

Regularized Dynamical Systems for Finding Zeros of Generalized Monotone Operators

Pham Viet Hai¹, Trinh Ngoc Hai^{1,*}

1. Faculty of Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam

Abstract: We propose regularized dynamical systems for finding zeros of generalized monotone operators in both continuous- and discrete-time settings. We establish strong convergence of the trajectories generated by such systems. Our results not only refine and extend the existing results, but also highlight examples where our results are applicable where current theory does not. Examples indicate that our methods exhibit varying convergence behaviors, with speeds that can be either faster or slower than the existing methods, depending on particular problems.

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1. Introduction

In this paper, $\mathcal{H}$ is a real Hilbert space, equipped with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. For a given operator $B : \mathcal{H} \rightarrow \mathcal{H}$, the subject of our study is the following problem

$$\text{Find } x^* \in \mathcal{H} \text{ such that } B(x^*) = 0. \quad (1)$$

For the sake of brevity, the set of zeros of B is referred to as $Z(B)$. Despite its apparent simplicity, Problem (1) has proven relevant in numerous diverse areas. The typical approach to tackling Problem (1) requires the assumption that the operator B satisfies a monotonicity, as defined in the following senses.

Definition 1.1. Let γ be a positive constant. An operator $B : \mathcal{H} \rightarrow \mathcal{H}$ is called

1. γ -strongly monotone if $\langle B(u) - B(v), u - v \rangle \geq \gamma \|u - v\|^2 \quad \forall u, v \in \mathcal{H}$.

2. monotone if $\langle B(u) - B(v), u - v \rangle \geq 0 \quad \forall u, v \in \mathcal{H}$. □

A natural first thought to solve Problem (1) might be the iterative scheme $x(k+1) = x(k) - B(x(k))$. However, this approach is problematic: fails to converge under the condition that the operator B is monotone; for instance, consider $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $B[x_1, x_2]^T = [-x_2, x_1]^T$ with initialization $x(0) \neq 0$. When faced with the limitation of monotonicity condition, the Tikhonov regularization method is often the approach taken by researchers. Originally designed for inverse problems, the regularization method has a rich history in optimization (see [2, 3, 12, 18]). Consider the regularized problem

$$\text{Find } x^* \in \mathcal{H} \text{ such that } B(x^*) + \epsilon F(x^*) = 0, \quad (2)$$

where $F : \mathcal{H} \rightarrow \mathcal{H}$ is a strongly monotone and Lipschitz continuous operator, $\epsilon > 0$ is a regularization parameter. It was proven in [11, 13] that Problem (2) has a unique solution x_ϵ and it strongly converges to a solution of (1) as ϵ tends to 0. Based on this idea, the aforementioned iterative scheme becomes:

$$x(k+1) = x(k) - s(k)B(x(k)) - s(k)\epsilon(k)F(x(k)). \quad (3)$$

*Corresponding author.

E-mail addresses: P. V. Hai: hai.phamviet@hust.edu.vn; T. N. Hai*: hai.trinhngoc@hust.edu.vn
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Under some certain conditions on the regularization parameter, the scheme (3) achieves strong convergence, as demonstrated in [4, 12]. These conditions impose limitations on the allowable scope of parameters. In this paper, we not only extend the scheme (3) to a broader class of operators, but also propose more relaxed conditions on the regularization parameter compared to the existing results.

The application of ordinary differential equations (ODEs) to the formulation of optimization algorithms is a subject of increasing scholarly attention, with the continuous gradient method $y'(t) = -\nabla\phi(y(t))$ for the unconstrained minimization of the differentiable convex function ϕ representing an early example. This method suggests tackling Problem (1) through the first-order ODE

$$y'(t) = -B(y(t)) \quad (4)$$

with starting time $t_0 \geq 0$ and initial conditions $y(t_0) = y^0$. Although ergodic convergence is a feature of the trajectories of (4) when the operator B is monotone, ordinary convergence is not ensured (see [6, 9]). The stronger condition of B being cocoercive leads to a significant change in convergence, resulting in weak convergence of the trajectory $y(t)$ and strong convergence of the residual $\|B(y(t))\|$ for the system (4) [8]. Like the discrete-time setting, the regularization method is highly effective in overcoming the disadvantages of the monotonicity condition (see [5, 8, 9]). While the papers [5, 8, 9], following this reasoning, introduce dynamical systems that possess strong convergence and fast speed, their analysis is limited to the case $F = I$ (with I denoting the identity operator), specifically for the system

$$x'(t) = -B(x(t)) - \varepsilon(t)x(t) \quad (5)$$

with initial time $t_0 \geq 0$ and initial conditions $x(t_0) = x^0$. Our paper differs itself by considering a more general case, focusing on the system

$$x'(t) = -\beta(t)[B(x(t)) + \varepsilon(t)F(x(t))]. \quad (6)$$

In practice, the operator B does not behave according to the monotonicity conditions set forth in Definition 1.1. A case in point is the operator $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$W \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{7x_1+x_2} - 1 + x_1 + 13x_2 \\ e^{x_1+13x_2} - 1 - (7x_1 + x_2) \end{bmatrix}. \quad (7)$$

The existing methods [5, 8, 9] are no longer applicable to the operator W . In the paper, we propose ODEs to address Problem (1), where the operator B is characterized by a property generalizing Definition 1.1 and concurrently containing the operator W . The monotone operators under consideration in this work are defined in the following manner.

Definition 1.2. [16] Let γ be a positive constant. A pair (A, B) is called

1. γ -strongly monotone if $\langle B(u) - B(v), Au - Av \rangle \geq \gamma\|u - v\|^2 \quad \forall u, v \in \mathcal{H}$.
2. monotone if $\langle B(u) - B(v), Au - Av \rangle \geq 0 \quad \forall u, v \in \mathcal{H}$. □

Example 2.4 and the works [1, 14] show the surprisingly large scope of operators satisfying Definition 1.2, a class that properly contains those meeting Definition 1.1.

The following outlines the organization of the subsequent sections. Section 2 prepares for the main results by defining key terms and establishing some initial observations.

Section 3 focuses on the regularized dynamical system (6), which extends the system studied in [5, 8, 9]. Our extension involves two generalizations. *Firstly*, the monotonicity property, initially defined for the pair (I, B) , is extended to the pair (A, B) . *Secondly*, the regularization $\varepsilon(t)x(t)$, which became so familiar in (5), is generalized to encompass a more comprehensive form $\varepsilon(t)F(x(t))$. Our analysis in Section 3 yields three main results.

- Theorem 3.5 shows that strong convergence of the trajectory is guaranteed under the condition that the set $Z(B) \neq \emptyset$. Note that Theorem 3.5 encompasses the classical result of trajectory convergence to the minimum-norm solution [5, 8, 9] as a special case.

- Theorem 3.7 proves the reverse of Theorem 3.5; in detail, bounded trajectories imply that the set $Z(B) \neq \emptyset$.

• Theorem 3.10 delivers a convergence rate for the residual, given certain additional conditions on the regularization coefficient. The flexibility to adjust the regularization operator F is found in Theorem 3.10, providing a way to enhance the efficiency of the regularized dynamical system (6) (see Remark 3.13).

Section 4 studies the discrete counterpart (3) of (6). Obtained here is the development of discrete-time counterparts to Theorems 3.5 and 3.7, with relaxed conditions compared to [12] (see Example 4.5). Note that the existing literature, specifically papers [5, 8, 9], has only studied the dynamical system (6) with $F = A = I$, leaving its discretizing unexplored. Our paper fills this gap with two contributions. First, we discretize the dynamical system (6) to produce a novel algorithm. Second, our paper not only provides a concrete algorithm but also establishes a strong two-way link between the algorithm's convergence and the condition $Z(B) \neq \emptyset$. Specifically, we prove that convergence requires this condition, and conversely, that the boundedness of the generated trajectories implies it, thereby completing the convergence analysis in discrete setting.

The theoretical results are empirically validated in Section 5 through illustrative examples, which specifically confirm the observations made in Remark 3.13. These examples indicate that the convergence rate of our method, in comparison to the existing ones, depends on particular problems; it can be either faster or slower.

2. Preliminaries

Key concepts, specifically monotone operators and ordinary and difference equations, are recalled in this section as a basis for the main results.

2.1. Monotone operators

We first review the concepts of monotone operators to provide the necessary context for the main results.

Assumption 2.1. Assume that the operators $A, B, F : \mathcal{H} \rightarrow \mathcal{H}$ satisfy the conditions (i)–(iii):

- (i) The pair (A, B) is monotone.
- (ii) The pair (A, F) is γ -strongly monotone.
- (iii) The operators B, F are Lipschitz with constants L_b, L_f , respectively.

Assumption 2.2. Assume that conditions (i)–(iii) hold.

- (i) The operator A is linear.
- (ii) There exist positive constants λ, L such that

$$\|Ag\| \leq L\|g\| \quad \forall g \in \mathcal{H}, \quad \text{and} \quad \langle Ag, g \rangle \geq \lambda\|g\|^2 \quad \forall g \in \mathcal{H}.$$

- (iii) The operator A is self-adjoint.

Remark 2.3. Recall from [17] that a bounded linear operator $R : \mathcal{H} \rightarrow \mathcal{H}$ is said to be *positive* if $\langle Rx, x \rangle \geq 0$ for all $x \in \mathcal{H}$, while it is classified as *strictly positive* if the inequality holds strictly for all non-zero x . It is known from Assumption 2.2(iii) and [17, Corollary 5.5] that there is only one positive self-adjoint operator T that squares to give A . This operator is the *positive square root* of A , denoted by $\sqrt{A}$. $\square$

Example 2.4. Let $\mathcal{H} = \mathbb{R}^n$, $\mathcal{A} = (a_{ij})_{n \times n}$ be a strictly positive matrix with $a_{21} > 0$, $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 3, \dots, n$ be Lipschitz continuous and monotone increasing functions. The operators $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are defined as

$$A(x) = \mathcal{A}x, \quad B(x) = \left[-\sum_{j=1}^n a_{2j}x_j, \sum_{j=1}^n a_{1j}x_j, f_3\left(\sum_{j=1}^n a_{3j}x_j\right), \dots, f_n\left(\sum_{j=1}^n a_{nj}x_j\right) \right]^T.$$

For all $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} \langle B(x) - B(y), A(x) - A(y) \rangle &= \\ &= \sum_{i=3}^n \left\{ f_i\left(\sum_{j=1}^n a_{ij}x_j\right) - f_i\left(\sum_{j=1}^n a_{ij}y_j\right) \right\} \left\{ \sum_{j=1}^n a_{ij}x_j - \sum_{j=1}^n a_{ij}y_j \right\} \geq 0. \end{aligned}$$

The pair (A, B) is monotone; but (I, B) is not, since

$$\langle B(e^1) - B(0), e^1 - 0 \rangle = -a_{21} < 0,$$

where $e^1 = [1, 0, \dots, 0]^T$.

2.2. Ordinary differential equations

The following discussion prepares the reader for the basic concepts regarding ordinary differential equations.

Definition 2.5. The function $h : [0, \infty) \rightarrow \mathcal{H}$ is said to be *locally absolutely continuous* if, within every finite interval $[c, d]$ of its domain, it is possible to identify an integrable function g mapping $[c, d]$ to $\mathcal{H}$, which satisfies the following equation

$$h(t) = h(c) + \int_c^t g(s) ds \quad \forall t \in [c, d]. \quad \square$$

Remark 2.6. Locally absolutely continuous functions are almost everywhere differentiable, with derivatives equal to their distributional derivatives almost everywhere.

