order parameter is required to match a prescribed datum $\psi \in C^1(\mathbb{R}^d;\mathbb{R}^m)$ on the perforations. We consider functionals of the form

$$F_\varepsilon(u) = \int_{\mathbb{R}^d}\int_{\Omega_\varepsilon(\xi)} f\left(\xi, \frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right)dx\,d\xi, \qquad u=\psi \text{ on } P_\varepsilon,$$

where P_ε is a δ_ε-periodic array of balls of radius r_ε.

Assuming $\varepsilon = o(r_\varepsilon)$ and $r_\varepsilon/\delta_\varepsilon^{d/(d-p)} \to \beta > 0$, we show that, with respect to the strong L^p-topology,

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{\hom}(\nabla u)\,dx + \beta^{d-p}\int_\Omega \varphi(u-\psi)\,dx, & u\in W^{1,p}(\Omega;\mathbb{R}^m),\\ +\infty, & \text{otherwise,}\end{cases}$$

where $f_{\hom}$ is the homogenized density defined through a 'suitable non-local version of the asymptotic homogenization cell formula' and φ is identified as a 'suitable nonlocal capacitary density'

$$\varphi(z) = \inf\left\{\int_{\mathbb{R}^d} f_{\hom}(\nabla v)\,dx : v - z \in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m),\ v\equiv 0 \text{ in } B_1\right\}.$$

The additional term $\int_\Omega \varphi(u-\psi)\,dx$ represents the energetic cost of enforcing the bilateral constraint, extending to the inhomogeneous case the results of [3]. Depending on the scaling of r_ε, one recovers the classical dichotomy: negligible constraint when $\beta = 0$ and total pinning when $\beta = +\infty$.

Our analysis connects naturally with the recent work by Braides, Noselli, and Vincini [10], who studied the equilibrium of hyperelastic bodies subjected to general kinematic constraints on many small regions. In their model, the displacement u is required to satisfy $u(x) \in F_x$, where F_x is a closed set possibly depending on x, and the constraint is imposed on a fine distribution of small "perforations". They proved that, under suitable assumptions on the strain-energy density σ and on F_x, the Γ-limit functional takes the form

$$\int_\Omega \sigma(\nabla u)\,dx + \int_\Omega \varphi(x,u(x))\,dx,$$

where $\varphi(x,\cdot)$ arises from a capacitary-type cell problem depending on F_x and the recession function of σ. Their framework includes both homogeneous and inhomogeneous, unilateral, or even nonlinear constraints, and provides a unified mechanical interpretation of micro-scale pinning in hyperelastic media. The present

work can thus be viewed as a nonlocal analogue of the asymptotic theory developed in [10], specialized to the case of affine Dirichlet data $F_x = \{\psi(x)\}$ and convolution-type nonlocal energies.

From a conceptual standpoint, the results proved here clarify how the capacitary penalization mechanism arises universally in both local and nonlocal regimes, whether the constraint is homogeneous, inhomogeneous, or nonlinear, and how the nonlocal effects vanish in the subcritical regime $\varepsilon = o(r_\varepsilon)$. Our approach relies on a nonlocal joining lemma, a truncation argument, and the asymptotic characterization of capacitary densities via Γ-convergence, but the main novelty lies in the uniform control of energies in presence of nonzero Dirichlet data.

Finally, we remark that this study opens several perspectives: the extension to the critical case $p = d$ and to non-homogeneous kernels, the inclusion of randomly distributed perforations, and the treatment of anisotropic or nonlinear constraint sets in the spirit of [10]. Such developments would further bridge the fine asymptotic analysis of nonlocal functionals with the mechanical modeling of materials endowed with complex micro-constraints.

2. Notation

In what follows $d, m \in \mathbb{N}$ will be fixed natural numbers denoting the dimension of the reference and target spaces of the functions we consider, respectively. Given $t \in \mathbb{R}$, $\lfloor t \rfloor$ denotes the integer part of t; for $x \in \mathbb{R}^d$, $r > 0$, $B_r(x)$ (if $x = 0$, simply B_r) stands for the open ball of centre x and radius r, $Q_r(x)$ (if $x = 0$, simply Q_r) stands for the cube $x + (-\frac{r}{2}, \frac{r}{2})^d$. We denote by S^{d-1} the unit sphere in $\mathbb{R}^d$. If A is a subset of $\mathbb{R}^d$ then $\operatorname{dist}(x, A) = \inf\{|y - x| : y \in A\}$. $\mathcal{A}^{\mathrm{reg}}(A)$ is the subfamily of open subsets with Lipschitz boundary. By $A \subset\subset B$ we mean that the closure of A is a compact subset of B. If $A \subset\subset B$, a *cut-off function* between A and B is a (smooth) function φ with $0 \le \varphi \le 1$, $\varphi = 0$ on ∂B and $\varphi = 1$ on A. Given a real function $h(\cdot)$, we use the symbols $o(h)$ to denote a generic function g such that $\lim_{t\to 0}(g(t)/h(t)) = 0$. If E is a measurable subset of $\mathbb{R}^d$ we denote by $|E|$ its Lebesgue measure. We use standard notation for Lebesgue and Sobolev spaces. If $p < d$, $p^* := (pd)/(d - p)$ is the Sobolev exponent of p. If u is an integrable function on a measurable set $E \subset \mathbb{R}^d$ with $|E| > 0$,

$$u_E := \frac{1}{|E|}\int_E u(x)dx$$

denotes the average of u on E. We use also standard notation for Γ-convergence [8, 14], indicating the topology with respect to which it is performed. Unless otherwise stated, the letter C denotes a generic strictly positive constant. Relevant dependencies on parameters will be as usual emphasised by putting them in parentheses.

