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On Proper Preordering of Convex Sets

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Abstract: The paper deals with convex sets of infinite-dimensional real vector spaces X with no topology. We endow each convex set $Q \subset X$ with a preorder relation $\trianglelefteq_Q$, called the dominance relation, the definition of which is based on the geometric properties of each particular set Q, and therefore, the dominance relation $\trianglelefteq_Q$ is specific to each Q. The results presented in the paper demonstrate that the view of a convex set Q as a preordered set $(Q, \trianglelefteq_Q)$ can be effectively used in studying the geometric structure of both the convex set Q itself and its faces.

Keywords: Convex sets, faces, cones, halfspaces, preorder relations, upper semilattice.
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1. Introduction

This paper continues the study of convex sets begun by the author in [8] and [9].

Here, as before in [8, 9], we assume that the convex sets in question belong to an abstract (generally, infinite-dimensional) real vector space X that has no topology. We endow each convex set $Q \subset X$ with a preorder relation $\trianglelefteq_Q$, the definition of which is based on the geometric properties of each particular set Q, and therefore, this relation $\trianglelefteq_Q$ is specific to each Q. We refer to the preorder relation $\trianglelefteq_Q$ as the dominance relation on the set Q. In the paper we demonstrate that the definition of the preorder relation $\trianglelefteq_Q$ on a convex set Q is fruitful in studying geometric structure of the set Q and its faces. As elementary components of geometric structure of a convex set Q we consider equivalence classes of the quotient set $\mathcal{O}(Q) := Q/\diamond_Q$ of the set Q by the equivalence relation $\diamond_Q$, where $\diamond_Q$ is the symmetric part of the dominance relation $\trianglelefteq_Q$. It follows from general properties of quotient sets that the family $\mathcal{O}(Q)$ of equivalence classes is a partition of Q, that is $Q = \bigcup\{E \mid E \in \mathcal{O}(Q)\}$, and $E_1 \bigcap E_2 = \varnothing$ for all $E_1, E_2 \in \mathcal{O}(Q), E_1 \neq E_2$. It is proved that each equivalence class $E \in \mathcal{O}(Q)$ is a relatively algebraically open subset of Q and for this reason we refer to subsets E from $\mathcal{O}(Q)$ as open components of Q. The dominance relation $\trianglelefteq_Q$ induces a partial order relation $\trianglelefteq_Q^*$ on the family $\mathcal{O}(Q)$ of all open components of the set Q with respect to which the partially ordered family $(\mathcal{O}(Q), \trianglelefteq_Q^*)$ is an upper semilattice. For halfspaces (convex sets H whose complements are also convex), the dominance relation is total on H and the corresponding upper semilattice $(\mathcal{O}(H), \trianglelefteq_H^*)$ is a linearly ordered set. Thus, geometric structure of a convex set Q is completely characterized by the upper semilattice $(\mathcal{O}(Q), \trianglelefteq_Q^*)$: the convex set Q is the disjoint union of the family of open components $\mathcal{O}(Q)$, and in Q its open components are arranged with respect to each other in accordance with the partial order $\trianglelefteq_Q^*$.

The relationship between faces of a convex set Q and open components of Q is describe as follows: a face F of a convex set Q, which has a common point with some open component E of Q, contains this open component entirely, that is

$$F \bigcap E \neq \varnothing \implies E \subset F.$$

This implies that every face F of a convex set Q is the disjunctive union of open components of Q which belong to F. Another noteworthy observation is that every face of a convex set Q is an initial segment of the preordered set $(Q, \trianglelefteq_Q)$. Moreover, when $(Q, \trianglelefteq_Q)$ is a weakly ordered set, in particular, when Q is a halfspace, then the converse statement is also true: every initial segment of the weakly ordered set $(Q, \trianglelefteq_Q)$ is a face of Q.

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The paper is organized as follows. Section 1 is the introduction, where the main results of the paper are shortly described. Section 2 presents the preliminaries on preordered sets. In Section 3 on every convex set Q we define the preorder $\trianglelefteq_Q$, called the dominance relation, that allows us to consider any convex set Q as the preordered set $(Q, \trianglelefteq_Q)$. In the particular cases, when Q is a convex cone, and also when Q is a halfspace, other, more simple, characterizations of the dominance relations are established. Section 4 is devoted to the description of the geometrical structure of convex sets via their open components. In Section 5 the relationship between faces of a convex set and its open components is discussed. At last, in Section 6 we establish the relationship between faces of a convex set Q and the initial segments of the preordered set $(Q, \trianglelefteq_Q)$. A short conclusion completes the paper.

2. Preliminaries on preordered sets

In this section we will recall the necessary concepts related to binary relations and, in particular, to ordering relations, and also specify the terminology adopted in this paper. More details on orderings can be found, for example, in [3, 4, 7, 12, 20].

Let Y be a set and let $\preceq \subset Y \times Y$ be a binary relation on Y which is supposed to be a reflexive ($y \preceq y$ for all $y \in Y$) and transitive ($y \preceq z, z \preceq w \Longrightarrow y \preceq w$ for all $y, z, w \in Y$). Such relations are usually called preorder (or quasiorder) relations. A preorder $\preceq$ is called *a weak order* if it is total, that is if for any $y, z \in Y$ at least one (or even both) of the relations, either $y \preceq z$ or $z \preceq y$, holds. An preorder $\preceq$ which is antisymmetric, is called *a partial order*. Recall that a binary relation $\preceq$ (not necessarily a preorder) is *antisymmetric* if for all $y, z \in Y$, for which both the relations $y \preceq z$ and $z \preceq y$ hold, one has $y = z$. A total partial order $\preceq$ is called *a linear order*. A weak order $\preceq$ is a linear order if and only if it is antisymmetric.

A set Y which is endowed with a preorder (partial order) relation $\preceq$ is called *a preordered (partially ordered) set* and is denoted as the pair $(Y, \preceq)$. When $\preceq$ is a weak order or a linear order a pair $(Y, \preceq)$ is called *a weakly ordered set* and *a linearly ordered set*, respectively.

Let $(Y, \preceq)$ be a preordered set. A subset $D \subset Y$ is called *an initial segment* of a preordered set $(Y, \preceq)$ if for any $y \in D$ and $z \in Y$ such that $z \preceq y$ one has that $z \in D$.