Remark 2.7. Let $f : [a, b] \rightarrow \mathcal{H}$ be an absolutely continuous function and $U : \mathcal{H} \rightarrow \mathcal{H}$. Then, $g = Uf$ is also absolutely continuous. Additionally, g is almost everywhere differentiable, and its derivative satisfies $\|g'(t)\| \leq L_u \|f'(t)\|$ almost everywhere, where L_u is the Lipschitz constant of U . $\square$

Consider the dynamical system

$$\begin{cases} x'(t) = -\beta(t)[B(x(t)) + \varepsilon(t)F(x(t))], \\ x(t_0) = x^0 \in \mathcal{H}, \end{cases} \quad (8)$$

where the starting time $t_0 \geq 0$, the functions $\beta, \varepsilon : [t_0, \infty) \rightarrow (0, \infty)$ and operators $B, F : \mathcal{H} \rightarrow \mathcal{H}$.

The solution of dynamical system (8) is understood in the following sense.

Definition 2.8. A function x is called a *strong global solution* of equation (8) if it holds:

(1) The function $x : [t_0, \infty) \rightarrow \mathcal{H}$ is locally absolutely continuous.

(2) $x'(t) = -\beta(t)[B(x(t)) + \varepsilon(t)F(x(t))]$ for almost every $t \geq t_0$ and $x(t_0) = x^0$. $\square$

Remark 2.9. To ensure the existence and uniqueness of the trajectory $x(t)$ generated by (8), we require that B and F satisfy Lipschitz conditions and that β and ε are locally integrable. See [10, Proposition 6.2.1] for details. $\square$

2.3. Difference equations

This section serves as a prerequisite for the main results in discrete time by revisiting the concepts of difference equations and proving some observations.

Operators employed in the discretization of ODEs include $y^\Delta(n) = y(n+1) - y(n)$. Proposition 2.10 supports our convergence analysis.

Proposition 2.10. *We always have*

$$[\langle a(n), b(n) \rangle]^\Delta = \langle a^\Delta(n), b(n) \rangle + \langle a(n), b^\Delta(n) \rangle + \langle a^\Delta(n), b^\Delta(n) \rangle.$$

Lemma 2.11. Let $\mu, y, f : \mathbb{N} \rightarrow (0, \infty)$ and

$$\mu(n+1) \leq \mu(n) + y(n)\sqrt{\mu(n)} + f(n) \quad \forall n.$$

Then it holds that $\sqrt{\mu(m+\kappa)} \leq \sqrt{\mu(\kappa)} + \sum_{j=\kappa}^{m+\kappa-1} \left(\sqrt{f(j)} + y(j) \right)$.

Proof. By taking square roots and subsequently utilizing $\sqrt{p+q} \leq \sqrt{p} + \frac{q}{2\sqrt{p}}$, we arrive at the following inequalities

$$\sqrt{\mu(n+1)} \leq \sqrt{\mu(n) + f(n)} + \frac{y(n)\sqrt{\mu(n)}}{2\sqrt{\mu(n) + f(n)}} \leq \sqrt{\mu(n)} + \sqrt{f(n)} + y(n).$$

Summing both sides from $n = \kappa$ to $n = m + \kappa - 1$ yields the desired result. $\square$

3. Regularized dynamical systems from continuous perspective

The monotonicity condition of the pair (A, B) is insufficient to guarantee the convergence of the trajectories generated by the dynamical system $x'(t) = -B(x(t))$, as demonstrated below.

Example 3.1. Let $\mathcal{H} = \mathbb{R}^2$, and define the operators $A, B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 16x_1 + x_2 \\ x_1 + 49x_2 \end{bmatrix}, \quad B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 - 49x_2 \\ 16x_1 + x_2 \end{bmatrix}.$$

It can easily be checked that the pair (A, B) is monotone, but the pair (I, B) is not. The trajectories generated by the dynamical system $x'(t) = -B(x(t))$ fail to converge to 0. $\square$

In this section, we consider the dynamical system (6) under Assumptions 2.1, 2.2 and 3.2, which extends the system studied in [5, 8, 9].

$$x'(t) = -\beta(t)[B(x(t)) + \varepsilon(t)F(x(t))].$$

This extension is twofold. We extend the applicability of the monotonicity property beyond the specific case of (I, B) to the more general case of (A, B) , with the operator A subject to Assumption 2.2. Furthermore, the regularization is generalized to encompass a more general form, $\varepsilon(t)F(x(t))$, where the operator F and regularization coefficient $\varepsilon(t)$ satisfy Assumptions 2.1 and 3.2, respectively.

Assumption 3.2. The regularization coefficient $\varepsilon : [t_0, \infty) \rightarrow (0, \infty)$ is assumed to be continuously differentiable, fulfilling (9) and (10)

$$\lim_{t \rightarrow \infty} \varepsilon(t) = 0, \tag{9}$$

$$\int_{t_0}^{\infty} \beta(s)\varepsilon(s) ds = \infty. \tag{10}$$

Remark 3.3. We list some of the terminology employed in the proofs.

(i) Since the operator A is self-adjoint, for all $x, y \in \mathcal{H}$, we have

$$\langle x - y, AB(x) - AB(y) \rangle = \langle A(x) - A(y), B(x) - B(y) \rangle \geq 0.$$

The operator AB is monotone and continuous and hence $Z(AB)$ is convex. Moreover, under Assumption 2.2, it is easily seen that $Z(B) = Z(AB)$. The convexity of $Z(B)$ and strong monotonicity of (A, F) ensure that the variational inequality $VI(AF, Z(B))$ possesses a single solution, denoted by ϕ , which fulfills

$$\begin{cases} \phi \in Z(B), \\ \langle AF(\phi), y - \phi \rangle \geq 0 \quad \forall y \in Z(B). \end{cases}$$

(ii) Denote $\Gamma(t, r) = \exp \left(\int_r^t \frac{2\gamma}{L} \beta(s)\varepsilon(s) ds \right)$. $\square$

Having made the necessary preparations, we are now in a position to analyze the convergence of the trajectories generated by (6), with the aid of the following lemma.

Lemma 3.4. For $z \in Z(B)$, denote

$$v(t) = \frac{1}{2} \langle A(x(t) - z), x(t) - z \rangle, \quad (11)$$

$$\omega(t) = \frac{1}{2} \langle A(B(x(t)) + \varepsilon(t)F(x(t))), B(x(t)) + \varepsilon(t)F(x(t)) \rangle. \quad (12)$$

Along with Assumptions 2.1 and 2.2(i)–(iii), assume that $Z(B) \neq \emptyset$. Then conclusions (i)–(iii) are valid:

(i) It holds that

$$\begin{aligned} v'(t) + \frac{2\gamma}{L} \beta(t) \varepsilon(t) v(t) &\leq -\beta(t) \varepsilon(t) \langle x(t) - z, AF(z) \rangle \\ &\leq \frac{\gamma}{L} \beta(t) \varepsilon(t) v(t) + \frac{L}{2\gamma} \beta(t) \varepsilon(t) \|\sqrt{AF}(z)\|^2. \end{aligned} \quad (13)$$

(ii) The trajectory $x(t)$ and $F(x(t))$ are bounded, meaning that

$$C = \sup_{t \geq t_0} \|F(x(t))\| < \infty.$$

(iii) It holds that

$$\omega'(t) + \frac{2\gamma}{L} \beta(t) \varepsilon(t) \omega(t) \leq -\frac{\varepsilon'(t)}{\beta(t)} \langle F(x(t)), Ax'(t) \rangle. \quad (14)$$

Proof. (i) We have

$$\begin{aligned} v'(t) &= \langle x'(t), Ax(t) - Az \rangle \\ &= -\beta(t) \langle B(x(t)), Ax(t) - Az \rangle - \beta(t) \varepsilon(t) \langle F(x(t)), Ax(t) - Az \rangle \\ &\leq 0 - \beta(t) \varepsilon(t) \langle F(x(t)) - F(z), Ax(t) - Az \rangle - \beta(t) \varepsilon(t) \langle F(z), Ax(t) - Az \rangle \\ &\leq -\frac{2\gamma}{L} \beta(t) \varepsilon(t) v(t) - \beta(t) \varepsilon(t) \langle AF(z), x(t) - z \rangle \end{aligned}$$

which gives

$$\begin{aligned} v'(t) + \frac{2\gamma}{L} \beta(t) \varepsilon(t) v(t) &\leq \beta(t) \varepsilon(t) \left\langle -\sqrt{AF}(z), \sqrt{A}(x(t) - z) \right\rangle \\ &\leq \frac{\gamma}{L} \beta(t) \varepsilon(t) v(t) + \frac{L}{2\gamma} \beta(t) \varepsilon(t) \|\sqrt{AF}(z)\|^2. \end{aligned}$$

(ii) By Assumptions 2.1(iii) and 2.2(ii), it is enough to show that the function $v(t)$ is bounded. It follows from (13) that

$$v'(t) + \frac{\gamma}{L} \beta(t) \varepsilon(t) v(t) \leq \frac{L}{2\gamma} \beta(t) \varepsilon(t) \|\sqrt{AF}(z)\|^2.$$

Multiply both sides by $\theta(t) = \exp\left(\frac{\gamma}{L} \int_{t_0}^t \beta(s) \varepsilon(s) ds\right)$ to get

$$\frac{d}{dt} (\theta(t) v(t)) \leq \frac{L}{2\gamma} \theta(t) \beta(t) \varepsilon(t) \|\sqrt{AF}(z)\|^2 = \frac{L^2 \theta'(t)}{2\gamma^2} \|\sqrt{AF}(z)\|^2.$$

After integrating $t \in [t_0, s]$, we obtain

$$\theta(s) v(s) - \theta(t_0) v(t_0) \leq \frac{L^2}{\gamma^2} [\theta(s) - \theta(t_0)] \cdot \|\sqrt{AF}(z)\|^2,$$

which means

$$\begin{aligned} v(s) &\leq \frac{\theta(t_0)v(t_0)}{\theta(s)} + \frac{L^2}{\gamma^2} \left(1 - \frac{\theta(t_0)}{\theta(s)}\right) \|\sqrt{A}F(z)\|^2 \\ &\leq \theta(t_0)v(t_0) + \frac{L^2}{\gamma^2} \|\sqrt{A}F(z)\|^2. \end{aligned}$$

(iii) We establish the conclusion by using (15)–(16), as proven below. The monotonicity property of the pair (A, B) gives

$$\left\langle \frac{B(x(s)) - B(x(t))}{s - t}, \frac{A(x(s)) - A(x(t))}{s - t} \right\rangle \geq 0,$$

which implies, by letting $s \rightarrow t$, that

$$\langle [B(x(t))]', Ax'(t) \rangle \geq 0 \quad \forall t \geq t_0. \quad (15)$$

The strong monotonicity of the pair (A, F) yields

$$\left\langle \frac{F(x(s)) - F(x(t))}{s - t}, \frac{A(x(s)) - A(x(t))}{s - t} \right\rangle \geq \gamma \left\| \frac{x(s) - x(t)}{s - t} \right\|^2$$

and then taking the limit as $s \rightarrow t$ gives

$$\langle [F(x(t))]', Ax'(t) \rangle \geq \gamma \|x'(t)\|^2 \quad \forall t \geq t_0. \quad (16)$$

Having established (15)–(16), we proceed to prove the conclusion (iii).