3. Setting of the problem and main results

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let us fix a function $\psi \in C^1(\mathbb{R}^d; \mathbb{R}^m)$, and $1 < p < d$. Let $\varepsilon, \delta_\varepsilon, r_\varepsilon$ be given positive parameters with $\delta_\varepsilon > r_\varepsilon \gg \varepsilon > 0$. We set

$$P_\varepsilon := \bigcup_{i\in\mathbb{Z}^d} B_{r_\varepsilon}(\delta_\varepsilon i), \tag{1}$$

and, given $f : \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty)$ Borel function, we introduce the non-local functionals $F_\varepsilon : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined as

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^d}\int_{\Omega_\varepsilon(\xi)} f\left(\xi, D^\xi_\varepsilon u(x)\right)dx\,d\xi & \text{if } u(x) = \psi(x) \text{ a.e. on } P_\varepsilon \cap \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{2}$$

where

$$D^\xi_\varepsilon u(x) := \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon} \tag{3}$$

and for every subset A of $\mathbb{R}^d$ we set

$$A_\varepsilon(\xi) := \{x \in A \,|\, x + \varepsilon\xi \in A\}. \tag{4}$$

We consider the following set of assumptions on the function f above:

(H) (positive p - homogeneity) $f(\xi, tz) = t^p f(\xi, z)$ for every $(\xi, z) \in \mathbb{R}^d \times \mathbb{R}^m$ and $t > 0$;

(G) (growth) the functions $m(\xi) := \inf_{z\in S^{m-1}} f(\xi,z)$ and $M(\xi) := \sup_{z\in S^{m-1}} f(\xi,z)$ satisfy:

(G0) there exist $\lambda_0, r_0 > 0$ such that $m(\xi) \geq \lambda_0$ if $|\xi| \leq r_0$;

(G1) $\int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\, d\xi < +\infty$;

(L) (p - Lipschitz continuity) there exists $C > 0$ such that for every $\xi \in \mathbb{R}^d$

$$|f(\xi,w) - f(\xi,z)| \leq CM(\xi)(|z|^{p-1} + |w|^{p-1})|w - z| \quad \forall z, w \in \mathbb{R}^m.$$

The main results of the paper are provided by the following theorems.

Theorem 3.1. *Let F_ε be defined by (2), with f satisfying assumptions* (**H**), (**G**) *and* (**L**). *Assume moreover that*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_\varepsilon} = 0 \tag{5}$$

and

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\delta_\varepsilon^{\frac{d}{d-p}}} = \beta \tag{6}$$

for some real number $\beta > 0$. Then

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\, dx + \beta^{d-p} \int_\Omega \varphi(u(x) - \psi(x))\, dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_{hom}(S)$ is defined for $S \in \mathbb{R}^{m\times d}$ as

$$f_{hom}(S) := \lim_{R\to\infty} \frac{1}{R^d} \inf \left\{ \int_{Q_R} \int_{Q_R} f(y - x, v(y) - v(x))dx\, dy : v \in \mathcal{D}^S(Q_R) \right\}, \tag{7}$$

with

$$\mathcal{D}^S(Q_R) := \left\{ u \in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m) : u(x) = Sx \text{ for a.e. } x \in \mathbb{R}^d,\ \mathrm{dist}(x, \mathbb{R}^d\backslash Q_R) < 1 \right\}.$$

and for every $z \in \mathbb{R}^m$

$$\varphi(z) := \inf \left\{ \int_{\mathbb{R}^d} f_{hom}(\nabla v)\, dx : v - z \in L^{p*}(\mathbb{R}^d;\mathbb{R}^m),\ v \equiv 0 \text{ in } B_1, v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m) \right\}. \tag{8}$$

Remark 3.2. Note that the positive p - homogeneity assumption (**H**) is inherited by f_{hom} and φ, that is,

$$f_{hom}(tS) = t^p f_{hom}(S) \quad \forall t > 0,\ \forall S \in \mathbb{R}^{m\times d},$$

$$\varphi(tz) = t^p \varphi(z) \quad \forall t > 0,\ \forall z \in \mathbb{R}^m.$$

Moreover assumption (**G**) easily yields that

$$m_0|S|^p \leq f_{hom}(S) \leq M_0|S|^p \quad \forall S \in \mathbb{R}^{m\times d} \tag{9}$$

for two suitable strictly positive constants m_0, M_0.

On the other hand, the density φ can be also characterized as follows

$$\varphi(z) = \inf \left\{ \int_{\mathbb{R}^d} f_{hom}(\nabla u)\, dx : u \equiv -z \text{ in } B_1,\ u \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m) \begin{array}{c}\text{compactly}\\ \text{supported}\end{array} \right\}. \tag{10}$$

According to the estimate (9) and formula (10), taking into account also characterization b) of the p-capacity of a set in $\mathbb{R}^d$ of Proposition 4.1, and (18), we also deduce that

$$m_0 \operatorname{cap}_p(B_1)|z|^p \le \varphi(z) \le M_0 \operatorname{cap}_p(B_1)|z|^p \quad \forall z \in \mathbb{R}^m. \tag{11}$$

Thanks to (9) and (11) we eventually get that, for any $u \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)$, it holds

$$C_1\Big(\int_\Omega |\nabla u|^p\,dx + \operatorname{cap}_p(B_1)\int_\Omega |u(x)-\psi(x)|^p\,dx\Big) \le \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u)$$

and

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) \le C_2\Big(\int_\Omega |\nabla u|^p\,dx + \operatorname{cap}_p(B_1)\int_\Omega |u(x)-\psi(x)|^p\,dx\Big),$$

for some suitable strictly positive constants C_1, C_2. □

Remark 3.3. If $f(\xi,\cdot)$ is convex for every $\xi \in \mathbb{R}^d$, the asymptotic formula (7) reduces to

$$f_{hom}(S) = \int_{\mathbb{R}^d} f(\xi, S\xi)\,d\xi$$

(see [2, Theorem 6.2]). Note moreover that, under this convexity assumption, hypothesis (**L**) is implied by assumptions (**H**) and (**G**). □

The next results concern the asymptotic behaviour of energies of type (2) when the scaling among the periodicity scale and the radius of the perforations does not satisfy (6).

Theorem 3.4. *Let F_ε be defined by (2), with f satisfying assumptions* (**H**), (**G**) *and* (**L**). *Assume moreover that*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_\varepsilon} = 0 \tag{12}$$

and

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\delta_\varepsilon^{\frac{d}{d-p}}} = +\infty. \tag{13}$$

Then $\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) < +\infty$ if and only if $u \equiv \psi$ in Ω and

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_\Omega f_{hom}(\nabla\psi(x))\,dx$$

where $f_{hom}(S)$ is defined for $S \in \mathbb{R}^{m\times d}$ as in (7).

Theorem 3.5. *Let F_ε be defined by (2), with f satisfying assumptions* (**H**), (**G**) *and* (**L**). *Assume moreover that*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_\varepsilon} = 0 \tag{14}$$

and

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\delta_\varepsilon^{\frac{d}{d-p}}} = 0. \tag{15}$$

Then

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_{hom}(S)$ is defined for $S \in \mathbb{R}^{m\times d}$ as in (7).

4. Preliminary results

In this section we collect some results that will be used in Section 5.