Obviously, for any preordered set $(Y, \preceq)$ the set Y itself, considered as a subset of Y, is an initial segment in $(Y, \preceq)$. By convention the empty subset $\varnothing$ is also considered to be an initial segment of any preordered set $(Y, \preceq)$.

Here are some examples of nontrivial initial segments.

Example 2.1. (1) Let $Y = \mathbb{N}$ be the set of nonnegative integer numbers, and $\preceq$ be the conventional relation 'less or equal' $\leq$ on $\mathbb{N}$. For any $n \in \mathbb{N}$ the set $D_n = \{0, 1, 2, \ldots, n\}$ is an initial segment of $(\mathbb{N}, \leq)$.

(2) Let $Y = \mathbb{R}^n$ and let $\preceq$ be the componentwise ordering $\leq$ of points from $\mathbb{R}^n$, that is $x \leq y$ for $x, y \in \mathbb{R}^n \Leftrightarrow x_i \leq y_i$, $i = 1, 2, \ldots, n$. The sets $D_1 = \{y \in Y \mid \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n < 0\}$ and $D_2 = \{y \in Y \mid \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n \leq 0\}$, where $\alpha_i > 0$ for all $i = 1, 2, \ldots, n$, are initial segments of $(\mathbb{R}^n, \leq)$. □

It is easy to verify that the union and the intersection of any family of initial segments of a preordered set $(Y, \preceq)$ is also an initial segment of $(Y, \preceq)$.

Proposition 2.2. *Let $(Y, \preceq)$ be an arbitrary preordered set and A be a subset from Y. Then the subset $D_\preceq(A) := \{y \in Y \mid y \preceq a$ for some $a \in A\}$ is the minimal (with respect to inclusion) initial segment of $(Y, \preceq)$ which contains A. In particular, for any $a \in Y$ the subset $D_\preceq(a) := \{y \in Y \mid y \preceq a\}$ is the minimal initial segment of $(Y, \preceq)$ containing the point a.*

Proof. Let $y \in D_\preceq(A)$ and let $a \in A$ be such that $y \preceq a$. Then for any $z \in Y$ such that $z \preceq y$ we have due to transitivity of $\preceq$ that $z \preceq a$ and, hence, $z \in D_\preceq(A)$. This proves that $D_\preceq(A)$ is an initial segment of $(Y, \preceq)$.

Clearly, the minimal (with respect to inclusion) initial segment of $(Y, \preceq)$ containing A is the intersection of all those initial segments of $(Y, \preceq)$ which contain A. Let D be an arbitrary initial segment of $(Y, \preceq)$ containing A. It is simply verified that $D_\preceq(A) \subset D$ and, hence, $D_\preceq(A)$ is the minimal initial segment of $(Y, \preceq)$ containing A. □

Proposition 2.3. *Let $(Y, \preceq)$ be a weakly ordered set and let D be an initial segment in $(Y, \preceq)$. Then $y \prec z$ for $y \in D$ and $z \in Y \setminus D$. Here $y \prec z \Leftrightarrow y \preceq z, z \not\preceq y$.*

Proof. For any $y, z \in Y$ only one of the following two relations holds: either $z \preceq y$, or $y \prec z$. If $y \in D$ and $z \preceq y$ then by the definition of an initial segment z also belongs to D. Hence, if $y \in D$ and $z \in Y \setminus D$, then $y \prec z$. □

Proposition 2.4. *The family of initial segments of a weakly ordered set $(Y, \preccurlyeq)$ is linearly ordered by inclusion relation, i.e., whatever the two initial segments of a weakly ordered set, one of them is a subset of the other.*

Proof. Let D_1 and D_2 be two distinct initial segments of a weakly ordered set $(Y, \preccurlyeq)$. Assume that $D_1 \not\subset D_2$. Then there exists $y \in D_1$ such that $y \notin D_2$. Since the set Y is weakly ordered by the relation $\preccurlyeq$, for any $z \in D_2$ either $z \preccurlyeq y$ or $y \preccurlyeq z$ holds. If the relation $y \preccurlyeq z$, were true, then it would follow from this, since D_2 is the initial segment in $(Y, \preccurlyeq)$, that $y \in D_2$, and this contradicts the choice of y. Hence, $z \prec y$ for all $z \in D_2$. From $y \in D_1$ and D_1 is an initial segment, we conclude that $z \in D_1$. Thus, in this case $D_2 \subset D_1$.

Similarly, assuming that $D_2 \not\subset D_1$, we come to the conclusion that $D_1 \subset D_2$. □

3. Preordering of a convex set by the dominance relation

In what follows X is nontrivial ($X \neq \{0\}$) real vector space which is supposed to be infinite-dimensional, unless otherwise stated. Besides, it is not supposed that some other structures (in particular, the topological one) are defined on X.

Let us begin with recalling the notion of the (relative) algebraical interior of subsets of real vector spaces [5, 13, 14, 15, 18, 21, 23].

The algebraical interior (or *the core*) of a set $Q \subset X$ is called the subset consisting of all points $x \in Q$ such that the intersection of Q with each straight line of X passing through x contains an open interval around x; the algebraic interior of the set Q is denoted by $\mathrm{cr}Q$.

The relative algebraical interior (or *the intrinsic core*) of a subset $Q \subset X$, denoted by $\mathrm{icr}Q$, is the subset of Q consisting of all points $x \in Q$ such that the intersection of Q with every straight line laying in $\mathrm{aff}Q$ ($\mathrm{aff}Q$ is the affine hull of Q) and passing through x contains an open interval around x.

More formally, $x \in \mathrm{cr}Q$ if and only if for any $y \in X \setminus \{x\}$ there exists a positive real number $\delta > 0$ such that $x + t(y - x) \in Q$ for all $t \in (-\delta, \delta)$, while $x \in \mathrm{icr}Q$ if and only if for any $y \in \mathrm{aff}Q \setminus \{x\}$ there exists a positive real number $\delta > 0$ such that $x + t(y - x) \in Q$ for all $t \in (-\delta, \delta)$.