Indeed, since the operator A is self-adjoint, we have

$$\begin{aligned} \omega'(t) &= \langle [B(x(t))]' + [\varepsilon(t)F(x(t))]', AB(x(t)) + \varepsilon(t)AF(x(t)) \rangle \\ &= \frac{1}{\beta(t)} \langle [B(x(t))]' + [\varepsilon(t)F(x(t))]', -Ax'(t) \rangle \\ &\stackrel{(15)}{\leq} 0 - \frac{\varepsilon(t)}{\beta(t)} \langle [F(x(t))]', Ax'(t) \rangle - \frac{\varepsilon'(t)}{\beta(t)} \langle F(x(t)), Ax'(t) \rangle, \end{aligned}$$

which, as

$$\begin{aligned} \langle [F(x(t))]', Ax'(t) \rangle &\stackrel{(16)}{\geq} \gamma \|x'(t)\|^2 = \gamma \beta(t)^2 \|B(x(t)) + \varepsilon(t)x(t)\|^2 \\ &\geq \frac{\gamma}{L} \beta(t)^2 \langle B(x(t)) + \varepsilon(t)x(t), AB(x(t)) + \varepsilon(t)Ax(t) \rangle \\ &= \frac{2\gamma}{L} \beta(t)^2 \omega(t), \end{aligned}$$

gives (14). □

3.1. Strong convergence

Theorem 3.5. Consider the dynamical system (6), under Assumptions 2.1 and 2.2. Assume that the regularization coefficient $\varepsilon : [t_0, \infty) \rightarrow (0, \infty)$, in addition to Assumption 3.2, satisfies

$$\text{either} \quad \lim_{t \rightarrow \infty} \frac{|\varepsilon'(t)|}{\beta(t)\varepsilon(t)} = 0, \quad (17)$$

$$\text{or} \quad \int_{t_0}^{\infty} \left(\beta(s) + \frac{1}{\beta(s)} \right) |\varepsilon'(s)| ds < \infty. \quad (18)$$

Then the trajectory $x(t)$ generated by (6) converges strongly to ϕ , the unique solution of the problem $\text{VI}(AF, Z(B))$ as $t \rightarrow \infty$, provided that $Z(B) \neq \emptyset$.

Proof. Claim: $\lim_{t \rightarrow \infty} B(x(t)) = 0$.

Case 1. Condition (17) holds.

Recall that the function ω is defined in (12). By (14) and Assumption 2.2(ii), we have

$$\begin{aligned} & \omega'(t) + \frac{2\gamma}{L}\beta(t)\varepsilon(t)\omega(t) \\ & \leq \frac{\gamma\lambda}{2L} \cdot \frac{\varepsilon(t)}{\beta(t)} \|x'(t)\|^2 + \frac{L^3}{2\gamma\lambda} \cdot \frac{|\varepsilon'(t)|^2}{\beta(t)\varepsilon(t)} \|F(x(t))\|^2 \\ & = \frac{\gamma\lambda}{2L}\beta(t)\varepsilon(t)\|B(x(t)) + \varepsilon(t)F(x(t))\|^2 + \frac{L^3}{2\gamma\lambda} \cdot \frac{|\varepsilon'(t)|^2}{\beta(t)\varepsilon(t)} \|F(x(t))\|^2 \\ & \leq \frac{\gamma}{L}\beta(t)\varepsilon(t)\omega(t) + \frac{L^3}{2\gamma\lambda} \cdot \frac{|\varepsilon'(t)|^2}{\beta(t)\varepsilon(t)} C^2, \end{aligned}$$

which implies, by [9, Lemma 1] and (17), that

$$\limsup_{t \rightarrow \infty} \omega(t) \leq \limsup_{t \rightarrow \infty} \left(\frac{|\varepsilon'(t)|}{\beta(t)\varepsilon(t)} \right)^2 = 0.$$

Here, the constant C is specified in Lemma 3.4. Thus, by Assumption 2.2(ii), the limit of $B(x(t)) + \varepsilon(t)F(x(t))$ as $t \rightarrow \infty$ is zero. Consequently, applying Lemma 3.4(ii) along with (9), it follows that the limit of $B(x(t))$ is also zero.

Case 2. Suppose that (18) holds.

The product rule, applied to the left side of (14) after multiplying by $\Gamma(t, t_0)$ as specified in Remark 3.3, gives us

$$\begin{aligned} \frac{d}{dt} \left(\Gamma(t, t_0)\omega(t) \right) & \leq -\frac{\varepsilon'(t)}{\beta(t)} \Gamma(t, t_0) \langle AF(x(t)), x'(t) \rangle \\ & \leq \frac{|\varepsilon'(t)|}{\beta(t)} \Gamma(t, t_0) (L^2 C^2 + \|x'(t)\|^2), \end{aligned}$$

with the last inequality due to Lemma 3.4(ii). Note that

$$\|x'(t)\|^2 \stackrel{(6)}{=} \beta(t)^2 \|B(x(t)) + \varepsilon(t)x(t)\|^2 \stackrel{\text{Assumption 2.2(iii)}}{\leq} \frac{2}{\lambda} \beta(t)^2 \omega(t).$$

Thus, we get

$$\frac{d}{dt} (\Gamma(t, t_0)\omega(t)) \leq \frac{|\varepsilon'(t)|}{\beta(t)} \Gamma(t, t_0) \left(L^2 C^2 + \frac{2}{\lambda} \beta(t)^2 \omega(t) \right),$$

which implies, after integrating $t \in [t_1, t_2]$, that

$$\begin{aligned} & \Gamma(t_2, t_0)\omega(t_2) \leq \\ & \leq \Gamma(t_1, t_0)\omega(t_1) + L^2 C^2 \int_{t_1}^{t_2} \frac{|\varepsilon'(t)|}{\beta(t)} \Gamma(t, t_0) dt + \frac{2}{\lambda} \int_{t_1}^{t_2} \beta(t) |\varepsilon'(t)| \Gamma(t, t_0) \omega(t) dt. \end{aligned}$$

[15, Theorem 1.3.1], applied to the inequality above, leads to the following

$$\Gamma(t_2, t_0)\omega(t_2) \leq \left(\Gamma(t_1, t_0)\omega(t_1) + L^2 C^2 \int_{t_1}^{t_2} \frac{|\varepsilon'(t)|}{\beta(t)} \Gamma(t, t_0) dt \right) e^{\frac{2}{\lambda} \int_{t_1}^{t_2} \beta(t) |\varepsilon'(t)| dt}.$$

Dividing the line above by $\Gamma(t_2, t_0)$ yields

$$\begin{aligned} \omega(t_2) & \leq \left(\frac{\Gamma(t_1, t_0)\omega(t_1)}{\Gamma(t_2, t_0)} + L^2 C^2 \int_{t_1}^{t_2} \frac{|\varepsilon'(t)|}{\beta(t)} \cdot \frac{\Gamma(t, t_0)}{\Gamma(t_2, t_0)} dt \right) e^{\frac{2}{\lambda} \int_{t_1}^{t_2} \beta(t) |\varepsilon'(t)| dt} \\ & \leq \left(\frac{\Gamma(t_1, t_0)\omega(t_1)}{\Gamma(t_2, t_0)} + L^2 C^2 \int_{t_1}^{t_2} \frac{|\varepsilon'(t)|}{\beta(t)} dt \right) e^{\frac{2}{\lambda} \int_{t_1}^{t_2} \beta(t) |\varepsilon'(t)| dt}, \end{aligned}$$

which implies, after letting $t_2 \rightarrow \infty$ and using (10), that

$$\limsup_{t_2 \rightarrow \infty} \omega(t_2) \leq \left(L^2 C^2 \int_{t_1}^{\infty} \frac{|\varepsilon'(t)|}{\beta(t)} dt \right) e^{\frac{2}{\lambda} \int_{t_1}^{\infty} \beta(t) |\varepsilon'(t)| dt}.$$

Note that by (18),

$$\infty > \int_{t_0}^{\infty} \left(\beta(s) + \frac{1}{\beta(s)} \right) |\varepsilon'(s)| ds > \int_{t_0}^{\infty} (\beta(s) + 0) |\varepsilon'(s)| ds,$$

which leads to

$$\int_{t_1}^{\infty} \beta(t) |\varepsilon'(t)| dt = \int_{t_0}^{\infty} \beta(t) |\varepsilon'(t)| dt - \int_{t_0}^{t_1} \beta(t) |\varepsilon'(t)| dt.$$

Hence,

$$\lim_{t_1 \rightarrow \infty} \int_{t_1}^{\infty} \beta(t) |\varepsilon'(t)| dt = 0.$$

Similarly,

$$\lim_{t_1 \rightarrow \infty} \int_{t_1}^{\infty} \frac{|\varepsilon'(t)|}{\beta(t)} dt = 0.$$

Now, letting $t_1 \rightarrow \infty$ in the lim sup estimate of $\omega(t_2)$ yields $\lim_{t \rightarrow \infty} \omega(t) = 0$.

As a consequence of Lemma 3.4(ii) and (9), it follows that the limit of $B(x(t))$ as t approaches infinity is zero.

To recapitulate, the satisfaction of either condition (17) or (18) implies

$$\lim_{t \rightarrow \infty} B(x(t)) = 0.$$

Applying (13) with the function $\psi(t)$ defined as $\frac{1}{2} \langle A(x(t) - \phi), x(t) - \phi \rangle$, where ϕ is an element of $Z(B)$ as specified in Remark 3.3, we obtain

$$\psi'(t) + \frac{2\gamma}{L} \beta(t) \varepsilon(t) \psi(t) \leq \beta(t) \varepsilon(t) \langle x(t) - \phi, -AF(\phi) \rangle,$$

which implies, by (10) and [9, Lemma 1], that

$$\limsup_{t \rightarrow \infty} \psi(t) \leq \limsup_{t \rightarrow \infty} \langle x(t) - \phi, -AF(\phi) \rangle. \quad (19)$$

We will now demonstrate that, on the condition that $B(x(t)) \rightarrow 0$ as $t \rightarrow \infty$, every weak cluster point of $x(\cdot)$ belongs to $Z(B)$. The existence of such cluster points is guaranteed by the boundedness of the trajectory. Let $x_{\odot}$ be an arbitrary weak cluster point of $x(t)$. Consequently, there exists a sequence $\{x(s_m)\}_{m \geq 0}$ such that $x(s_m)$ converges weakly to $x_{\odot}$ as $m \rightarrow \infty$. Moreover, $B(x(s_m)) \rightarrow 0$ as $m \rightarrow \infty$, which implies by Assumption 2.2, that $AB(x(s_m)) \rightarrow 0$, too. Given that AB is maximally monotone, its graph is sequentially closed in the weak $\times$ strong topology of $\mathcal{H} \times \mathcal{H}$. This implies that $AB(x_{\odot}) = 0$ and so $B(x_{\odot}) = 0$. This result, in conjunction with (19), allows us to conclude that

$$\limsup_{t \rightarrow \infty} \psi(t) \leq \langle x_{\odot} - \phi, -AF(\phi) \rangle = -\langle x_{\odot} - \phi, AF(\phi) \rangle \leq 0,$$

which is equivalent to the fact that $\lim_{t \rightarrow \infty} x(t) = \phi$. □

Example 3.6. The functions $\varepsilon(t) = \frac{1}{\sqrt[3]{1+t}}$ and $\beta(t) = \frac{1}{\sqrt{1+t}}$ are shown to fulfill the conditions specified in both Assumption 3.2 and Theorem 3.5. □

We end this section by noting that for $A = F = I$ and $\beta(t) \equiv 1$, Theorem 3.5 simplifies to the classical finding that the trajectory generated by (6) converges to the minimum-norm solution, as established in [5, 8, 9].