Capacity

Let us recall the notion of p-capacity. Given an open set $A \subset \mathbb{R}^d$ and a set $E \subset\subset A$ we define the *relative p-capacity* of E in A as

$$\mathrm{cap}_p(E,A) = \inf\left\{\int_A |\nabla u|^p\,dx : u \in C_c^1(A),\ u \geq 1 \text{ in a neighbourhood of } E\right\}.$$

If $A = \mathbb{R}^d$, we simply write $\mathrm{cap}_p(E)$. It follows by the very definition that the set function $\mathrm{cap}_p(E,A)$ is increasing in the variable E and decreasing in the variable A. More precisely,

$$\begin{aligned} &\mathrm{cap}_p(E_1,A) \leq \mathrm{cap}_p(E_2,A) \quad \text{if } E_1 \subseteq E_2;\\ &\mathrm{cap}_p(E,A_1) \leq \mathrm{cap}_p(E,A_2) \quad \text{if } A_1 \supseteq A_2. \end{aligned} \tag{16}$$

Note that it can be easily shown that $\mathrm{cap}_p(E) > 0$ if $|E| > 0$ and it is well known that the following properties hold true.

Proposition 4.1. *For any $E \subset\subset A$ open set it holds*

(a) $\mathrm{cap}_p(E,A) = \inf\left\{\int_A |\nabla u|^p\,dx : u \in W_0^{1,p}(A),\ u = 1 \text{ a.e. in } E\right\}$;

(b) $\mathrm{cap}_p(E) = \inf\left\{\int_{\mathbb{R}^d} |\nabla u|^p\,dx : u \in W^{1,p}_{loc}(\mathbb{R}^d) \cap L^{p^*}(\mathbb{R}^d),\ u = 1 \text{ a.e. in } E\right\}$;

(c) $\mathrm{cap}_p(E) = \lim_{R\to+\infty} \mathrm{cap}_p(E,B_R) = \inf_{R>0} \mathrm{cap}_p(E,B_R)$.

Remark 4.2. A straightforward computation shows that

$$\inf\left\{\int_A |\nabla u|^p\,dx : u \in C_c^1(A),\ u = t \text{ in a neighbourhood of } E\right\} = |t|^p \mathrm{cap}_p(E,A).$$

Moreover, one may consider, for $z \in \mathbb{R}^m$, the vectorial infimum problems

$$\inf\left\{\int_A |\nabla u|^p\,dx : u \in C_c^1(A;\mathbb{R}^m),\ u = z \text{ in a neighbourhood of } E\right\}. \tag{17}$$

Note that, thanks to the rotational invariance of (17), it can be proved that

$$\inf\left\{\int_A |\nabla u|^p\,dx : u \in C_c^1(A;\mathbb{R}^m),\ u = z \text{ in a neighbourhood of } E\right\} = \mathrm{cap}_p(E,A),$$

for any $z \in \mathbb{R}^m$ such that $|z| = 1$. Indeed, up to a rotation, it is enough to prove the equality for $z = (1,0,\dots,0)$. In this case we can confine the infimum in (17) to functions $v \in W_0^{1,p}(A;\mathbb{R}^m)$ such that $v = (v_1,0,\dots,0)$. Hence, we can conclude that, for any $z \in \mathbb{R}^m$,

$$\inf\left\{\int_A |\nabla u|^p\,dx : u \in W_0^{1,p}(A;\mathbb{R}^m),\ u = z \text{ a.e. in } E\right\} = \mathrm{cap}_p(E,A)|z|^p. \tag{18}$$

Convolution-type energies

For any open set $A \subset \mathbb{R}^d$ and $u \in L^p(A;\mathbb{R}^m)$, with $p > 1$, we define

$$\mathcal{F}_\varepsilon(u,A) := \int_{\mathbb{R}^d} \int_{A_\varepsilon(\xi)} f\left(\xi, D_\varepsilon^\xi u(x)\right) dx\,d\xi, \tag{19}$$

with $f : \mathbb{R}^d \times \mathbb{R}^m \to [0,+\infty)$ a Borel function satisfying hypotheses (**H**) and (**G**).

Note that $\mathcal{F}_\varepsilon(\cdot,\Omega)$ is exactly the functional $F_\varepsilon(\cdot)$ on functions $u \in L^p(\Omega;\mathbb{R}^m)$ that satisfy the constraint $u = \psi$ on $P_\varepsilon \cap \Omega$.

Let $r > 0$ and define $f_r(\xi, z) = \mathbb{1}_{B_r}(\xi)|z|^p$. For such a choice of $f_r(\xi, z)$ we denote the associated functional in (19) as

$$G_\varepsilon^{r,p}(u, A) = \int_{B_r} \int_{A_\varepsilon(\xi)} \left| D_\varepsilon^\xi u(x) \right|^p \, dx \, d\xi, \tag{20}$$

where $D_\varepsilon^\xi u(x)$ and $A_\varepsilon(\xi)$ are defined by (3) and (4), respectively.

The nonlocal functionals $G_\varepsilon^{r,p}(u, \Omega)$ will have a relevant role in the sequel as they provide a natural lower bound for F_ε. Indeed, taking into account hypothesis (**G**), it holds

$$F_\varepsilon(u) \geq \lambda_0 G_\varepsilon^{r_0,p}(u, \Omega) \quad \forall u \in L^p(\Omega; \mathbb{R}^m). \tag{21}$$

Estimate (21) will be crucial to identify the domain of any $\Gamma(L^p)$-$\lim_{\varepsilon \to 0} F_\varepsilon$ in view of the following compactness result in the strong L^p-topology of sequences of functions with uniformly bounded energy (see [2, Theorem 4.2]).

Theorem 4.3. *Let A be any bounded open Lipschitz set of $\mathbb{R}^d$ and assume that $\{u_\varepsilon\}_\varepsilon \subset L^p(A; \mathbb{R}^m)$ be such that for some $r > 0$*

$$\sup_{\varepsilon > 0} \left\{ \|u_\varepsilon\|_{L^p(A;\mathbb{R}^m)} + G_\varepsilon^{r,p}(u_\varepsilon, A) \right\} < +\infty.$$

Then, for any $\varepsilon_j \to 0$, $\{u_{\varepsilon_j}\}_j$ is relatively compact in $L^p(A; \mathbb{R}^m)$ and every limit of a converging subsequence lies in $W^{1,p}(A; \mathbb{R}^m)$.

The last result on convolution type energies that will be used in the sequel is the following Γ-convergence result (see [2, Theorem 6.1]).