A set $Q \subset Y$ is called *(relatively) algebraically open* if it coincides with its (relative) algebraical interior, i.e., if $Q = \mathrm{cr}Q$ ($Q = \mathrm{icr}Q$).

A set $Q \subseteq X$ is called [19] *convex*, if for any $x, y \in Q$ the closed line segment $[x, y] := \{\alpha x + (1 - \alpha)y \mid \alpha \in [0, 1]\}$ is entirely contained in Q.

Proposition 3.1. [23, p. 3] *If a set $Q \subset X$ is convex then $x \in \mathrm{icr}Q$ if and only if for any $y \in Q \setminus \{x\}$ there exists a positive real number $\delta > 0$ such that $x - \delta(y - x) \in Q$.*

3.1. Dominance relation on convex sets

Let $Q \subset X$ be a convex set. For a pair of points $x, y \in Q$ we say [9] that *a point $x \in Q$ dominates a point $y \in Q$* if there exists a positive real number $\delta > 0$ such that $x - \delta(y - x) \in Q$. Geometrically, a point $x \in Q$ dominates a point $y \in Q, y \neq x$ in a convex set Q if there exists a point $z \in Q$ such that $x \in (z, y)$, where $(z, y) := \{\alpha z + (1 - \alpha)y \mid \alpha \in (0, 1)\}$ is an open line segment whose endpoints are z and y.

The binary relation defined on Q in such way is called [9] *the dominance relation* and is denoted by $\trianglelefteq_Q$. It is worth noting that the dominance relation $\trianglelefteq_Q$ is specific for each convex set Q on which it is defined.

The dominance relation $\trianglelefteq_Q$ is a preorder (i.e., a reflexive and transitive binary relation) on Q. By the symbol $\lessgtr_Q$ we denote the symmetric part of $\trianglelefteq_Q$ defined by $x_1 \lessgtr_Q x_2 \Leftrightarrow x_1 \trianglelefteq_Q x_2, x_2 \trianglelefteq_Q x_1$. It is easy to see that $x_1 \lessgtr_Q x_2$ if and only if there exists a closed line segment $[y_1, y_2] := \{\alpha y_1 + (1-\alpha)y_2 \mid \alpha \in [0, 1]\}$ in Q such that $x_1, x_2 \in (y_1, y_2)$. The asymmetric part of $\trianglelefteq_Q$ denoted by $\triangleleft_Q$ is defined by $x_1 \triangleleft_Q x_2 \Leftrightarrow x_1 \trianglelefteq_Q x_2, x_2 \not\trianglelefteq_Q x_1$. The relation $\triangleleft_Q$ is a strict partial order, while $\lessgtr_Q$ is an equivalence relation.

Remark 3.2. In fact, the dominance relation $\trianglelefteq_Q$ on a convex set Q was considered earlier by Rubinstein in [21, p. 187] (see also [15]). □

The next assertion follows immediately from Proposition 3.1 and the definition of the dominance relation $\trianglelefteq_Q$.

Proposition 3.3. *Let the set $Q \subset X$ be convex and a point $x \in Q$. Then $x \in \mathrm{icr}Q$ if and only if $y \trianglelefteq_Q x$ for all $y \in Q$.*

Thus, we can consider each convex set $Q \subset X$ as a preordered set $(Q, \trianglelefteq_Q)$, where $\trianglelefteq_Q$ is the dominance relation on Q generated by the set Q itself. Note, that the dominance relation $\trianglelefteq_Q$ is specific for every set Q: for two different convex sets Q_1 and Q_2, the relations $\trianglelefteq_{Q_1}$ and $\trianglelefteq_{Q_2}$ are also different.

3.2. Dominance relation on convex cones

A subset $K \subset X$ is called *a cone* if $x \in K, \lambda > 0 \Longrightarrow \lambda x \in K$. Geometrically, a subset $K \subseteq X$ is a cone if along with each point $x \in K$ the subset K contains the entire ray emanating from the origin and passing through the point x. Note, that the origin not necessarily belongs to a cone K.

A cone $K \subset X$ is convex if and only $x + y \in K$ whenever $x, y \in K$ or, equivalently, if and only if $K + K = K$.

A cone $K \subset X$ is said to be *asymmetric*, if $K \bigcap(-K) = \varnothing$. A convex cone $K \subset X$ is asymmetric if and only if $0 \notin K$.

Proposition 3.4. *For the dominance relation $\trianglelefteq_K$ defined on a convex cone $K \subset X$ the following equivalence is true: $y \trianglelefteq_K x$ for $x, y \in K$ if and only if there exists a positive real number $\mu > 0$ such that $x - \mu y \in K$.*

Proof. Due to the definition of the dominance relation, we have that $x \trianglelefteq_K y$ if and only if there exists a real number $\delta > 0$ such that $x - \delta(y - x) \in K$ or, equivalently, such that $(1 + \delta)x - \delta y \in K$. Since K is a cone it follows from the last condition that $x - \mu y \in K$, where $\mu = \frac{\delta}{1+\delta}$. □

Remark 3.5. In fact, the characterization of the dominance relation on convex cones given in Proposition 3.4 was used in [8] as the definition of the dominance relation on convex cones. □

3.3. Dominance relation on halfspaces

A convex subset $H \subset X$ of a real vector space X is called [16] *a convex halfspace in X* or simply [10, 11] *a halfspace in X* if its complement $X \setminus H$ is also a convex set. Clearly, if $H \subset X$ is a halfspace in X then $X \setminus H$ is also a halfspace in X.

A halfspace which is a cone is called *a conical halfspace*.

It was shown in [10, 11] that

(a) a non-asymmetric ($K \bigcap(-K) \neq \varnothing$) convex cone $K \subset X$ is a conical halfspace in X if and only if $K \bigcup(-K) = X$;

(b) an asymmetric ($K \bigcap(-K) = \varnothing$) convex cone $K \subset X$ is a conical halfspace in X if and only if $X \setminus (K \bigcup(-K))$ is a linear subspace in X;

(c) a subset $H \subset X$ is a halfspace in X if and only if for any straight line $l \subset X$ just one of the following three alternatives holds:

$$(1)\ l \bigcap H = \varnothing, \quad (2)\ l \bigcap H = l, \quad (3)\ l \bigcap H \text{ is a halfline of } l.$$

Proposition 3.6. *For the dominance relation $\trianglelefteq_H$ defined on a halfspace $H \subset X$ the following equivalence is true:*

$$y \trianglelefteq_H x \text{ for } x, y \in H \iff (x = y) \text{ or } (x \text{ is not a boundary point of } l(x, y) \bigcap H),$$

where $l(x, y)$ stands for the straight line passing through x and y.