3.2. Non-emptiness of the zero set

Theorem 3.7. Consider the dynamical system (6), under Assumptions 2.1 and 2.2. Assume that the regularization coefficient $\varepsilon : [t_0, \infty) \rightarrow (0, \infty)$, in addition to Assumption 3.2, satisfies

$$\lim_{t \rightarrow \infty} \frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)} = 0 \quad (20)$$

$$\text{or} \quad \int_{t_0}^{\infty} \left(|(\beta\varepsilon)'(t)| + |\beta'(t)| \right) dt < \infty. \quad (21)$$

Then the zero set $Z(B) \neq \emptyset$, provided that the trajectory $x(t)$ generated by (6) is bounded.

Proof. Let $s \geq 0$ and $t_2 \geq t \geq t_1 \geq t_0$. Consider the function

$$g(t) = \frac{1}{2} \langle x(t+s) - x(t), Ax(t+s) - Ax(t) \rangle.$$

A direct computation gives

$$\begin{aligned} g'(t) &= \langle x'(t+s) - x'(t), Ax(t+s) - Ax(t) \rangle \\ &= -\beta(t) \langle B(x(t+s)) - B(x(t)), Ax(t+s) - Ax(t) \rangle \\ &\quad + [\beta(t) - \beta(t+s)] \langle B(x(t+s)), Ax(t+s) - Ax(t) \rangle \\ &\quad - \beta(t)\varepsilon(t) \langle F(x(t+s)) - F(x(t)), Ax(t+s) - Ax(t) \rangle \\ &\quad + [\beta(t)\varepsilon(t) - \beta(t+s)\varepsilon(t+s)] \langle F(x(t+s)), Ax(t+s) - Ax(t) \rangle \\ &\leq 0 + [\beta(t) - \beta(t+s)] \left\langle \sqrt{A}B(x(t+s)), \sqrt{A}x(t+s) - \sqrt{A}x(t) \right\rangle \\ &\quad - \gamma\beta(t)\varepsilon(t) \|x(t+s) - x(t)\|^2 \\ &\quad + [\beta(t)\varepsilon(t) - \beta(t+s)\varepsilon(t+s)] \left\langle \sqrt{A}F(x(t+s)), \sqrt{A}x(t+s) - \sqrt{A}x(t) \right\rangle. \end{aligned}$$

Estimating the term $\|x(t+s) - x(t)\|^2$ using Assumption 2.2(ii), moving the resulting expression to the left-hand side, and applying the Cauchy-Schwarz inequality to the right-hand side, we obtain

$$\begin{aligned} g'(t) + \frac{2\gamma}{L} \beta(t)\varepsilon(t)g(t) &\leq |\beta(t)\varepsilon(t) - \beta(t+s)\varepsilon(t+s)| \cdot \|\sqrt{A}F(x(t+s))\| \cdot \|\sqrt{A}x(t+s) - \sqrt{A}x(t)\| \\ &\quad + |\beta(t) - \beta(t+s)| \cdot \|\sqrt{A}B(x(t+s))\| \cdot \|\sqrt{A}x(t+s) - \sqrt{A}x(t)\|. \end{aligned}$$

Applying the product rule on the left side, after multiplying by $\Gamma(t, t_1)$ (as specified in Remark 3.3), yields

$$\begin{aligned} \frac{d}{dt} (\Gamma(t, t_1)g(t)) &\leq |\beta(t)\varepsilon(t) - \beta(t+s)\varepsilon(t+s)| \cdot \sqrt{2} \|\sqrt{A}F(x(t+s))\| \sqrt{\Gamma(t, t_1)g(t)} \\ &\quad + |\beta(t) - \beta(t+s)| \cdot \sqrt{2} \|\sqrt{A}B(x(t+s))\| \sqrt{\Gamma(t, t_1)g(t)}, \end{aligned}$$

which implies, by [5, Lemma 3.1], that

$$\begin{aligned} &\sqrt{\Gamma(t_2, t_1)g(t_2)} \\ &\leq \sqrt{g(t_1)} + \int_{t_1}^{t_2} |\beta(t)\varepsilon(t) - \beta(t+s)\varepsilon(t+s)| \cdot \sqrt{2} \|\sqrt{A}F(x(t+s))\| \sqrt{\Gamma(t, t_1)} dt \\ &\quad + \int_{t_1}^{t_2} |\beta(t) - \beta(t+s)| \cdot \sqrt{2} \|\sqrt{A}B(x(t+s))\| \sqrt{\Gamma(t, t_1)} dt \\ &\leq \sqrt{g(t_1)} + D \int_{t_1}^{t_2} \{ |\beta(t)\varepsilon(t) - \beta(t+s)\varepsilon(t+s)| + |\beta(t) - \beta(t+s)| \} e^{\int_{t_1}^t \frac{\gamma}{L} \beta(r)\varepsilon(r) dr} dt, \end{aligned}$$

where $D = \sqrt{2} \sup_{t \geq t_0} \{\|\sqrt{A}B(x(t))\| + \|\sqrt{A}F(x(t))\|\} < \infty$ by virtue of Lemma 3.4(ii). The last inequality, upon dividing both sides by s and then taking the limit as $s \rightarrow 0$, transforms into

$$\begin{aligned} & e^{\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \\ & \leq \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + D\sqrt{2} \int_{t_1}^{t_2} \{ |(\beta\varepsilon)'(t)| + |\beta'(t)| \} e^{\int_{t_1}^t \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} dt. \end{aligned} \quad (22)$$

Claim:

$$\lim_{t \rightarrow \infty} x'(t) = 0. \quad (23)$$

Case 1. Condition (20) is met.

The inequality (22) gives

$$\begin{aligned} & e^{\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \\ & \leq \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \sup_{t_1 \leq t \leq t_2} \frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)} \int_{t_1}^{t_2} \frac{\gamma}{L} \beta(t) \varepsilon(t) e^{\int_{t_1}^t \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} dt \\ & = \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \sup_{t_1 \leq t \leq t_2} \frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)} \int_{t_1}^{t_2} \frac{d}{dt} \left(e^{\int_{t_1}^t \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \right) dt \\ & = \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \sup_{t_1 \leq t \leq t_2} \frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)} \left(e^{\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} - 1 \right), \end{aligned}$$

which implies, after dividing both sides by $\exp\left(\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr\right)$, that

$$\begin{aligned} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} & \leq e^{-\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} \\ & \quad + \frac{LD\sqrt{2}}{\gamma} \sup_{t_1 \leq t \leq t_2} \frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)} \left(1 - e^{-\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \right) \\ & \leq e^{-\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \sup_{t_1 \leq t \leq t_2} \frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)}. \end{aligned}$$

By (20), for every $\eta > 0$ there exists $t_\star = t_\star(\eta) \geq t_0$ such that

$$\frac{|(\beta\varepsilon)'(t)| + |\beta'(t)|}{\beta(t)\varepsilon(t)} < \eta \quad \forall t \geq t_\star.$$

For $t_2 \geq t_1 \geq t_\star$, we have

$$\sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \leq e^{-\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \eta,$$

which implies, after letting $t_2 \rightarrow \infty$, that

$$\begin{aligned} \limsup_{t_2 \rightarrow \infty} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} & \leq e^{-\int_{t_1}^{\infty} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \eta \\ & \leq \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \eta. \end{aligned} \quad (24)$$

Passing to the limit as $t_1 \rightarrow \infty$ yields the inequality

$$\limsup_{t_2 \rightarrow \infty} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \leq \liminf_{t_1 \rightarrow \infty} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \eta.$$

Since this inequality holds for an arbitrary η , we can conclude that $\sqrt{\langle x'(t), Ax'(t) \rangle}$ converges to a limit as $t \rightarrow \infty$. Thus, by (24), we get

$$\lim_{t_2 \rightarrow \infty} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \leq e^{-\int_{t_1}^{\infty} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + \frac{LD\sqrt{2}}{\gamma} \eta. \quad (25)$$

Note that by (10):

$$\int_{t_1}^{\infty} \beta(r) \varepsilon(r) dr = \int_{t_0}^{\infty} \beta(r) \varepsilon(r) dr - \int_{t_0}^{t_1} \beta(r) \varepsilon(r) dr,$$

which gives

$$\lim_{t_1 \rightarrow \infty} \int_{t_1}^{\infty} \beta(r) \varepsilon(r) dr = 0. \quad (26)$$

Through letting $t_1 \rightarrow \infty$ in (25) and then using (26), we obtain

$$0 \leq \lim_{t_2 \rightarrow \infty} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \leq 0 + \frac{LD\sqrt{2}}{\gamma} \eta,$$

where the non-negativity of the limit is derived from Assumption 2.2(ii). The limit as $\eta \rightarrow 0$, taken in conjunction with Assumption 2.2(ii), gives (23).

Case 2. Condition (21) is met.

Dividing both sides of (22) by $\exp\left(\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr\right)$ yields

$$\begin{aligned} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} &\leq e^{-\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} \\ &\quad + D\sqrt{2} \int_{t_1}^{t_2} \{ |(\beta\varepsilon)'(t)| + |\beta'(t)| \} e^{-\int_t^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} dt \\ &\leq e^{-\int_{t_1}^{t_2} \frac{\gamma}{L} \beta(r) \varepsilon(r) dr} \sqrt{\langle x'(t_1), Ax'(t_1) \rangle} + D\sqrt{2} \int_{t_1}^{t_2} \{ |(\beta\varepsilon)'(t)| + |\beta'(t)| \} dt. \end{aligned} \quad (27)$$

Taking the limit as $t_2 \rightarrow \infty$, along with applying the result from (10), gives

$$\limsup_{t_2 \rightarrow \infty} \sqrt{\langle x'(t_2), Ax'(t_2) \rangle} \leq D\sqrt{2} \int_{t_1}^{\infty} \{ |(\beta\varepsilon)'(t)| + |\beta'(t)| \} dt.$$

Consequently, (23) follows by taking the limit as $t_1 \rightarrow \infty$ and using Assumption 2.2(ii).

Having made the necessary preparations, we are ready to analyze the convergence of the trajectories generated by (6), with the aid of (23). Note that weak cluster points of the trajectory $x(t)$ always exist due to the boundedness of $x(t)$. Let (t_n) be a sequence such that

$$t_n \rightarrow \infty, \quad x(t_n) \rightharpoonup x_o.$$

By (6), (23) and (9), we have

$$B(x(t_n)) = -x'(t_n) - \varepsilon(t_n)F(x(t_n)) \rightarrow 0,$$

which implies by Assumption 2.2, that $AB(x(t_n)) \rightarrow 0$, too. Given that AB is maximally monotone, its graph is sequentially closed in the weak $\times$ strong topology of $\mathcal{H} \times \mathcal{H}$. This implies that $AB(x_o) = 0$ and so by Assumption 2.2 $B(x_o) = 0$. $\square$

3.3. The case when $\beta(t)$ is a constant function

When $\beta(t)$ is a function with a constant value, we combine Theorems 3.5 and 3.7 to provide a characterization, as follows.