Theorem 4.4. *Let $\mathcal{F}_\varepsilon(\cdot, \Omega)$ be defined by (19), with f satisfying assumptions (**H**) and (**G**). Then*

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u, \Omega) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u) \, dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise}, \end{cases} \tag{22}$$

where f_{hom} is defined in (7).

5. Supporting results

In this section we present some key results instrumental for the proofs of Theorems 3.1, 3.4, and 3.5. Their proofs can be found in [3].

A Gagliardo-Nirenberg-Sobolev type inequality and a compactness property

We state here a nonlocal version of the classical Gagliardo-Nirenberg-Sobolev inequality in Sobolev spaces. This result will be crucial to prove a closure and compactness theorem in $L^{p^*}(\mathbb{R}^d; \mathbb{R}^m)$ for sequences with bounded nonlocal energies (20) (see Proposition 5.2). In turn Proposition 5.2 will allow us to prove the convergence of the infimum problems defining the approximating capacitary energy densities.

Theorem 5.1. (Gagliardo-Nirenberg-Sobolev type inequality) *Let $p \in [1, d)$. Then there exists a constant $C = C(p, d, r) > 0$ such that for every $u \in L^p(\mathbb{R}^d; \mathbb{R}^m)$*

$$\left(\int_{\mathbb{R}^d} |T_\varepsilon u(x)|^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq C G_\varepsilon^{r,p}(u, \mathbb{R}^d), \tag{23}$$

where $T_\varepsilon u(x) := \dfrac{1}{(\tilde{r}\varepsilon)^d} \displaystyle\int_{\tilde{r}\varepsilon k + [0, \tilde{r}\varepsilon)^d} u(y) \, dy$ on $\tilde{r}\varepsilon k + [0, \tilde{r}\varepsilon)^d$, $k \in \mathbb{Z}^d$, with $\tilde{r} := \frac{r}{\sqrt{d+3}}$.

Proposition 5.2. *Let $p \in (1, d)$ and $\{u_\varepsilon\}_\varepsilon \subset L^p(\mathbb{R}^d; \mathbb{R}^m)$ be such that for some $r > 0$*

(i) $\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^p(K;\mathbb{R}^m)} < +\infty$ *for every compact set $K \subset \mathbb{R}^d$;*

(ii) $\sup_{\varepsilon > 0} G_\varepsilon^{r,p}(u_\varepsilon, \mathbb{R}^d) < +\infty$.

Then, for any $\varepsilon_j \to 0$, $\{u_{\varepsilon_j}\}_j$ is relatively compact in $L^p_{loc}(\mathbb{R}^d;\mathbb{R}^m)$ and every limit of a converging subsequence lies in $W^{1,p}_{loc}(\mathbb{R}^d;\mathbb{R}^m) \cap L^{p^*}(\mathbb{R}^d;\mathbb{R}^m)$.

A joining lemma

Here we state the analog of Lemma 3.1 in [4] for our nonlocal functionals. It allows to restrict our attention in our Γ-convergence analysis to sequences of converging functions that are constant on suitable annuli surrounding the perforations. The proof of this result is essentially identical to that of Lemma 3.1 in [4] so we omitted it (for more details see [4]).

Lemma 5.3. *Let $\delta_j \to 0$ as $j \to +\infty$. Let $T \geq r_0$ be fixed and let $0 < \varepsilon_j < \rho_j < \frac{\delta_j}{2}$ with $\varepsilon_j = o(\rho_j)$ as $j \to +\infty$. Let the sequence u_j converge to u in $L^p(\Omega,\mathbb{R}^m)$ with $\sup_j \mathcal{F}^T_{\varepsilon_j}(u_j) < +\infty$. Set*

$$Z_j(\Omega) := \{i \in \mathbb{Z}^d : \operatorname{dist}(i\delta_j, \mathbb{R}^d \setminus \Omega) > \delta_j\},$$

then, given $N \in \mathbb{N}$, for every $i \in Z_j(\Omega)$ there exists $k_i \in \{1,\cdots,N-1\}$, a sequence w_j still converging to u in $L^p(\Omega,\mathbb{R}^m)$ and $j_0 \in \mathbb{N}$, such that for $j \geq j_0$

$$w_j = u_j \text{ on } \Omega \setminus \bigcup_{i\in Z_j(\Omega)} A^i_j, \quad w_j = u^i_j \text{ on } \partial B_{\rho^i_j}(i\delta_j) + B_{T\varepsilon_j},$$

$$\left|\mathcal{F}^T_{\varepsilon_j}(w_j,\Omega) - \mathcal{F}^T_{\varepsilon_j}(u_j,\Omega)\right| \leq \frac{C}{N}. \tag{24}$$

where, $i \in Z_j(\Omega)$, set

$$A^i_j := \{x \in \Omega : 2^{-k_i-1}\rho_j < |x - i\delta_j| < 2^{-k_i}\rho_j\}, \tag{25}$$

$$u^i_j := \fint_{A^i_j} u_j \qquad \rho^i_j := \tfrac{3}{4}2^{-k_i}\rho_j. \tag{26}$$

Truncation Lemma

By the composition with a suitable Lipschitz function, the following technical lemma allows us to replace a given sequence u_j with equibounded energies and L^p-norms, by a new sequence uniformly bounded in L^∞ and with a small gap in energy. As already quoted for Lemma 5.3, we skip the proof and refer to [4].

Lemma 5.4. *Let $\{u_j\}$ with $\sup_j(\mathcal{F}_{\varepsilon_j}(u_j) + \|u_j\|_{L^p(\Omega;\mathbb{R}^m)}) < +\infty$. Then for every $\eta > 0$ and $M > 1$ there exist $R_M > M > 0$ and a sequence of Lipschitz functions $\Phi_{j,M} : \mathbb{R}^m \to \mathbb{R}^m$ with $Lip(\Phi_{j,M}) = 1$, $\Phi_{j,M}(z) = z$ if $|z| < M$ and $\Phi_{j,M}(z) = 0$ if $|z| > R_M$, such that it holds*

$$\mathcal{F}_{\varepsilon_j}(\Phi_{j,M}(u_j),\Omega) \leq \mathcal{F}_{\varepsilon_j}(u_j,\Omega) + \eta$$

for every $j \in \mathbb{N}$ such that $\varepsilon_j < \varepsilon_0$, with ε_0 depending on Ω. Moreover we can extract a subsequence (j_k) such that $\Phi_{j_k,M} =: \Phi_M$ do not depend on $k \in \mathbb{N}$.