Proof. For $x = y, x, y \in H$ the equivalence is evident.

Let $x, y \in H$, $x \neq y$. By the definition $y \trianglelefteq_H x$ if and only if there exists $\delta > 0$ such that $z := x - \delta(y - x) \in H$. From this we see that the point x is an interior point of the open segment (y, z), which belongs the straight line $l(x, y)$. Through convexity of H we have $x \in (y, z) \subset l(x, y) \bigcap H$ and, hence, x can not be a boundary point of $l(x, y) \bigcap H$.

Conversely, assume that points $x, y \in H, x \neq y$ are such that the point x is not a boundary point of $l(x, y) \bigcap H$. Since H is a halfspace in X then either $l(x, y) \bigcap H = l(x, y)$ or $l(x, y) \bigcap H$ is a halfline of $l(x, y)$. In both cases x is an interior point of $l(x, y) \bigcap H$. This implies that there is a point $z \in l(x, y) \bigcap H$ such that x belongs to the open interval $(z, y) \subset l(x, y) \bigcap H$. It shows that, $y \trianglelefteq_H x$. □

The dominance relations defined on halfspaces have an additional property of totality.

Recall that a binary relation $\preccurlyeq$ defined on a set Y is called *total*, if for any $y_1, y_2 \in Y$ at least one from the two relations $y_1 \preccurlyeq y_2$ and $y_2 \preccurlyeq y_1$ holds.

Proposition 3.7. *The dominance relation $\trianglelefteq_H$ defined on a halfspace $H \subset X$ is total.*

Proof. To prove this proposition we use the following characterization of halfspaces:

a convex subset $H \subset X$ of a real vector space X is a halfspace in X [10, 11] if and only if its recession cone $0^+H := \{y \in X \mid x + ty \in H\ \forall x \in H \text{ and } \forall t > 0\}$ is a conical halfspace.

Let $H \subset X$ be a halfspace in X. Since the recession cone 0^+H corresponding H is a conical halfspace and $0 \in 0^+H$ then $0^+H \bigcup(-0^+H) = X$. Hence, for any $x, y \in X$ either $x - y \in 0^+H$, or $y - x \in 0^+H$, or both are satisfied. If $x - y \in 0^+H$, then by the definition of a recession cone we have $x - t(y - x) \in H \ \forall \ t > 0$ and, hence, $y \trianglelefteq_H x$. Similarly, if $y - x \in 0^+H$, then $x \trianglelefteq_H y$. Thus, for any $x, y \in H$ at least one of the two relations $y \trianglelefteq_H x$ and $x \trianglelefteq_H y$ holds, and, consequently, the dominance relation $\trianglelefteq_H$ is total on H. □

In the following sections we will demonstrate that the dominance relation $\trianglelefteq_Q$ is a fruitful tool for characterizing the geometric structure of a convex set Q and its faces.

4. Description of the geometrical structure of a convex set via its open components

In this section, along with new results, we provide a brief overview of some of the necessary results of the article [9] written in Russian, in order to make them accessible to a wider audience of readers.

Let $Q \subset X$ be a convex set.

As it was noted above the dominance relation $\trianglelefteq_Q$ defined on Q generates the equivalence relation $<>_Q$ on Q which is defined as follows: $x_1 <>_Q x_2 \Leftrightarrow x_1 \trianglelefteq_Q x_2, x_2 \trianglelefteq_Q x_1$. We use the partition of Q which corresponds to $<>_Q$ for description of geometrical structure of the set Q.

Let $\mathcal{O}(Q) := Q/<>_Q$ be the quotient set of the convex set Q by the equivalence relation $<>_Q$. The set Q is the disjunctive union of all equivalence classes from $\mathcal{O}(Q)$, i.e., $Q = \bigcup\{E \mid E \in \mathcal{O}(Q)\}$, and $E_1 \bigcap E_2 = \varnothing$ for all $E_1, E_2 \in \mathcal{O}(Q), E_1 \neq E_2$. The quotient set $\mathcal{O}(Q)$ is equipped with the partial order $\trianglelefteq^*_Q$ which is defined as follows: $E_1 \trianglelefteq^*_Q E_2$ holds for $E_1, E_2 \in \mathcal{O}(Q)$ if and only if $x_1 \trianglelefteq_Q x_2$ for all (some) $x_1 \in E_1$ and all (some) $x_2 \in E_2$.

The class equivalence from $\mathcal{O}(Q)$ containing a point $x \in Q$ will be denoted by E_x, thus $E_x := \{y \in Q \mid y <>_Q x\}$.

Proposition 4.1. [9] *Every equivalence class E from $\mathcal{O}(Q)$ is relatively algebraically open convex subset of the set Q. Moreover,*

$$E = \mathrm{icr}F_Q(E), \text{ where } F_Q(E) := \bigcup\{\tilde{E} \in \mathcal{O}(Q) \mid \tilde{E} \trianglelefteq^*_Q E\} \ .$$

Proof. Consider an arbitrary equivalence class $E \in \mathcal{O}(Q)$ and an arbitrary point $x \in E$. Then $E = E_x := \{y \in Q \mid y <>_Q x\}$. Note that $E_x = F_Q(x) \bigcap G_Q(x)$, where $F_Q(x) := \{y \in Q \mid y \trianglelefteq_Q x\}$ and $G_Q(x) := \{y \in Q \mid x \trianglelefteq_Q y\}$. Since $x \in F_Q(x)$ and $x \in G_Q(x)$, both of these sets, $F_Q(x)$ and $G_Q(x)$, are nonempty and, moreover, they are convex. To prove the convexity of the subset $F_Q(x)$ we choose arbitrary points $y_1, y_2 \in F_Q(x)$. By the definition of $\trianglelefteq_Q$ for $y_1, y_2 \in F_Q(x)$ there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $x - \delta_1(y_1 - x) \in Q$ and $x - \delta_2(y_2 - x) \in Q$. For $\delta = \min\{\delta_1, \delta_2\}$ we have also $x - \delta(y_1 - x) \in Q$ and $x - \delta(y_2 - x) \in Q$. From the convexity of Q for every $\alpha \in (0, 1)$ we obtain

$$\begin{aligned}\alpha(x - \delta(y_1 - x)) + (1 - \alpha)(x - \delta(y_2 - x)) &= x + \delta x - \delta(\alpha y_1 + (1 - \alpha)y_2) \\ &= x - \delta((\alpha y_1 + (1 - \alpha)y_2) - x) \in Q.\end{aligned}$$