Corollary 3.8. Consider the dynamical system (6), under the Assumptions 2.1 and 2.2. Let $\beta(t) = 1$ for all $t \geq t_0$. Assume that the regularization coefficient $\varepsilon : [t_0, \infty) \rightarrow (0, \infty)$ satisfies, in addition to Assumption 3.2,

$$\text{either } \lim_{t \rightarrow \infty} \frac{|\varepsilon'(t)|}{\varepsilon(t)} = 0 \quad \text{or} \quad \int_{t_0}^{\infty} |\varepsilon'(t)| dt < \infty.$$

Then the following assertions are equivalent:

- (i) The trajectory $x(t)$ generated by (6) converges strongly to a zero of the operator B as $t \rightarrow \infty$.
- (ii) The trajectory $x(t)$ generated by (6) is bounded
- (iii) the zero set $Z(B)$ is non-empty.

Proof. When β is identically constant, we have the following equivalences:

$$\begin{aligned} (17) &\iff (20) \iff \lim_{t \rightarrow \infty} \frac{|\varepsilon'(t)|}{\varepsilon(t)} = 0 \\ \text{and } (18) &\iff (21) \iff \int_{t_0}^{\infty} |\varepsilon'(t)| dt < \infty. \end{aligned}$$

We can now apply Theorems 3.5 and 3.7 to establish the equivalence of assertions (i), (ii), and (iii). The implication (i) $\implies$ (ii) is immediate, as convergence inherently implies boundedness. The remaining implications (ii) $\implies$ (iii) and (iii) $\implies$ (i) follow from Theorems 3.7 and 3.5, respectively. $\square$

3.4. Convergence rate

This section establishes a convergence rate of the dynamical system (6) under Assumption 3.9. The malleability of the regularization operator F is a key property of this method, as detailed in Theorem 3.10, thus providing a means to improve computational efficiency.

Assumption 3.9. The operator F meets the requirements of (i) and regularization coefficient $\varepsilon(t)$ is subject to the conditions of (ii).

- (i) The operator $F : \mathcal{H} \rightarrow \mathcal{H}$ is assumed to be a bounded, linear operator satisfying the condition that the product AF is self-adjoint and positive.
- (ii) There exists $t_1 \geq t_0$ such that

$$\left\{ \begin{array}{l} \varepsilon(t) \text{ is twice differentiable on } [t_1, \infty), \\ \frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} + \frac{\gamma}{L} \varepsilon(t)^2 \right) \text{ has a sign that does not switch on } [t_1, \infty), \\ \varepsilon'(t) \leq 0 \text{ on } [t_1, \infty). \end{array} \right.$$

Theorem 3.10. Consider the dynamical system (6), under the Assumptions 2.1 and 2.2. If Assumption 3.9 holds, then it holds that

$$\|B(x(t))\| \leq M_1 \max \left\{ \sqrt{\frac{-\varepsilon'(t)}{\beta(t)}}, \exp \left(-\int_{t_0}^t \frac{\gamma}{L} \beta(s) \varepsilon(s) ds \right) \right\} + M_2 \varepsilon(t) \quad \forall t \geq t_1, \quad (28)$$

where M_1, M_2 are some positive constants.

Proof. Assumption 2.2 and Lemma 3.4(ii) give

$$\|B(x(t))\| \leq \|B(x(t)) + \varepsilon(t)F(x(t))\| + \varepsilon(t)\|F(x(t))\| \leq \sqrt{\frac{2\omega(t)}{\lambda}} + \varepsilon(t)C. \quad (29)$$

By (14), we have $\omega'(t) + \frac{2\gamma}{L} \beta(t) \varepsilon(t) \omega(t) \leq -\frac{\varepsilon'(t)}{2\beta(t)} \frac{d}{dt} \|\sqrt{AF}x(t)\|^2$.

Using the product rule on the left-hand side, after multiplying by $\Gamma(t, t_0)$ (as specified in Remark 3.3), results in the following inequality:

$$\frac{d}{dt} \left(\Gamma(t, t_0) \omega(t) \right) \leq -\frac{\varepsilon'(t)}{2\beta(t)} \Gamma(t, t_0) \frac{d}{dt} \|\sqrt{AF}x(t)\|^2.$$

We integrate with respect to $t \in [t_1, t_2]$ and apply integration by parts on the right-hand side of the inequality above

$$\begin{aligned} & \Gamma(t_2, t_0) \omega(t_2) + \frac{\varepsilon'(t_2)}{2\beta(t_2)} \Gamma(t_2, t_0) \|\sqrt{AF}x(t_2)\|^2 \\ & \leq \underbrace{\Gamma(t_1, t_0) \omega(t_1) + \frac{\varepsilon'(t_1)}{2\beta(t_1)} \Gamma(t_1, t_0) \|\sqrt{AF}x(t_1)\|^2}_{C_0} \\ & \quad + \frac{1}{2} \int_{t_1}^{t_2} \|\sqrt{AF}x(t)\|^2 \frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} \Gamma(t, t_0) \right) dt. \end{aligned} \quad (30)$$

Note that $\frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} \Gamma(t, t_0) \right) = \Gamma(t, t_0) \frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} + \frac{\gamma}{L} \varepsilon(t)^2 \right)$.

The subsequent analysis will be conducted under two distinct cases, predicated on the sign of the derivative $\frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} + \frac{\gamma}{L} \varepsilon(t)^2 \right)$.

Case 1: $\frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} + \frac{\gamma}{L} \varepsilon(t)^2 \right) \geq 0$ for $t \geq t_1$.

The estimation of (30) by C_1 , where C_1 is defined as $\sup_{t \geq t_1} \|\sqrt{AF}x(t)\|^2$, leads to

$$\begin{aligned} & \Gamma(t_2, t_0) \omega(t_2) + \frac{\varepsilon'(t_2)}{2\beta(t_2)} \Gamma(t_2, t_0) \|\sqrt{AF}x(t_2)\|^2 \\ & \leq C_0 + \frac{C_1}{2} \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} \Gamma(t, t_0) \right) dt \\ & = C_0 + \frac{C_1}{2} \left(\frac{\varepsilon'(t_2)}{\beta(t_2)} \Gamma(t_2, t_0) - \frac{\varepsilon'(t_1)}{\beta(t_1)} \Gamma(t_1, t_0) \right). \end{aligned}$$

Dividing both sides by $\Gamma(t_2, t_0)$ yields

$$\begin{aligned} & \omega(t_2) + \frac{\varepsilon'(t_2)}{2\beta(t_2)} \|\sqrt{AF}x(t_2)\|^2 \\ & \leq \frac{C_1}{2} \cdot \frac{\varepsilon'(t_2)}{\beta(t_2)} + \left(C_0 - \frac{C_1}{2} \cdot \frac{\varepsilon'(t_1)}{\beta(t_1)} \Gamma(t_1, t_0) \right) \frac{1}{\Gamma(t_2, t_0)}. \end{aligned}$$

On the left-hand side of the inequality, we evaluate $\|\sqrt{AF}x(t_2)\|^2 \leq C_1$ and combine it with $\varepsilon'(t_2) \leq 0$ to obtain

$$\omega(t_2) \leq \left(C_0 - \frac{C_1}{2} \cdot \frac{\varepsilon'(t_1)}{\beta(t_1)} \Gamma(t_1, t_0) \right) \frac{1}{\Gamma(t_2, t_0)}. \quad (31)$$

Case 2: $\frac{d}{dt} \left(\frac{\varepsilon'(t)}{\beta(t)} + \frac{\gamma}{L} \varepsilon(t)^2 \right) \leq 0$ for $t \geq t_1$. Thanks to (30), we deduce

$$\omega(t_2) + \frac{\varepsilon'(t_2)}{2\beta(t_2)} \|\sqrt{AF}x(t_2)\|^2 \leq \frac{C_0}{\Gamma(t_2, t_0)}.$$

On the left-hand side of the inequality, we evaluate $\|\sqrt{AF}x(t_2)\|^2 \leq C_1$ and combine it with $\varepsilon'(t_2) \leq 0$ to get

$$\omega(t_2) \leq \frac{C_0}{\Gamma(t_2, t_0)} - \frac{C_1}{2} \cdot \frac{\varepsilon'(t_2)}{\beta(t_2)}. \quad (32)$$

Combining (29) with (31), and then with (32), yields the desired result. $\square$

Example 3.11. Let $\mathcal{H} = \mathbb{R}^2$, and the operator $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by (7). The pair (I, W) lacks monotonicity, rendering the existing theoretical framework [5, 8, 9] inapplicable to find a zero of the operator W . However, our Theorems 3.5 and 3.10 overcome this limitation by leveraging the monotonicity of the pair (C, W) , where the operator $C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 + x_2 \\ x_1 + 13x_2 \end{bmatrix}.$$

Consider the system $x'(t) + W(x(t)) + \frac{\epsilon_0}{t^m} x(t) = 0$, $m \in (0, 1]$, $\epsilon_0 > 0$. Use Theorem 3.10 to obtain the following convergence rate

$$\|W(x(t))\| = O\left(\frac{1}{t^{\frac{m+1}{2}}}\right) \quad \text{as } t \rightarrow \infty.$$

3.5. Selection of parameters

Theorem 3.10 offers the basis for our examples, aimed at accelerating the convergence of the proposed method. Although the convergence rate (28) is influenced by three parameters $\varepsilon(t)$, $\beta(t)$, and γ , the interdependence of $\varepsilon(t)$ and $\beta(t)$, imposed by the conditions of Assumption 3.9, limits the ability to effectively accelerate the convergence of the residual $B(x(t))$ through their individual adjustment. The inequality (28) reveals that the convergence speed of $B(x(t))$ towards zero is directly proportional to γ . This connection, in conjunction with γ being independent of both $\varepsilon(t)$ and $\beta(t)$, suggests that maximizing γ increases the convergence rate. Compared to the identity operator $F = I$, which corresponds to $\gamma = \lambda$, we can obtain an improved convergence by selecting the new operator F with $\gamma > \lambda$, as below.

Corollary 3.12. Consider the system $x'(t) = -\beta(t)[B(x(t)) + \varepsilon(t)pA^m x(t)]$, where the operators A , B and the functions $\beta(t)$, $\varepsilon(t)$ satisfy all the conditions in Theorem 3.10, m is a positive integer, p is a positive real number. Then it holds that

$$\|B(x(t))\| \leq M_1 \max \left\{ \sqrt{\frac{-\varepsilon'(t)}{\beta(t)}}, \exp \left(-\int_{t_0}^t \frac{p\lambda^m}{L} \beta(s)\varepsilon(s) ds \right) \right\} + M_2 \varepsilon(t) \quad \forall t \geq t_1, \quad (33)$$

where M_1, M_2 are some positive constants.