Approximating capacitary energy densities

For any $\varepsilon > 0$, $T > r_0$, $R \geq 2 + T\varepsilon$, and $z \in \mathbb{R}^m$, let us define

$$\varphi^T_{\varepsilon,R}(z) := \inf\left\{\mathcal{F}^T_\varepsilon(v,B_R) : \begin{array}{l} v - z \in L^p(\mathbb{R}^d;\mathbb{R}^m),\ v \equiv 0 \text{ in } B_1, \\ v \equiv z \text{ on } \mathbb{R}^d \setminus B_{R-\varepsilon T} \end{array}\right\}, \tag{27}$$

where we denote $\mathcal{F}^T_\varepsilon$ the functionals in (19) with $f(\xi,z) = 0$ for any $\xi \in \mathbb{R}^d$, such that $|\xi| > T$.

In [3] many fine properties of the densities $\varphi^T_{\varepsilon,R}$ are established in order to determine the pointwise and locally uniform limit of $\varphi^T_{\varepsilon,R}(\cdot)$ when the parameters R, T go to $+\infty$, and ε goes to 0. We summarize these properties in Proposition 5.5 for future reference. We remark that these ingredients clarify the nature of the energetic contribution near the perforations, leading to the appearance of the density functions φ defined by (8).

Proposition 5.5. *Let f satisfy assumptions* (**H**), (**G**) *and* (**L**). *Then there exist constants $C, C_1, C_2 > 0$ independent of ε, T and R such that for every $z, w \in \mathbb{R}^m$ we have*

$$C_1|z|^p \le \varphi^T_{\varepsilon,R}(z) \le C_2|z|^p \quad \forall z \in \mathbb{R}^m$$

$$|\varphi^T_{\varepsilon,R}(w) - \varphi^T_{\varepsilon,R}(z)| \le C(|z|^{p-1} + |w|^{p-1})|w - z|.$$

In addition, if $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it holds

$$\lim_{\varepsilon\to 0} \varphi^T_{\varepsilon,R_\varepsilon}(z) = \varphi^T(z)$$

uniformly on compact sets, where

$$\varphi^T(z) := \inf\left\{ \int_{\mathbb{R}^d} f^T_{hom}(\nabla v)\,dx : \begin{array}{l} v - z \in L^{p*}(\mathbb{R}^d;\mathbb{R}^m),\ v \equiv 0 \text{ in } B_1, \\ v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m) \end{array} \right\} \tag{28}$$

being f^T_{hom} the limit energy density of $\Gamma(L^p)$-$\lim_{\varepsilon\to 0} f^T_{hom}$ (see Theorem 4.4).

The properties of the densities $\varphi^T_{\varepsilon,R}$ quoted above allow us to prove the following result about the L^1-convergence of suitable Riemann sums to the capacitary densities φ^T.

Proposition 5.6. *Let $\varepsilon_j \to 0$ and $R_j \to +\infty$ as $j \to +\infty$ and let (u_j) be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$ such that $\sup_j \mathcal{F}^T_{\varepsilon_j}(u_j) < +\infty$ and $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$, for some $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Let A^i_j and u^i_j, $i \in Z_j(\Omega)$, be as in (25) and (26), respectively, with $\rho_j = O(\delta_j)$ and k_i arbitrarily chosen. Let $\Psi^T_j : \Omega \to \mathbb{R}$ be defined by*

$$\Psi^T_j(x) = \sum_{i\in Z_j(\Omega)} \varphi^T_{\varepsilon_j,R_j}(u^{i,k_i}_j)\mathbb{1}_{Q_{\delta_{\varepsilon_j}}(\delta_{\varepsilon_j} i)}(x),$$

where

$$Q_{\delta_{\varepsilon_j}}(\delta_{\varepsilon_j} i) = \delta_{\varepsilon_j} i + \delta_{\varepsilon_j} Q_1.$$

Then $\Psi^T_j \to \varphi^T(u)$ in $L^1(\Omega)$.

6. Proof of Theorem 3.1

We divide the proof into four steps: we first show that it suffices to prove the theorems for the truncated functionals F^T_ε (Step 1), and, for what concerns the Γ-lim inf, for sequences in $L^p(\Omega;\mathbb{R}^m)$ that are equibounded in $L^\infty(\Omega;\mathbb{R}^m)$ (Step 2). Eventually, we deal separately with the Γ-lim inf (Step 3) and the Γ-lim sup inequalities (Step 4).

Step 1. We claim that it is not restrictive to prove both theorems under the additional assumption that there exists $T > 0$ such that $f(\xi, z) = 0$ if $|\xi| > T$.

Indeed, given $f(\xi, z)$ a Borel function satisfying (**H**), (**G**) and (**L**), associated to F_ε, we consider, for $T > 0$, $f^T(\xi, z) = \mathbb{1}_{B_T}(\xi) f(\xi, z)$ and the associated F^T_ε.

If we assume that for every $T > 0$

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F^T_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f^T_{hom}(\nabla u)\,dx + \beta^{d-p}\int_\Omega \varphi^T(u-\psi)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{29}$$

where f^T_{hom} and φ^T are defined by (7) and (8), respectively. Arguing as in the proof of [2, Lemma 5.1], we can easily infer that the result stated there holds even in the constrained case, that is

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \lim_{T\to+\infty} \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F^T_\varepsilon(u).$$

Hence, by the Monotone Convergence Theorem, the statement follows once we prove that for every $S \in \mathbb{R}^{d\times m}$ and $z \in \mathbb{R}^m$

$$\lim_{T\to+\infty} f^T_{hom}(S) = f_{hom}(S), \qquad \lim_{T\to+\infty} \varphi^T(z) = \varphi(z).$$

The equalities above are, in turn, again a straightforward consequence of the Monotone Convergence Theorem.