Thus, $\alpha y_1 + (1 - \alpha)y_2 \trianglelefteq_Q x$ for all $\alpha \in (0, 1)$ and, hence, the subset $F_Q(x)$ is convex.

The convexity of the subset $G_Q(x)$ is proved similarly.

Hence, the quotient class E_x being the intersection of two convex sets $F_Q(x)$ and $G_Q(x)$ is also a convex subset of Q.

Since $y <>_Q x$ for all $y \in E_x$, it follows from convexity of E_x and Proposition 3.3 that $x \in \mathrm{icr}E_x$. Due to the arbitrary choice of $E \in \mathcal{O}(Q)$ and the point $x \in E$, we conclude, taking into account the equality $E = E_x$, that each equivalence class E is a relatively algebraically open set.

Let us prove now the equality $E = \mathrm{icr}F_Q(E)$. Note, that for every $x \in E$ the equality $F_Q(E) = F_Q(x)$ holds. Let $y \in \mathrm{icr}F_Q(x)$ for some $x \in E$. Since $F_Q(x)$ is a convex set, it follows from Proposition 3.1 that for each $z \in F_Q(x)$ there exists $\delta_z > 0$ such that $y - \delta_z(z - y) \in F_Q(x)$. Hence, for $x \in F_Q(x)$ there exists $\delta_x > 0$ such that $y - \delta_x(x - y) \in F_Q(x) \subset Q$. This implies that $x \trianglelefteq_Q y$. In addition, since $y \in F_Q(x)$, we have that $y \trianglelefteq_Q x$. Thus, $y <>_Q x$, i.e., $y \in E_x$, and we have proved that $\mathrm{icr}F_Q(x) \subset E_x$.

To prove the reverse inclusion we observe that $y \trianglelefteq_Q x$ for all $y \in F_Q(x)$. From this, through Proposition 3.3, we conclude that $x \in \mathrm{icr}F_Q(x)$ and, consequently, $E = E_x \subset \mathrm{icr}F_Q(x) = \mathrm{icr}F_Q(E)$. □

In what follows we will refer to equivalence classes from $\mathcal{O}(Q)$ as *relatively algebraically open components* or, shortly, *open components* of the convex set Q.

Proposition 4.2. [open components of convex cones]. *Let $K \subset X$ be a convex cone in X. Then each open component of K is a relatively algebraically open convex subcone of K.*

Proof. Due to Proposition 4.1 we only have to prove that every open component of a cone is also a cone. Let $E \in \mathcal{O}(K)$ and $x \in E$. Since $E = E_x = F_K(x) \bigcap G_K(x)$, where $F_K(x) := \{y \in K \mid y \trianglelefteq_K x\}$ and $G_K(x) := \{y \in K \mid x \trianglelefteq_K y\}$, it is sufficient to prove that $F_K(x)$ and $G_K(x)$ are cones.

Let $y \in F_K(x)$. Then $y \trianglelefteq_K x$ and by Proposition 3.4 it is equivalent to the existence of a real number $\mu > 0$ such that $x - \mu y \in K$. Since K is a cone for any $\lambda > 0$ we have $\lambda y \in K$ and $x - \frac{\mu}{\lambda}(\lambda y) \in K$. Hence, $\lambda y \trianglelefteq_K x$ and, consequently, $\lambda y \in F_K(x)$. It proves that $F_K(x)$ is a cone. The proof that $G_K(x)$ is a cone is similar. □

Now we will discuss the geometric shape of open components of halfspaces. For this we need the following generalization of the concept of a halfspace.

Let $M \subset X$ be an affine manifold in a real vector space X. A subset $C \subset M$ is called a *halfspace in* M if both C and $M \setminus C$ are convex sets.

Proposition 4.3. [open components of halfspaces]. *Let $H \subset X$ be a halfspace in X. Then each open component $E \in \mathcal{O}(H)$ of the halfspace H is a relatively algebraically open halfspace in* aff(E), *where* aff(E) *is the affine hull of E.*

For the case when H is a conical halfspace the proof of the assertion of this proposition was given in [8, Theorem 27]. For an arbitrary halfspace it was proven in [11, Section 5].

To formulate the following statement we need the concept of an upper semilattice.

Recall [3], that a partial ordered set $(Z, \preceq)$ is called *an upper semilattice*, if every pair of elements $\{z_1, z_2\}$ from Z has the least upper bound in Z, which is usually denoted by $z_1 \vee z_2$.

Theorem 4.4. *The family $\mathcal{O}(Q)$ of open components of a convex set Q, ordered by the partial order $\trianglelefteq_Q^*$, is an upper semilattice, and for any $x, y \in Q$ and any number $\alpha \in (0,1)$ the equality $E_x \vee E_y = E_{\alpha x+(1-\alpha)y}$ holds, in particular, $E_x \vee E_y = E_{\frac{x+y}{2}}$.*

Proof. We consider arbitrary open components E_x and E_y from $\mathcal{O}(Q)$. For any $\alpha \in (0,1)$ the point $x_\alpha := \alpha x + (1-\alpha)y$ belongs to the interior of the line segment $[x, y]$. For sufficiently small $\delta_1 > 0$ and $\delta_2 > 0$ the points $x_\alpha - \delta_1(x - x_\alpha)$ and $x_\alpha - \delta_2(y - x_\alpha)$ also belong to the segment $[x, y]$. Since a set Q is convex, the segment $[x, y]$ is contained in Q and, consequently, the points $x_\alpha - \delta_1(x - x_\alpha)$ and $x_\alpha - \delta_2(y - x_\alpha)$ belong to Q. By the definition of the dominance relation $\trianglelefteq_Q$ we conclude that $x \trianglelefteq_Q x_\alpha$ and $y \trianglelefteq_Q x_\alpha$. This implies that $E_x \trianglelefteq_Q^* E_{x_\alpha}$ and $E_y \trianglelefteq_Q^* E_{x_\alpha}$. Thus, in the partial ordered set $(\mathcal{O}(Q), \trianglelefteq_Q^*)$ the open component $E_{\alpha x+(1-\alpha)y}$ is an upper bound for the pair of open components $\{E_x, E_y\}$.