Proof. This follows from Theorem 3.10, where $F = pA^m$ and $\gamma = p\lambda^m$. □

Remark 3.13. Setting $p = 1$ transforms (33) into the following

$$\|B(x(t))\| \leq M_1 \max \left\{ \sqrt{\frac{-\varepsilon'(t)}{\beta(t)}}, \exp \left(-\int_{t_0}^t \frac{\lambda^m}{L} \beta(s)\varepsilon(s) ds \right) \right\} + M_2 \varepsilon(t) \quad \forall t \geq t_1, \quad (34)$$

while setting $m = 1$ results in the inequality below

$$\|B(x(t))\| \leq M_1 \max \left\{ \sqrt{\frac{-\varepsilon'(t)}{\beta(t)}}, \exp \left(-\int_{t_0}^t \frac{p\lambda}{L} \beta(s)\varepsilon(s) ds \right) \right\} + M_2 \varepsilon(t) \quad \forall t \geq t_1. \quad (35)$$

Observation of the rate (34) suggests that setting $F = A^m$ offers an advantage in the convergence rate over setting $F = I$. The rate (34) outperforms (35) in convergence when $\lambda^{m-1} > p$, while (35) converges better than (34) when $\lambda^{m-1} < p$.

4. Regularized dynamical systems from the discrete perspective

4.1. The discretized system

Using the forward difference $y^\Delta(n) = y(n+1) - y(n)$, the dynamical system (6) is discretized to form the following difference equation

$$y^\Delta(n) = -s(n)[B(y(n)) + \epsilon(n)F(y(n))]. \quad (36)$$

Assumption 4.1. The regularization coefficient $\epsilon : \mathbb{N} \rightarrow (0, \infty)$ and the step size $s : \mathbb{N} \rightarrow (0, \infty)$ satisfy conditions (37)–(39)

$$s(n) \leq \frac{\gamma\lambda}{2L^2} \cdot \min \left\{ \frac{\epsilon(n)}{(L_b + \epsilon(n)L_f)^2}, \frac{\epsilon(n+1)}{(L_b + \epsilon(n+1)L_f)^2} \right\}, \quad (37)$$

$$\lim_{n \rightarrow \infty} \epsilon(n) = 0, \quad (38)$$

$$\sum_{n=0}^{\infty} s(n)\epsilon(n) = \infty. \quad (39)$$

Remark 4.2. Our proofs utilize the inequalities presented below, which are consequences of (37)

$$s(n)\epsilon(n) \leq \frac{\gamma\lambda}{2L^2} \cdot \frac{\epsilon(n)^2}{(0 + \epsilon(n)L_f)^2} = \frac{\gamma\lambda}{2L_f^2 L^2}, \quad (40)$$

$$s(n)\epsilon(n+1) \leq \frac{\gamma\lambda}{2L^2} \cdot \frac{\epsilon(n+1)^2}{(0 + \epsilon(n+1)L_f)^2} = \frac{\gamma\lambda}{2L^2 L_f^2}. \quad (41)$$

Like as the continuous counterpart, our analysis is conducted with the help of the following sequences

$$z(n) = \langle A(y(n) - z), y(n) - z \rangle, \quad (42)$$

$$x(n) = \langle A(B(y(n)) + \epsilon(n)F(y(n))), B(y(n)) + \epsilon(n)F(y(n)) \rangle, \quad (43)$$

where $z \in Z(B)$.

Lemma 4.3. Along with Assumptions 2.1 and 2.2, assume that $Z(B) \neq \emptyset$. Then conclusions (i)–(iii) are valid.

(i) For every $p_1 > 0$, it holds that

$$\begin{aligned} & z^\Delta(n) + \frac{2\gamma}{L} s(n)\epsilon(n)z(n) \\ & \leq -2s(n)\epsilon(n) \left\langle \sqrt{A}F(z), \sqrt{A}y(n) - \sqrt{A}z \right\rangle \end{aligned} \quad (44)$$

$$\begin{aligned} & + \frac{2L}{\lambda} s(n)^2 (L_b + \epsilon(n)L_f)^2 z(n) + 2Ls(n)^2 \epsilon(n)^2 \|F(z)\|^2 \\ & \leq \frac{p_1\gamma}{L} s(n)\epsilon(n)z(n) + \frac{2L}{\lambda} s(n)^2 (L_b + \epsilon(n)L_f)^2 z(n) \\ & + \frac{L}{p_1\gamma} s(n)\epsilon(n) \|\sqrt{A}F(z)\|^2 + 2Ls(n)^2 \epsilon(n)^2 \|F(z)\|^2. \end{aligned} \quad (45)$$

(ii) $y(n)$ is bounded.

(iii) For every $p_3 > 0$, it holds that

$$\begin{aligned} & x^\Delta(n) + \frac{p_3\gamma}{L} s(n)\epsilon(n+1)x(n) \\ & \leq -\frac{2\epsilon^\Delta(n)}{s(n)} \langle F(y(n)), Ay^\Delta(n) \rangle + 2L|\epsilon^\Delta(n)|^2 \|F(y(n))\|^2 \\ & + \frac{2L}{\lambda} \left(L_b + \epsilon(n+1)L_f \right)^2 s(n)^2 x(n) - \frac{(2-p_3)\gamma}{L} s(n)\epsilon(n+1)x(n). \end{aligned} \quad (46)$$

Proof. (i) We have

$$\begin{aligned} z^\Delta(n) &= 2 \langle y^\Delta(n), Ay(n) - Az \rangle + \langle y^\Delta(n), Ay^\Delta(n) \rangle \\ &= -2s(n) \langle B(y(n)) + \epsilon(n)F(y(n)), Ay(n) - Az \rangle \\ &+ s(n)^2 \langle B(y(n)) + \epsilon(n)F(y(n)), AB(y(n)) + \epsilon(n)AF(y(n)) \rangle. \end{aligned} \quad (47)$$

For the evaluation of $z^\Delta(n)$, we require the estimation of the two following inner products. For the first product, we observe

$$\begin{aligned} & \langle B(y(n)) + \epsilon(n)F(y(n)), Ay(n) - Az \rangle \\ &= \langle B(y(n)) - B(z), Ay(n) - Az \rangle \\ & \quad + \epsilon(n) \langle F(y(n)) - F(z), Ay(n) - Az \rangle + \epsilon(n) \langle F(z), Ay(n) - Az \rangle \\ & \geq 0 + \epsilon(n)\gamma \|y(n) - z\|^2 - \epsilon(n) \left\langle -\sqrt{A}F(z), \sqrt{A}y(n) - \sqrt{A}z \right\rangle \\ & \geq \frac{\gamma}{L} \epsilon(n)z(n) - \epsilon(n) \left\langle -\sqrt{A}F(z), \sqrt{A}y(n) - \sqrt{A}z \right\rangle \end{aligned} \quad (48)$$

$$\geq \frac{\gamma}{L} \epsilon(n)z(n) - \epsilon(n) \left(\frac{L}{p_1\gamma} \|\sqrt{A}F(z)\|^2 + \frac{p_1\gamma}{L} \|\sqrt{A}y(n) - \sqrt{A}z\|^2 \right). \quad (49)$$

Next,

$$\begin{aligned} & \langle B(y(n)) + \epsilon(n)F(y(n)), AB(y(n)) + \epsilon(n)AF(y(n)) \rangle \\ & \leq L \|B(y(n)) + \epsilon(n)F(y(n))\|^2 \\ & \leq 2L(L_b + \epsilon(n)L_f)^2 \|y(n) - z\|^2 + 2L\epsilon(n)^2 \|F(z)\|^2 \\ & \leq \frac{2L}{\lambda} (L_b + \epsilon(n)L_f)^2 z(n) + 2L\epsilon(n)^2 \|F(z)\|^2. \end{aligned} \quad (50)$$

Combining (47) with (50)–(48) leads to (44), and with (50)–(49) leads to (45).

(ii) Let p_1 such that $\max \{0, 1 - \frac{L}{\gamma}\} < p_1 < 1$. Using (45) and (37) yields

$$\begin{aligned} & z^\Delta(n) + \frac{2\gamma}{L} s(n)\epsilon(n)z(n) \\ & \leq \frac{(p_1+1)\gamma}{L} s(n)\epsilon(n)z(n) + \frac{L}{p_1\gamma} s(n)\epsilon(n) \|\sqrt{A}F(z)\|^2 + 2Ls(n)^2\epsilon(n)^2 \|F(z)\|^2, \end{aligned}$$

which leads to the following

$$\begin{aligned} & z^\Delta(n) + \frac{(1-p_1)\gamma}{L} s(n)\epsilon(n)z(n) \\ & \leq \frac{L}{p_1\gamma} s(n)\epsilon(n) \|\sqrt{A}F(z)\|^2 + 2Ls(n)^2\epsilon(n)^2 \|F(z)\|^2 \\ & \leq \left(\frac{L}{p_1\gamma} + \frac{\gamma\lambda}{L_f^2 L} \right) s(n)\epsilon(n) \left(\|\sqrt{A}F(z)\|^2 + \|F(z)\|^2 \right), \end{aligned}$$

where (40) is taken into account. Denote

$$C = \left(\frac{L}{p_1\gamma} + \frac{\gamma\lambda}{2L_f^2 L} \right) \left(\|\sqrt{A}F(z)\|^2 + \|F(z)\|^2 \right) \frac{L}{(1-p_1)\gamma}$$

and use the fact that $z^\Delta(n) = z(n+1) - z(n)$ to get

$$z(n+1) - \left(1 - \frac{(1-p_1)\gamma}{L} s(n)\epsilon(n) \right) z(n) \leq C \frac{(1-p_1)\gamma}{L} s(n)\epsilon(n). \quad (51)$$

Let us define $g : \mathbb{N} \rightarrow \mathbb{R}$ by setting $g(n) = \prod_{j=1}^{n-1} \left(1 - \frac{(1-p_1)\gamma}{L} s(j)\epsilon(j) \right)$.