Step 2. We may confine to prove the Γ-lim inf inequality for $u_\varepsilon \to u \in L^p(\Omega;\mathbb{R}^m)$ with u_ε, u equibounded in $L^\infty(\Omega;\mathbb{R}^m)$. Indeed, given $\varepsilon_j \to 0^+$ as $j \to +\infty$ and $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, let $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ be such that $\sup_j F^T_{\varepsilon_j}(u_j) < +\infty$. Using the fact that $\psi \in L^\infty(\Omega;\mathbb{R}^m)$ and up to passing to a subsequence (not relabelled), given $\eta > 0$ and $M > \|\psi\|_{L^\infty(\Omega;\mathbb{R}^m)}$, by applying Lemma 5.4, one may find $R_M > M$ and a Lipschitz function $\Phi_M : \mathbb{R}^m \to \mathbb{R}^m$, with $Lip(\Phi_M) = 1$, $\Phi_M(z) = z$ if $|z| < M$ and $\Phi_M(z) = 0$ if $|z| > R_M$ such that

$$F^T_{\varepsilon_j}(u_j) > F^T_{\varepsilon_j}(\Phi_M(u_j)) - \eta. \tag{30}$$

It is easy to check that $\Phi_M(u) \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\Phi_M(u_j) \to \Phi_M(u)$ in $L^p(\Omega;\mathbb{R}^m)$, so that assuming that

$$\Gamma(L^p)\text{-}\liminf_\varepsilon F^T_\varepsilon(\Phi_M(u)) \geq \int_\Omega f^T_{hom}(\nabla\Phi_M(u))\,dx + \beta^{d-p}\int_\Omega \varphi^T(\Phi_M(u)-\psi)\,dx,$$

by (30) we also get

$$\liminf F^T_{\varepsilon_j}(u_j) \geq \int_\Omega f^T_{hom}(\nabla\Phi_M(u))\,dx + \beta^{d-p}\int_\Omega \varphi^T(\Phi_M(u)-\psi)\,dx - \eta.$$

The conclusion follows letting $M \to +\infty$ and $\eta \to 0$.

Step 3. With fixed $T > r_0$, we now prove the validity of the Γ-lim inf inequality for F_ε assuming that the function f in (19) satisfies $f(\xi, z) = 0$ for any $\xi \in \mathbb{R}^d$ such that $|\xi| > T$, and testing the liminf inequality for a given u along sequences such that $\sup \|u_j\|_{L^\infty(\Omega,\mathbb{R}^m)} < +\infty$. In order to keep track of the bound on the energy density f we denote as usual F_ε with F^T_ε.

The main idea of the proof is to take advantage of Lemma 5.3 iteratively. More precisely, with $\varepsilon_j \to 0^+$ as $j \to +\infty$, let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and let $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ be such that $\sup_j F^T_{\varepsilon_j}(u_j) + \|u_j\|_{L^\infty} < +\infty$. Given $N \in \mathbb{N}$, let $\{w^1_j\}_j$ be the sequence constructed in Lemma 5.3 applied with $\rho_j = \delta_{\varepsilon_j}/4$ and $\{u_j\}_j$ as above.

Now we apply again Lemma 5.3 to $\{w^1_j\}_j$ with $\rho_j = (1 + o(1))r_{\epsilon_j}$ and obtain a new sequence $\{w_j\}_j$ with the property that, on any cube $\delta_\varepsilon i + [-\delta_\varepsilon, \delta_\varepsilon]^d$, w_j is constant both on an annulus centered on a spherical surface centered in $\delta_\varepsilon i$ and radius ρ^i_j of order δ_ε and amplitude εT, and both on an annulus centered on a spherical surface centered in $\delta_\varepsilon i$ and radius $\tilde\rho^i_j$of order r_ε and amplitude εT. Moreover, for the sequence $\{w_j\}_j$ the gap in energy with respect to the initial $\{u_j\}_j$ is still controlled as

$$|F_{\varepsilon_j}(u_j) - F_{\varepsilon_j}(w_j)| \leq 2\,\frac{C}{N}.$$

Set

$$F_j = \bigcup_{i \in Z_j(\Omega)} B_{\rho^i_j}(\delta_{\varepsilon_j} i),$$

where ρ^i_j is obtained in the first application of Lemma 5.3, and define

$$v_j(x) := \begin{cases} w_j(x) & \text{if } x \in \Omega \setminus F_j \\ u^i_j & \text{if } x \in B_{\rho^i_j}(\delta_{\varepsilon_j} i), \end{cases}$$

with u^i_j is defined in (26), relatively to ρ^i_j.

Notice that, by (**H**), (G0) and Lemma 5.3, $\sup_j G^{r_0,p}_{\varepsilon_j}(v_j, \Omega) < +\infty$. Hence, by Theorem 4.3, $\{v_j\}_j$ is relatively compact in $L^p(\Omega;\mathbb{R}^m)$. Arguing as in [5], we now show that $v_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$. Specifically, for $h \in \{0, \dots, N-1\}$ set

$$r_h := \tfrac{3}{4}2^{-h-2}, \quad \chi^h_j(x) := \chi^h\Big(\frac{x}{\delta_{\varepsilon_j}}\Big),$$

where χ^h coincides with $\mathbb{1}_{Q_1 \setminus B_{r_h}}$ on Q_1 and is extended Q_1-periodically in $\mathbb{R}^d$,

$$Z^h_j := \{i \in Z_j(\Omega) : k_i = h\}, \quad D^h_j := \bigcup_{i \in Z^h_j} \delta_{\varepsilon_j}\,(i + Q_1)\,, \quad \psi^h_j(x) := \mathbb{1}_{D^h_j}(x),$$

where $k_i \in \{0, \dots, N-1\}$ is the number selected for any $i \in Z_j(\Omega)$ by Lemma 5.3.

Recall that

$$\chi_j^h \overset{*}{\rightharpoonup} m_h := |Q_1 \setminus B_{r_h}| > 0 \quad \text{weakly* in } L^\infty(\mathbb{R}^d)$$

and note that,

$$\sum_{h=0}^{N-1} \psi_j^h \to 1 \text{ strongly in } L^1(\Omega).$$

Moreover, since $\chi_j^h \geq \chi_j^0$ for every $h \in \{0, \dots, N-1\}$, we have that

$$\mathbb{1}_{\Omega \setminus F_j} = \sum_{h=0}^{N-1} \psi_j^h \chi_j^h \geq \chi_j^0 \sum_{h=0}^{N-1} \psi_j^h \overset{*}{\rightharpoonup} m_0 > 0. \tag{31}$$

Let us consider a subsequence (not relabelled) such that $\mathbb{1}_{\Omega \setminus F_j} \overset{*}{\rightharpoonup} g$ in $L^\infty(\Omega)$ and $v_j \to v$ strongly in $L^p(\Omega; \mathbb{R}^m)$. Hence

$$\mathbb{1}_{\Omega \setminus F_j} v_j \rightharpoonup g\, v, \quad \mathbb{1}_{\Omega \setminus F_j} w_j \rightharpoonup g\, u \quad \text{weakly in } L^p(\Omega; \mathbb{R}^m).$$

Taking into account that $\mathbb{1}_{\Omega \setminus F_j} v_j = \mathbb{1}_{\Omega \setminus F_j} w_j$, we conclude that $v = u$ thanks to the lower bound on g ensured by (31).