Let us show that E_{x_α} is in fact the least upper bound for the open components $\{E_x, E_y\}$. Suppose that $E_x \trianglelefteq_Q^* E$ and $E_y \trianglelefteq_Q^* E$ for some open component $E \in \mathcal{O}(Q)$. Then $x \trianglelefteq_Q u$ and $y \trianglelefteq_Q u$ for some (any) point $u \in E$ and, consequently, $x, y \in F_Q(u)$. Since the set $F_Q(u)$ is convex, $\alpha x + (1-\alpha)y \in F_Q(u)$ for any $\alpha \in (0,1)$ and, hence, $E_{\alpha x+(1-\alpha)y} \trianglelefteq^* E$ for any $\alpha \in (0,1)$. □

It worth observing that from the equality $E_x \vee E_y = E_{\alpha x+(1-\alpha)y}$ which is satisfied for any $x, y \in Q$ and any $\alpha \in (0,1)$ follows that for any $x, y \in Q$ all points of the open segment (x, y) belong to the same open component of the set Q.

For halfspaces, the upper semilattice of their open components has a more perfect order structure.

Proposition 4.5. *The family of open components $\mathcal{O}(H)$ of any halfspace H is linearly ordered by the relation $\trianglelefteq_H^*$.*

Proof. Recall, that a binary relation $\preccurlyeq$, defined on a set Y, is called a linear order if it is a total partial order. Due to Proposition 3.7 the dominance relation $\trianglelefteq_H$ defined on a halfspace H is a total preorder and, hence, the relation $\trianglelefteq_Q^*$ induced by $\trianglelefteq_H$ on the quotient set $\mathcal{O}(H) = Q/\!<\!>_Q$ is a total partial order, that is a linear order. □

Since $Q = \bigsqcup\{E \mid E \in \mathcal{O}(Q)\}$, where $\bigsqcup$ denotes a disjoint union, we can, informally speaking, consider the open components of the family $\mathcal{O}(Q)$ as "building blocks" from which the convex set Q is "composed", and in Q these blocks are arranged relative to each other not arbitrarily, but in accordance with the structure of the upper semilattice $(\mathcal{O}(Q), \trianglelefteq_Q^*)$. Thus, the upper semilattice $(\mathcal{O}(Q), \trianglelefteq_Q^*)$ (its elements and their ordering by the relation $\trianglelefteq_Q^*$) completely characterizes the geometrical structure of the convex set Q.

Proposition 4.6. *Let $Q \subset X$ be a convex set. Then* icr$Q \neq \varnothing$ *if and only if the upper semilattice $(\mathcal{O}(Q), \trianglelefteq_Q^*)$ has the greatest element E^{sup}, and* icr$Q = E^{sup}$.

Proof. The proof follows from Proposition 3.3 and the definition of the relation $\trianglelefteq_Q^*$. □

Corollary 4.7. *A convex set $Q \subset X$ is relatively algebraically open if and only if the family $\mathcal{O}(Q)$ of its open components is a singleton, i.e., if and only if $\mathcal{O}(Q) = \{Q\}$.*

Example 4.8. Let X be a real vector space and let $\{e_i, i \in I\}$ be a Hamel basis for X. Consider the convex set Q in X which consists of non-zero vectors $x \in X$ whose components $\{x_i, i \in I\}$ in the given basis $\{e_i, i \in I\}$ is nonnegative. Since the basis $\{e_i, i \in I\}$ is infinite, every vector $x \in Q$ has zero components. Suppose that $x_j = 0$. Since $e_j \in Q$ and for any real $\delta > 0$ the vector $x - \delta e_j$ does not belong to Q, then $x \notin \mathrm{icr}Q$ and, hence, through Proposition 3.3, $\mathrm{icr}Q = \varnothing$ as x is an arbitrary vector from Q. This was proven earlier by Raikov [18].

For $x \in Q$ by the symbol $I(x)$ we denote the subset of those indices from I for which $x_i > 0$, that is $I(x) := \{i \in I \mid x_i > 0\}$. The subset $I(x)$ is nonempty and finite for every vector $x \in Q$. It is easily to see that $y \trianglelefteq_Q x \Leftrightarrow I(y) \subseteq I(x)$ and $y <\!>_Q x \Leftrightarrow I(y) = I(x)$. Thus, E is an open component of the set Q if and only if $E = \{x \in Q \mid I(x) = J\}$, where J is some a finite subset of I. Since every finite subset of I is a proper subset of another finite subset of I, then the family $\mathcal{O}(Q)$ of open components of the set Q does not have the greatest (with respect to $\trianglelefteq^*$) element.

5. Open components and faces of a convex set

Faces of convex sets have been and continue to be the subject of much research (see, for example, [1, 2, 5, 6, 17, 19, 22] and references therein).

Let Q be a convex set in a real vector space X.

A nonempty convex subset $F \subset Q$ is called *a face* of Q (see, for instance, [19]) if every segment $[x, y]$ of Q such that $(x, y) \bigcap F \neq \varnothing$ is entirely contained in F. More formally, F is a face of a convex set Q if for any $u, v \in Q$ for which there exists $\alpha \in (0, 1)$ such that $\alpha u + (1 - \alpha)v \in F$ we have $u, v \in F$.

It is evident that the set Q itself is its own face. The empty set is also considered to be a face of any convex set Q. The face F of Q is said to be *proper* if it is nonempty and does not coincide with Q.