By (37)–(38), we can assume without loss of generality that $g(n) > 0$. Upon dividing both sides of (51) by the expression $g(n+1)$, we arrive at

$$\frac{z(n+1)}{g(n+1)} - \frac{z(n)}{g(n)} \leq C \left(\frac{1}{g(n+1)} - \frac{1}{g(n)} \right).$$

Adding up the inequalities above from $n = 2$ to m gives

$$\begin{aligned} \frac{z(m+1)}{g(m+1)} - \frac{z(2)}{g(2)} &= \sum_{n=2}^m \left(\frac{z(n+1)}{g(n+1)} - \frac{z(n)}{g(n)} \right) \leq C \sum_{n=2}^m \left(\frac{1}{g(n+1)} - \frac{1}{g(n)} \right) \\ &= C \left(\frac{1}{g(m+1)} - \frac{1}{g(2)} \right), \end{aligned}$$

which implies, after multiplying both sides by $g(m+1)$, that

$$z(m+1) \leq \frac{z(2)}{g(2)}g(m+1) + C \left(1 - \frac{g(m+1)}{g(2)} \right) \leq \frac{z(2)}{g(2)} + C.$$

(iii) We have

$$\begin{aligned} x^\Delta(n) &= 2 \langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, AB(y(n)) + \epsilon(n)AF(y(n)) \rangle \\ &\quad + \langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, A[B(y(n))]^\Delta + A[\epsilon(n)F(y(n))]^\Delta \rangle. \end{aligned}$$

The first inner product on the right side is estimated, using (36), in the following manner

$$\begin{aligned} &\langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, AB(y(n)) + \epsilon(n)AF(y(n)) \rangle \\ &= -\frac{1}{s(n)} \langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, Ay^\Delta(n) \rangle \\ &\leq 0 - \frac{1}{s(n)} \langle [\epsilon(n)F(y(n))]^\Delta, Ay^\Delta(n) \rangle \\ &= -\frac{1}{s(n)} \langle \epsilon^\Delta(n)F(y(n)) + \epsilon(n+1)[F(y(n))]^\Delta, Ay^\Delta(n) \rangle, \end{aligned}$$

where Proposition 2.10 is employed. Using the strong monotonicity of the pair (A, F) , we get

$$\begin{aligned} &\langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, AB(y(n)) + \epsilon(n)AF(y(n)) \rangle \\ &\leq -\frac{1}{s(n)} \langle \epsilon^\Delta(n)F(y(n)), Ay^\Delta(n) \rangle - \frac{\gamma}{s(n)} \epsilon(n+1) \|y^\Delta(n)\|^2 \\ &= -\frac{1}{s(n)} \langle \epsilon^\Delta(n)F(y(n)), Ay^\Delta(n) \rangle - \gamma s(n) \epsilon(n+1) \|B(y(n)) + \epsilon(n)F(y(n))\|^2 \\ &\leq -\frac{1}{s(n)} \langle \epsilon^\Delta(n)F(y(n)), Ay^\Delta(n) \rangle - \frac{\gamma}{L} s(n) \epsilon(n+1) x(n). \end{aligned}$$

The estimation of the second product is likewise achieved through the application of (36) and Proposition 2.10, as follows

$$\begin{aligned} &\langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, A[B(y(n))]^\Delta + A[\epsilon(n)F(y(n))]^\Delta \rangle \\ &\leq L \| [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta \|^2 \\ &= L \| [B(y(n))]^\Delta + \epsilon(n+1)[F(y(n))]^\Delta + \epsilon^\Delta(n)F(y(n)) \|^2 \\ &\leq 2L \left((L_b + \epsilon(n+1)L_f)^2 \|y^\Delta(n)\|^2 + |\epsilon^\Delta(n)|^2 \|F(y(n))\|^2 \right). \end{aligned}$$

In the above inequality, we use (36) on the right side where the difference $y^\Delta(n)$ appears

$$\begin{aligned} &\langle [B(y(n))]^\Delta + [\epsilon(n)F(y(n))]^\Delta, A[B(y(n))]^\Delta + A[\epsilon(n)F(y(n))]^\Delta \rangle \\ &\leq 2Ls(n)^2 (L_b + \epsilon(n+1)L_f)^2 \|B(y(n)) + \epsilon(n)F(y(n))\|^2 + 2L|\epsilon^\Delta(n)|^2 \|F(y(n))\|^2 \\ &\leq \frac{2L}{\lambda} s(n)^2 (L_b + \epsilon(n+1)L_f)^2 x(n) + 2L|\epsilon^\Delta(n)|^2 \|F(y(n))\|^2. \end{aligned}$$

□

4.2. Strong convergence

Theorem 4.4. Consider the dynamical system (36), under the Assumptions 2.1 and 2.2. Assume that the regularization coefficient $\epsilon : \mathbb{N} \rightarrow (0, \infty)$, in addition to Assumption 4.1, satisfies

$$\lim_{n \rightarrow \infty} \frac{\epsilon^\Delta(n)}{s(n)\epsilon(n+1)} = 0. \quad (52)$$

Then the trajectory $y(n)$ generated by (36) converges strongly to a zero of the operator B as $n \rightarrow \infty$, provided that $Z(B) \neq \emptyset$.

Proof. Claim:

$$\lim_{n \rightarrow \infty} B(y(n)) = 0, \quad (53)$$

$$\sum_{n=0}^{\infty} s(n)\epsilon(n+1) = \infty. \quad (54)$$

Indeed, by (52), we have

$$\lim_{n \rightarrow \infty} \frac{s(n)\epsilon(n+1)}{s(n)\epsilon(n)} = \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{\epsilon^\Delta(n)}{s(n)\epsilon(n+1)} \cdot s(n)} = 1,$$

and employing the asymptotic comparison test for series gives (54). The estimation of (46), using (37) and $p_3 = 1$, is as follows

$$\begin{aligned} & x^\Delta(n) + \frac{\gamma}{L} s(n)\epsilon(n+1)x(n) - 2L|\epsilon^\Delta(n)|^2 \|F(y(n))\|^2 \\ & \leq \frac{2|\epsilon^\Delta(n)|}{s(n)} \|F(y(n))\| \cdot L \|y^\Delta(n)\| \\ & \leq \frac{2L^3 \|F(y(n))\|^2}{\gamma\lambda} \cdot \frac{|\epsilon^\Delta(n)|^2}{s(n)\epsilon(n+1)} + \frac{\gamma\lambda}{2L} s(n)\epsilon(n+1) \|B(y(n)) + \epsilon(n)F(y(n))\|^2 \\ & \leq \frac{2L^3 \|F(y(n))\|^2}{\gamma\lambda} \cdot \frac{|\epsilon^\Delta(n)|^2}{s(n)\epsilon(n+1)} + \frac{\gamma}{2L} s(n)\epsilon(n+1)x(n). \end{aligned}$$

Through moving the expression $2L|\epsilon^\Delta(n)|^2 \|F(y(n))\|^2$ to the right-hand side and then eliminating the expression $\frac{\gamma}{2L} s(n)\epsilon(n+1)x(n)$, we obtain

$$\begin{aligned} & x^\Delta(n) + \frac{\gamma}{2L} s(n)\epsilon(n+1)x(n) \\ & \leq 2L|\epsilon^\Delta(n)|^2 \|F(y(n))\|^2 + \frac{2L^3 \|F(y(n))\|^2}{\gamma\lambda} \cdot \frac{|\epsilon^\Delta(n)|^2}{s(n)\epsilon(n+1)} \\ & \leq \left(2L \cdot \frac{\gamma\lambda}{2L^2 L_f^2} \|F(y(n))\|^2 + \frac{2L^3 \|F(y(n))\|^2}{\gamma\lambda} \right) \frac{|\epsilon^\Delta(n)|^2}{s(n)\epsilon(n+1)} \\ & \leq \left(2L \cdot \frac{\gamma\lambda}{2L^2 L_f^2} \|F(y(n))\|^2 + \frac{2L^3 \|F(y(n))\|^2}{\gamma\lambda} \right) \frac{\gamma\lambda}{2L^2 L_f^2} \left(\frac{|\epsilon^\Delta(n)|}{s(n)\epsilon(n+1)} \right)^2, \end{aligned}$$

where usage of (41) is made. By [7, Lemma B.2]

$$\limsup_{n \rightarrow \infty} x(n) \leq \limsup_{n \rightarrow \infty} \left(\frac{|\epsilon^\Delta(n)|}{s(n)\epsilon(n+1)} \right)^2 = 0.$$

Thus, by Assumption 2.2(ii), the limit of $B(y(n)) + \epsilon(n)F(y(n))$ as $n \rightarrow \infty$ is zero. Consequently, applying Lemma 4.3(ii) along with (38), it follows that the limit of $B(y(n))$ is also zero. Applying (44) with the function $\mu(n)$ defined as $\frac{1}{2} \langle A(y(n) - \phi), y(n) - \phi \rangle$, where ϕ is an element of $Z(B)$ as specified in Remark 3.3, and then using (37),

$$\begin{aligned} & \mu^\Delta(n) + \frac{2\gamma}{L} s(n)\epsilon(n)\mu(n) \\ & \leq -2s(n)\epsilon(n) \left\langle \sqrt{A}F(\phi), \sqrt{A}y(n) - \sqrt{A}\phi \right\rangle \\ & \quad + 2Ls(n)^2 \epsilon(n)^2 \|F(\phi)\|^2 + \frac{\gamma}{L} s(n)\epsilon(n)\mu(n). \end{aligned}$$

After eliminating $\frac{\gamma}{L}s(n)\epsilon(n)\mu(n)$ from both sides of the inequality, we arrive at the following

$$\begin{aligned} & \mu^\Delta(n) + \frac{\gamma}{L}s(n)\epsilon(n)\mu(n) \\ & \leq s(n)\epsilon(n) \left[-2 \left\langle \sqrt{A}F(\phi), \sqrt{A}y(n) - \sqrt{A}\phi \right\rangle + 2Ls(n)\epsilon(n)\|F(\phi)\|^2 \right], \end{aligned}$$

which implies, by (38) and [7, Lemma B.2], that

$$\limsup_{n \rightarrow \infty} \mu(n) \leq \limsup_{n \rightarrow \infty} \left[2 \left\langle -AF(\phi), y(n) - \phi \right\rangle + 2Ls(n)\epsilon(n)\|F(\phi)\|^2 \right]. \quad (55)$$

We will now demonstrate that, under the condition that $B(y(n)) \rightarrow 0$ as $n \rightarrow \infty$, every weak cluster point of $y(\cdot)$ belongs to $Z(B)$. The existence of such cluster points is guaranteed by the boundedness of the trajectory. Let $x_\odot$ be an arbitrary weak cluster point of $y(n)$. Consequently, there exists a sequence $\{y(s_m)\}_{m \geq 0}$ such that $y(s_m)$ converges weakly to $x_\odot$ as $m \rightarrow \infty$. Moreover, $B(y(s_m)) \rightarrow 0$ as $m \rightarrow \infty$.

Given that AB is maximally monotone, its graph is sequentially closed in the weak $\times$ strong topology of $\mathcal{H} \times \mathcal{H}$. This implies $B(x_\odot) = 0$. This result, in conjunction with (55) and (38), allows us to conclude that

$$\limsup_{n \rightarrow \infty} \mu(n) \leq \langle x_\odot - \phi, -AF(\phi) \rangle = -\langle x_\odot - \phi, AF(\phi) \rangle \leq 0,$$

which is equivalent to the fact that $\lim_{n \rightarrow \infty} y(n) = \phi$. □

Consider the following example, which illustrates the conditions of Theorem 4.4.

Example 4.5. An example of $\epsilon(n)$ and $s(n)$ satisfying the conditions in Theorem 4.4 is $s(n) = \frac{1}{(n+1)^q}$, $\epsilon(n) = \frac{1}{(n+1)^p}$, where $0 < p < q < 1$ and $p+q \leq 1$. Compared to the paper [12], where $s(n) = \frac{2(\gamma-\ell)}{(n+1)^q(L_b+L_f)^2}$ and $\epsilon(n) = \frac{1}{(n+1)^p}$, $0 < \ell < \min\{1; \gamma\}$, $0 < p < q < 1$ and $2p+q < 1$, our conditions are more relaxed.

4.3. Non-emptiness of the zero set

Theorem 4.6. Consider the dynamical system (36), under Assumptions 2.1 and 2.2. Assume that the regularization coefficient $\epsilon : \mathbb{N} \rightarrow (0, \infty)$, in addition to Assumption 4.1, satisfies

$$\text{either} \quad \lim_{n \rightarrow \infty} \frac{|(s\epsilon)^\Delta(n)| + |s^\Delta(n)|}{s(n)\epsilon(n)} = 0, \quad (56)$$

$$\text{or} \quad \sum_{n=0}^{\infty} \left(|(s\epsilon)^\Delta(n)| + |s^\Delta(n)| \right) < \infty. \quad (57)$$

Then the zero set $Z(B) \neq \emptyset$, provided that the trajectory $y(n)$ generated by (36) is bounded.