By Lemma 5.3, we have

$$\begin{aligned} F_{\varepsilon_j}^T(u_j) &\geq F_{\varepsilon_j}^T(w_j) - 2\frac{C}{N} = \mathcal{F}_{\varepsilon_j}^T(w_j) - 2\frac{C}{N} \\ &\geq \mathcal{F}_{\varepsilon_j}^T(w_j, \Omega \setminus F_j) + \mathcal{F}_{\varepsilon_j}^T(w_j, F_j) - 2\frac{C}{N} \\ &\geq \mathcal{F}_{\varepsilon_j}^T(v_j, \Omega) + \mathcal{F}_{\varepsilon_j}^T(w_j, F_j) - 2\frac{C}{N} \end{aligned} \tag{32}$$

since by definition $w_j = v_j$ in $\Omega \setminus F_j$, and v_j is constant on each $\partial B_{\rho_j^i}(\delta_{\varepsilon_j} i) + B_{\varepsilon_j T}$.

By Theorem 4.4, it holds

$$\liminf_{j \to +\infty} \mathcal{F}_{\varepsilon_j}^T(v_j, \Omega) \geq \int_\Omega f_{hom}^T(\nabla u)\, dx. \tag{33}$$

We now turn to the estimate of the energy contribution on F_j: $\mathcal{F}_{\varepsilon_j}^T(w_j, F_j)$.

Set

$$W_j = \bigcup_{i \in Z_j(\Omega)} B_{\tilde{\rho}_j^i}(\delta_{\varepsilon_j} i)$$

where $\tilde{\rho}_j^i < r_{e_j}$ is obtained by the second iteration of Lemma 5.3, and define

$$\hat{v}_j(x) := \begin{cases} w_j(x) & \text{if } x \in F_j \setminus W_j \\ \tilde{w}_j^i & \text{if } x \in B_{\tilde{\rho}_j^i}(\delta_{\varepsilon_j} i), \end{cases}$$

where $\tilde{w}_j^i$ is defined in formula (26), relatively to $\tilde{\rho}_j^i$. Since w_j is constant on each $\partial B_{\tilde{\rho}_j^i}(\delta_{\varepsilon_j} i) + B_{\varepsilon_j T}$ we get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_j, F_j) &= \mathcal{F}_{\varepsilon_j}^T(w_j, F_j \setminus W_j) + \mathcal{F}_{\varepsilon_j}^T(w_j, W_j) \\ &\geq \mathcal{F}_{\varepsilon_j}^T(\hat{v}_j, F_j) = \sum_{i \in Z_j(\Omega)} \mathcal{F}_{\varepsilon_j}^T(\hat{v}_{j,i}, B_{\rho_j^i}(\delta_{\varepsilon_j} i)) \end{aligned}$$

With fixed $i \in Z_j(\Omega)$, let $v_{j,i}(y) := \hat{v}_j(\delta_{\varepsilon_j} i + r_{\varepsilon_j} y)$ be defined on the ball $B_{R_j^i}$, with $R_j^i := \rho_j^i r_{\varepsilon_j}^{-1}$ and extended to u_j^i outside this ball. Setting $s_j := \varepsilon_j r_{\varepsilon_j}^{-1}$, taking into account that $\hat{v}_{i,j}$ is equal to w_j^i on the unitary ball and equal to u_j^i outside $B_{R_j^i}$, we get

$$\mathcal{F}_{\varepsilon_j}^T(\hat{v}_{j,i}, B_{\rho_j^i}(\delta_{\varepsilon_j} i)) = r_{\delta_{\varepsilon_j}}^{d-p} \mathcal{F}_{s_j}^T(v_{j,i}, B_{R_j^i}) \geq r_{\delta_{\varepsilon_j}}^{d-p} \varphi_{s_j, R_j^i}^T(u_j^i - w_j^i). \tag{34}$$

We take

$$\Psi_j^T(x) = \sum_{i\in Z_j(\Omega)} \varphi^T_{s_j,R_j^i}(u_j^i - w_j^i)\mathbb{1}_{Q_{\delta_{\varepsilon_j}}(\delta_{\varepsilon_j} i)}(x).$$

By (34), we get

$$\mathcal{F}^T_{\varepsilon_j}(w_j, F_j) \geq \frac{r_{\delta_{\varepsilon_j}}^{d-p}}{\delta_{\varepsilon_j}^d} \int_\Omega \Psi_j^T(x)\,dx = \left(\frac{r_{e_j}}{\delta_{\varepsilon_j}^{\frac{d}{d-p}}}\right)^{d-p} \int_\Omega \Psi_j^T(x)\,dx. \tag{35}$$

We underline that, by construction, w_j coincides with ψ on each $B_{r_{\varepsilon_j}}(\delta_{\varepsilon_j} i)$ then

$$w_j^i = \fint_{\tilde{A}_j^i} \psi(x)\,dx, \tag{36}$$

where $\tilde{A}_j^i$ is the annulus determined by the second iteration of Lemma 5.3. Hence, by Proposition 5.6, applied to $\{u_j - \psi\}$, with s_j, R_j^i in place of ε_j, R_j, respectively, we have

$$\Psi_j^T \to \varphi^T(u-\psi) \text{ in } L^1(\Omega). \tag{37}$$

Hence, by (35), we deduce that

$$\liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(w_j, F_j) \geq \beta^{d-p} \int_\Omega \varphi^T(u-\psi)\,dx. \tag{38}$$

By (32), (33), and by the arbitrariness of $N \in \mathbb{N}$, we infer that

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j}(u_j) \geq \int_\Omega f^T_{hom}(\nabla u)\,dx + \beta^{d-p} \int_\Omega \varphi^T(u-\psi)\,dx.$$

We conclude that, under the assumptions of Theorem 3.1,

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j}(u_j) \geq \int_\Omega f^T_{hom}(\nabla u)\,dx + \beta^{d-p} \int_\Omega \varphi^T(u-\psi)\,dx.$$

Step 4. With fixed $T \geq r_0$, we now prove the validity of the Γ-lim sup inequality for F_ε^T. By a density argument it suffices to prove the inequality for $u \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$. For such a u, fixed an open set $\Omega' \in \mathcal{A}^{\rm reg}(\mathbb{R}^d)$ such that $\Omega' \supset\supset \Omega$, and given $\varepsilon_j \to 0$ as $j \to +\infty$, by Theorem 4.4 there exists a sequence $(\tilde{u}_j)$, converging in $L^p(\Omega';\mathbb{R}^m)$ to u, such that

$$\lim_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(\tilde{u}_j, \Omega') = \int_{\Omega'} f^T_{hom}(\nabla u)\,dx. \tag{39}$$