If F is a face of Q and G is a face of F, then G is also a face of Q. The intersection of any family of faces of a convex set Q is also a face of Q while the union of a family of faces of a convex set Q may not be a face of Q. However, the union of a family of faces which is linearly ordered by the inclusion relation is a face too (see for example, [5, Proposition 2.8]).

Let $C \subset Q$ be a (not necessarily convex) subset of a convex set $Q \subset X$. The least (with respect to set inclusion) face of Q which contains C is denoted by $F_Q(C)$ and is refer to as *the minimal face containing* C. Clearly, that $F_Q(C)$ is the intersection of all those faces of Q which contain C. When a set C is a singleton, i.e. $C = \{x\}$, we use the notation $F_Q(x)$ instead of $F_Q(\{x\})$.

Theorem 5.1. *Let $E \in \mathcal{O}(Q)$ be an open component of a convex set Q. Then*

$$F_Q(E) := \bigcup \{\tilde{E} \in \mathcal{O}(Q) \mid \tilde{E} \trianglelefteq^*_Q E\},$$

is the minimal face of Q containing E and, in addition, $\mathrm{icr}F_Q(E) = E$.

Proof. Let $E \in \mathcal{O}(Q)$. Since $F_Q(E) = F_Q(x)$ for all $x \in E$, then $F_Q(E)$ is a convex set (convexity of sets $F_Q(x)$ for any $x \in Q$ was proved in Proposition 4.1). To prove that $F_Q(E)$ is a face of Q we consider an arbitrary point $x \in F_Q(E)$ and assume that $x = \alpha u + (1-\alpha)v$ for some $\alpha \in (0, 1)$ and $u, v \in Q$. It follows from Theorem 4.4 that $E_x = E_u \vee E_v$. Since $E_x = E$, then $E_u \trianglelefteq^*_Q E$ and $E_v \trianglelefteq^*_Q E$. Hence, $u \in E_u \subset F_Q(E)$ and $v \in E_v \subset F_Q(E)$. This proves that $F_Q(E)$ is a face of Q.

To prove that $F_Q(E)$ is the minimal face of Q containing E we need to show that $F_Q(E) \subseteq F$ for any face F of the set Q such that $E \subset F$.

Let F be an arbitrary face of Q containing E, $x \in E \subset F$ and $y \in F_Q(E) = F_Q(x)$. Since $y \trianglelefteq_Q x$, by the definition of the dominance relation $\trianglelefteq_Q$ there exists $\delta > 0$ such that $z := x - \delta(y - x) \in Q$. It is directly verified that

$$x = \frac{\delta}{1+\delta} y + \frac{1}{1+\delta} z.$$

Therefore, $x \in (y, z)$, and $y, z \in Q$. Since the point x belongs to the face F of Q then it follows from the definition of a face that $y, z \in F$. Due to the arbitrary choice of the point y from the set $F_Q(E)$, we conclude that $F_Q(E) \subseteq F$.

Let us prove now the equality $E = \mathrm{icr}F_Q(E)$. Choose arbitrary points $x \in E$ and $y \in \mathrm{icr}F_Q(x)$. Recall that for every $x \in E$ the equality $F_Q(E) = F_Q(x)$ is true. Since the set $F_Q(x)$ is convex, it follows from Proposition 3.1 and the definition of the dominance relation $\trianglelefteq_Q$ that for every point $z \in F_Q(x)$ there exists $\delta_z > 0$ such that $y - \delta_z(z - y) \in F_Q(x)$. Hence, for $x \in F_Q(x)$ there exists $\delta_x > 0$ such that $y - \delta_x(x - y) \in F_Q(x) \subset Q$ and,

consequently, $x \trianglelefteq_Q y$. In addition, since $y \in F_Q(x)$, then $y \trianglelefteq_Q x$. Thus, $y <>_Q x$, and we have obtained that $y \in E_x$. Since y is an arbitrary point from $\mathrm{icr}F_Q(x)$, then $\mathrm{icr}F_Q(x) \subset E_x = E$.

It remains to prove the reverse inclusion. Since $y \trianglelefteq_Q x$ for all $y \in F_Q(x)$, it follows from Proposition 3.3 that $x \in \mathrm{icr}F_K(x)$. Consider an arbitrary point $z \in E_x$. Since $x <>_Q z$ and, in addition, $y \trianglelefteq_Q x$ for all $y \in F_Q(x)$, then from transitivity of $\trianglelefteq_Q$ we obtain that $y \trianglelefteq_Q z$ for all $y \in F_Q(x)$. Applying Proposition 3.3 again, we conclude that $z \in \mathrm{icr}F_Q(x)$ and hence $E_x \subseteq \mathrm{icr}F_Q(x)$. □

Corollary 5.2. *Let $Q \subset X$ be a convex set.*
For any point $x \in Q$ the set $F_Q(x) := \{y \in K \mid y \trianglelefteq_Q x\}$ is the minimal face of the set Q containing x, and $\mathrm{icr}F_K(x) = E_x \neq \varnothing$.

Theorem 5.3. *For every face F of a convex set Q such that $\mathrm{icr}F \neq \varnothing$ there exists a unique open component $E \in \mathcal{O}(Q)$ of the set Q such that $F = F_Q(E)$ and hence $\mathrm{icr}F = E$.*

Proof. Let F be a face of a convex set Q such that $\mathrm{icr}F \neq \varnothing$ and let $x \in \mathrm{icr}F$. Since the face F is a convex set, then due to Proposition 3.3 $y \trianglelefteq_Q x$ for all $y \in F$ and hence $F \subseteq F_Q(x) = F_Q(E_x)$. In order to prove the reverse inclusion we first show that $E_x \subseteq \mathrm{icr}F$ for all $x \in \mathrm{icr}F$. Indeed, if $z \in E_x$, then $x <>_Q z$, and since $y \trianglelefteq_Q x$ for all $y \in F$, it follows from transitivity of $\trianglelefteq_Q$ that $y \trianglelefteq_Q z$ for all $y \in F$. Applying Proposition 3.3 once more we conclude that $z \in \mathrm{icr}F$. Thus, we have $E_x \subseteq \mathrm{icr}F \subset F$, and, consequently, $F_Q(E_x) \subset F$.

So, we have that $F = F_Q(E_x)$. The equality $\mathrm{icr}F = E_x$ follows from Theorem 5.1.

The uniqueness of an open component $E \in \mathcal{O}(Q)$ satisfying for the face F the equalities $F = F_K(E)$ and $\mathrm{icr}F = E$ follows from the fact that different open components of the set Q do not intersect. □

Remark 5.4. It follows from Theorem 5.3 that the family of open components of a convex set Q coincides with the family of nonempty relatively algebraic interiors of faces of a set Q, i.e. $\mathcal{O}(Q) = \{\mathrm{icr}F \mid F \text{ is a face of } Q \text{ such that } \mathrm{icr}F \neq \varnothing\}$.

6. Faces of a convex set Q as initial segments of the preordered set $(Q, \trianglelefteq_Q)$

The results of this section show that there is a close relationship between the faces of a convex set $Q \subset X$ and the initial segments of a preordered set $(Q, \trianglelefteq_Q)$.

Proposition 6.1. *Let $D \subset Q$ be an initial segment of the preordered set $(Q, \trianglelefteq_Q)$. Then $E_x \subset D$ for any point $x \in D$.*

Proof. Let $x \in D$ and $y \in E_x$. Since $y <>_Q x$, we get, by the definition of an initial segment, that $y \in D$. □

Corollary 6.2. *Let $Q \subset X$ be a convex set in X. Any initial segment $D \subset Q$ of the preordered set $(Q, \trianglelefteq_Q)$ is the disjunctive union of open components of the convex set Q which belong to D.*

Theorem 6.3. *Any face of a convex set $Q \subset X$ is an initial segment of the preordered set $(Q, \trianglelefteq_Q)$. If $(Q, \trianglelefteq_Q)$ is weakly ordered set, then the converse is also true: any initial segment of $(Q, \trianglelefteq_Q)$ is a face of Q.*

Proof. Let F be a face of a convex set $Q \subset X$. For any point $x \in F$ we have $F_Q(x) \subset F$, where $F_Q(x) := \{y \in Q \mid y \trianglelefteq_Q x\}$ is the minimal face of Q containing x. If $z \in Q$ and $z \trianglelefteq_Q x$, then $z \in F_Q(x) \subset F$. This proves the first statement.

Assume that $(Q, \trianglelefteq_Q)$ is a weakly ordered set. Then $(\mathcal{O}(Q), \trianglelefteq_Q^*)$ is linearly ordered and, hence, any initial segment D of the ordered set $(Q, \trianglelefteq_Q)$ can be presented (see., Corollary 6.2) in the form $D = \bigsqcup\{E \mid E \in \mathcal{O}(Q), E \subset D\}$. Moreover, it follows from the last equality and the definition of an initial segment that

$$D = \bigcup\{F_Q(E) \mid E \mid E \in \mathcal{O}(Q), E \subset D\}.$$

Thus, D is the union of the linearly ordered (by inclusion) family of faces of the set Q and, consequently, due to [5, Proposition 2.8] D is a face of Q. □

Simple examples show that when a convex set is such that the preordered set $(Q, \trianglelefteq_Q)$ is not weakly ordered, then, in general, not every initial segment $D \subset (Q, \trianglelefteq_Q)$ is a face of Q.

Example 6.4. We consider the convex set $Q := \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_i| \leq 1, i = 1, 2.\}$. A subset $D = D_1 \cup D_2 \cup D_3$, where $D_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 1, |x_2| \leq 1\}$, $D_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 1, |x_1| \leq 1\}$, and $D_3 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = -1, |x_1| \leq 1\}$, is an initial segment of $(Q, \trianglelefteq_Q)$, however it is not convex and, consequently, is not a face of Q. Note, that the sets D_1, D_2 and D_3 are faces of the set Q. □

Thus, the family of faces of a convex set $Q \subset X$ is, in general, a proper subfamily of the family of initial segments of a preordered set $(Q, \trianglelefteq_Q)$, and at the same time these families coincide when the convex set Q is a halfspace in X (or even in some affine manifold from X).

Remark 6.5. It follows from Proposition 2.2 and Theorem 5.1 that for each open component $E \in Q$ the minimal initial segment $D_Q(E)$ containing E coincides with the minimal face $F_Q(E)$ containing E. In particular, the minimal initial segment $D_Q(x)$ containing a given point $x \in Q$ coincides with a minimal face $F_Q(x)$ containing this point x. □

7. Conclusion

Thus, each convex subset Q of a real vector space X is at the same time a preordered set $(Q, \trianglelefteq_Q)$ where the preorder $\trianglelefteq_Q$, interpreted as the dominance relation $\trianglelefteq_Q$ between elements of Q, is defined only by the geometric properties of the given set Q and, therefore, is specific for each set Q. The preorder $\trianglelefteq_Q$ generates two families of subsets of the set Q.

The subsets of the first family are equivalence classes of the quotient set $\mathcal{O}(Q) := Q/\!<\!>_Q$ of the set Q by the equivalence relation $<\!>_Q$ which is the symmetric part of the preorder $\trianglelefteq_Q$. Each equivalence class from $\mathcal{O}(Q)$ is relatively algebraically open subset, and the family $\mathcal{O}(Q)$ of all open components is a partition of Q. Moreover, open components from $\mathcal{O}(Q)$ are arranged with respect to each other in Q in accordance with the partial order generated on $\mathcal{O}(Q)$ by the preorder $\trianglelefteq_Q$.

The second family $\mathcal{D}(Q)$ consists of initial segments of the preordered set $(Q, \trianglelefteq_Q)$. It turns out that each face F of a convex set Q is an initial segment of $(Q, \trianglelefteq_Q)$ and, hence, $F \in \mathcal{D}(Q)$. In the particular case when $(Q, \trianglelefteq_Q)$ is linearly ordered the family of faces of a convex set Q coincides with the family of initial segments of $(Q, \trianglelefteq_Q)$. Along with this, examples shows that in the general case there are initial segments of $(Q, \trianglelefteq_Q)$ which are not faces of Q.

The results obtained in the paper demonstrate that the representation of a convex set Q as the preordered set $(Q, \trianglelefteq_Q)$ allows to establish new properties of convex sets and their faces.

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