Taking into account Lemma 5.4, up to replacing $\tilde{u}_j$ with a suitable truncated function, we may also assume that $\sup_j \|\tilde{u}_j\|_{L^\infty(\Omega';\mathbb{R}^m)} < +\infty$. In order to exhibit a suitable sequence that satisfies the constraint on the perforations, we introduce the functions $\hat{u}_j = \tilde{u}_j \mathbb{1}_{\Omega'\setminus P_{\varepsilon_j}} + \psi \mathbb{1}_{P_{\varepsilon_j}\cap\Omega'}$. Following the same strategy of Step 3, given $N \in \mathbb{N}$, we consider the sequence $\{w_j^1\}$ constructed in Lemma 5.3 applied with Ω' in place of Ω, $\rho_j = \delta_{\varepsilon_j}/4$ and $\hat{u}_j$ in place of u_j, and applying in turn Lemma 5.3 to $\{w_j^1\}$ with $\rho_j = 2r_{\varepsilon_j}$ we obtain a sequence $\{w_j\}$ such that

$$\mathcal{F}^T_{\varepsilon_j}(w_j, \Omega'\setminus E_j) \leq \mathcal{F}^T_{\varepsilon_j}(\tilde{u}_j, \Omega') + 2\,\frac{C}{N}, \tag{40}$$

where

$$E_j = \bigcup_{i\in Z_j(\Omega')} B_{\rho_j^i}(\delta_{\varepsilon_j} i).$$

We now pass to the estimate of the energetic contribution on the set E_j. With the same notation of Step 3, set

$$R_j^i := \frac{\rho_j^i}{r_{e_j}}, \quad s_j = \frac{\varepsilon_j}{r_{\varepsilon_j}},$$

and let u_j^i, w_j^i the average of w_j on the two annuli obtained by applying twice Lemma 5.3.

For $i \in Z_j(\Omega')$, let $\tilde{v}_{j,i} \in L^p(B_{R_j^i};\mathbb{R}^m)$ be such that

$$\mathcal{F}_{s_j}^T(\tilde{v}_{j,i}, B_{R_j^i}) = \varphi^T_{s_j,R_j^i}(u_j^i - w_j^i) + o(\varepsilon_j).$$

Then set

$$v_{j,i}(x) := \tilde{v}_{j,i}\Big(\frac{x - \delta_{\varepsilon_j} i}{r_{\varepsilon_j}}\Big), \quad x \in B_{\rho_j^i}(\varepsilon_j i),$$

and

$$u_j(x) := \begin{cases} w_j(x) & \text{if } x \in (\Omega' \setminus E_j) \cup W_j \\ v_{j,i}(x) & \text{if } x \in B_{\rho_j^i}(\delta_{\varepsilon_j} i) \setminus B_{\tilde{\rho}_j^i}(\delta_{\varepsilon_j} i), \end{cases}$$

where W_j is defined xactly as in Step 3. Note that

$$\mathcal{F}^T_{\varepsilon_j}(v_{j,i}, B_{\rho_j^i}(\delta_{\varepsilon_j} i)) = r^{d-p}_{\delta_{\varepsilon_j}} \mathcal{F}^T_{s_j}(\tilde{v}_{j,i}, B_{R_j^i}) = r^{d-p}_{\delta_{\varepsilon_j}} (\varphi^T_{s_j,R_j^i}(u_j^i - w_j^i) + o(\varepsilon_j)). \tag{41}$$

Moreover the sequence $\{u_j\}$ satisfies the constraint and, arguing as in Step 3, we also deduce that $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$. We finally pass to the estimate of the energy. By (41), we get

$$\begin{aligned} \mathcal{F}^T_{\varepsilon_j}(u_j) &\le \mathcal{F}^T_{\varepsilon_j}(w_j, \Omega') + \sum_{i \in Z_j(\Omega')} \mathcal{F}^T_{\varepsilon_j}(u_j, B_{\rho_j^i}(\delta_{\varepsilon_j} i) \setminus B_{\tilde{\rho}_j^i}(\delta_{\varepsilon_j} i)) + o(\varepsilon_j) \\ &= \mathcal{F}^T_{\varepsilon_j}(w_j, \Omega') + \sum_{i \in Z_j(\Omega')} r^{d-p}_{\delta_{\varepsilon_j}} (\varphi^T_{s_j,R_j^i}(u_j^i - w_j^i) + o(\varepsilon_j)) \end{aligned} \tag{42}$$

Applying Proposition 5.6 as in Step 3, we deduce that

$$\sum_{i \in Z_j(\Omega')} r^{d-p}_{\delta_{\varepsilon_j}} \varphi^T_{s_j,R_j^i}(u_j^i - w_j^i) \to \beta^{d-p} \int_{\Omega'} \varphi^T(u - \psi)\, dx. \tag{43}$$

Hence, by (39), (40), together with (42), (43), we get that

$$\limsup_{j \to +\infty} \mathcal{F}^T_{\varepsilon_j}(u_j) \le \int_{\Omega'} f^T_{hom}(\nabla u)\, dx + \beta^{d-p} \int_{\Omega'} \varphi^T(u - \psi)\, dx + \frac{C}{N}.$$

The conclusion follows by the arbitrariness of $N \in \mathbb{N}$ and letting $\Omega' \to \Omega$.

The proofs of Theorems 3.4 and 3.5 follow the same strategy and take advantage of the estimate established in the proof of Theorem 3.1. We skip all the details and we note only that under the hypothesis of Theorem 3.4, (35) allows to conclude immediately that the Γ-lim inf $F_\varepsilon < +\infty$ if and only if $\varphi(u - \psi)$ is equal to zero a.e. in Ω. One may argue analogously for the Γ-lim sup F_ε exploiting the estimate (42) under the hypothesis of Theorem 3.5.

Acknowledgments: The author is a member of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM. The research of M. S. Gelli has been supported by PRIN Project 2022E9CF89 "Geometric Evolution Problems and Shape Optimizations", and by PRIN PNRR Project P2022WJW9H "Magnetic skyrmions, skyrmionic bubbles and domain walls for spintronic applications". PRIN projects are part of PNRR Italia Domani, financed by European Union through NextGenerationEU.

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