Proof. Let us define $h(n) = \langle A(y(n+1) - y(n)), y(n+1) - y(n) \rangle$. By Proposition 2.10, we have

$$\begin{aligned} h^\Delta(n) &= 2 \langle y^\Delta(n+1) - y^\Delta(n), Ay(n+1) - Ay(n) \rangle \\ &\quad + \langle y^\Delta(n+1) - y^\Delta(n), Ay^\Delta(n+1) - Ay^\Delta(n) \rangle. \end{aligned} \quad (58)$$

For the evaluation of $h^\Delta(n)$, we require the estimation of the following two inner products. For the first product, we observe

$$\begin{aligned} & \langle y^\Delta(n+1) - y^\Delta(n), Ay(n+1) - Ay(n) \rangle \\ &= -s(n) \langle B(y(n+1)) - B(y(n)), Ay(n+1) - Ay(n) \rangle \\ &\quad - s^\Delta(n) \langle \sqrt{A}B(y(n+1)), \sqrt{A}y(n+1) - \sqrt{A}y(n) \rangle \\ &\quad - s(n)\epsilon(n) \langle F(y(n+1)) - F(y(n)), Ay(n+1) - Ay(n) \rangle \\ &\quad - (s\epsilon)^\Delta(n) \langle \sqrt{A}F(y(n+1)), \sqrt{A}y(n+1) - \sqrt{A}y(n) \rangle, \end{aligned}$$

which implies, by Assumption 2.1, that

$$\begin{aligned} & \langle y^\Delta(n+1) - y^\Delta(n), Ay(n+1) - Ay(n) \rangle \\ & \leq 0 + |s^\Delta(n)| \cdot \|\sqrt{A}B(y(n+1))\| \cdot \|\sqrt{A}y(n+1) - \sqrt{A}y(n)\| \\ & \quad - \frac{\gamma}{L} s(n)\epsilon(n)h(n) + |(s\epsilon)^\Delta(n)| \cdot \|\sqrt{A}F(y(n+1))\| \cdot \|\sqrt{A}y(n+1) - \sqrt{A}y(n)\|. \end{aligned}$$

Hence,

$$\begin{aligned} & \langle y^\Delta(n+1) - y^\Delta(n), Ay(n+1) - Ay(n) \rangle \\ & \leq -\frac{\gamma}{L} s(n)\epsilon(n)h(n) + (|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)Q \|\sqrt{A}y(n+1) - \sqrt{A}y(n)\|, \end{aligned} \quad (59)$$

where $Q = \sup_{n \geq 0} \{\|\sqrt{A}B(y(n+1))\| + \|\sqrt{A}F(y(n+1))\|\}$, which is finite. The second product is estimated as

$$\begin{aligned} & \langle y^\Delta(n+1) - y^\Delta(n), Ay^\Delta(n+1) - Ay^\Delta(n) \rangle \\ & \leq L \|y^\Delta(n+1) - y^\Delta(n)\|^2 \\ & = L \|s^\Delta(n)B(y(n+1)) + (s\epsilon)^\Delta(n)F(y(n+1)) \\ & \quad + s(n)[B(y(n+1)) - B(y(n))] - s(n)\epsilon(n)[F(y(n+1)) - F(y(n))]\|^2, \end{aligned}$$

which implies, by the Lipschitz property of F and B , that

$$\begin{aligned} & \langle y^\Delta(n+1) - y^\Delta(n), Ay^\Delta(n+1) - Ay^\Delta(n) \rangle \\ & \leq L \left[(|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)(\|B(y(n+1))\| + \|F(y(n+1))\|) \right. \\ & \quad \left. + s(n)(L_b + \epsilon(n)L_f)\|y(n+1) - y(n)\| \right]^2 \\ & \leq 2L(|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)^2(\|B(y(n+1))\| + \|F(y(n+1))\|)^2 \\ & \quad + \frac{2L}{\lambda} s(n)^2(L_b + \epsilon(n)L_f)^2 h(n), \end{aligned}$$

where the Cauchy-Schwarz inequality and Assumption 2.2(ii) are taken into account. Consequently, by (37), we get

$$\begin{aligned} & \langle y^\Delta(n+1) - y^\Delta(n), Ay^\Delta(n+1) - Ay^\Delta(n) \rangle \\ & \leq 2L(|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)^2(\|B(y(n+1))\| + \|F(y(n+1))\|)^2 + \frac{\gamma}{L} s(n)\epsilon(n)h(n). \end{aligned} \quad (60)$$

Using (59) and (60), we can estimate (58) as follows

$$\begin{aligned} & h^\Delta(n) + \frac{\gamma}{L} s(n)\epsilon(n)h(n) \\ & \leq 2(|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)Q \|\sqrt{A}y(n+1) - \sqrt{A}y(n)\| \\ & \quad + 2L(|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)^2(\|B(y(n+1))\| + \|F(y(n+1))\|)^2. \end{aligned} \quad (61)$$

Let us define $g(n)$ by setting $g(n) = \prod_{j=1}^{n-1} (1 - \frac{\gamma}{L} s(j)\epsilon(j))$ and by (39) the limit below holds

$$\lim_{n \rightarrow \infty} g(n) = 0. \quad (62)$$

Dividing both sides of (61) by $g(n+1)$ gives

$$\begin{aligned} \frac{h(n+1)}{g(n+1)} - \frac{h(n)}{g(n)} & \leq \frac{2(|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)Q}{(1 - \frac{\gamma}{L} s(n)\epsilon(n))\sqrt{g(n)}} \cdot \frac{\|\sqrt{A}y(n+1) - \sqrt{A}y(n)\|}{\sqrt{g(n)}} \\ & \quad + \frac{2L}{g(n+1)} (|s^\Delta(n)| + |(s\epsilon)^\Delta(n)|)^2 Q^2. \end{aligned}$$

Note that by conditions (37) and (38) there exists a constant $M > 0$ subject to

$$\frac{1}{1 - \frac{\gamma}{L} s(n)\epsilon(n)} \leq M \quad \forall n.$$

We use Lemma 2.11 to get

$$\begin{aligned}
& \sqrt{\frac{h(m+\kappa)}{g(m+\kappa)}} - \sqrt{\frac{h(\kappa)}{g(\kappa)}} \\
& \leq \sum_{j=\kappa}^{m+\kappa-1} \left(\frac{2}{(1 - \frac{\gamma}{L}s(j)\epsilon(j))\sqrt{g(j)}} + \sqrt{\frac{2L}{g(j+1)}} \right) (|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|)Q \\
& \leq \sum_{j=\kappa}^{m+\kappa-1} \left(\frac{2M + \sqrt{2L}}{\sqrt{g(j+1)}} \right) (|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|)Q.
\end{aligned}$$

Upon multiplying both sides by $\sqrt{g(m+\kappa)}$, we obtain

$$\begin{aligned}
\sqrt{h(m+\kappa)} & \leq \sqrt{\frac{h(\kappa)}{g(\kappa)}} \cdot \sqrt{g(m+\kappa)} + \\
& + \frac{P\gamma}{L} \sum_{j=\kappa}^{m+\kappa-1} \frac{\sqrt{g(m+\kappa)}}{\sqrt{g(j+1)}} (|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|),
\end{aligned} \tag{63}$$

where $P = \frac{L}{\gamma}(2M + \sqrt{2L})Q$.

Claim:

$$\lim_{n \rightarrow \infty} y^\Delta(n) = 0. \tag{64}$$

Case 1. Condition (56) is met. By (63)

$$\begin{aligned}
& \sqrt{h(m+\kappa)} - \sqrt{\frac{h(\kappa)}{g(\kappa)}} \cdot \sqrt{g(m+\kappa)} \leq \\
& \leq P\sqrt{g(m+\kappa)} \sup_{\kappa \leq j \leq m+\kappa-1} \frac{|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|}{s(j)\epsilon(j)} \sum_{j=\kappa}^{m+\kappa-1} \frac{\frac{\gamma}{L}s(j)\epsilon(j)}{\sqrt{g(j+1)}} = \\
& = P\sqrt{g(m+\kappa)} \sup_{\kappa \leq j \leq m+\kappa-1} \frac{|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|}{s(j)\epsilon(j)} \times \\
& \quad \times \sum_{j=\kappa}^{m+\kappa-1} \frac{\sqrt{g(j)} - \sqrt{g(j+1)}}{\sqrt{g(j)}\sqrt{g(j+1)}} \cdot \frac{\sqrt{g(j)} + \sqrt{g(j+1)}}{\sqrt{g(j)}},
\end{aligned}$$

which implies, after using $\frac{\sqrt{g(j)} + \sqrt{g(j+1)}}{\sqrt{g(j)}} \leq 2$, that

$$\begin{aligned}
& \sqrt{h(m+\kappa)} - \sqrt{\frac{h(\kappa)}{g(\kappa)}} \cdot \sqrt{g(m+\kappa)} \\
& \leq 2P\sqrt{g(m+\kappa)} \sup_{\kappa \leq j \leq m+\kappa-1} \frac{|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|}{s(j)\epsilon(j)} \sum_{j=\kappa}^{m+\kappa-1} \left(\frac{1}{\sqrt{g(j+1)}} - \frac{1}{\sqrt{g(j)}} \right) \\
& = 2P \sup_{\kappa \leq j \leq m+\kappa-1} \frac{|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|}{s(j)\epsilon(j)} \left(1 - \frac{\sqrt{g(m+\kappa)}}{\sqrt{g(\kappa)}} \right).
\end{aligned}$$

The limit as $m \rightarrow \infty$, taken in conjunction with (56) and (62), gives (64).

Case 2. Condition (57) is met. By (63)

$$\sqrt{h(m+\kappa)} \leq \sqrt{\frac{h(\kappa)}{g(\kappa)}} \cdot \sqrt{g(m+\kappa)} + \frac{P\gamma}{L} \sum_{j=\kappa}^{m+\kappa-1} (|s^\Delta(j)| + |(s\epsilon)^\Delta(j)|).$$

The inequality above, when taking the limit as $m \rightarrow \infty$, becomes

$$\limsup_{i \rightarrow \infty} h(i) \leq \frac{P\gamma}{L} \sum_{j=\kappa}^{\infty} (|s^{\Delta}(j)| + |(s\epsilon)^{\Delta}(j)|).$$

Consequently, (64) follows by taking the limit as $\kappa \rightarrow \infty$ and using (57).

Using arguments similar to Theorem 3.7, we get the desired conclusion. $\square$

Examples 4.7 and 4.8 demonstrate parameters that satisfy the assumptions of Theorem 4.6. These examples provide a clear demonstration of two distinct scenarios: Example 4.7 details a case where the parameters meet condition (56) but fail to meet condition (57), and Example 4.8 describes a complementary situation in which condition (57) is met while condition (56) is not.

Example 4.7. (Meets (56) but not (57).) Let

$$\begin{cases} h_n = \frac{1}{n} \left(1 + \frac{(-1)^n}{\ln n} \right), \\ s(n) = h_n \sqrt{\ln n}, \quad \epsilon(n) = \frac{1}{\sqrt{\ln n}}. \end{cases}$$

Note that condition (37) holds for sufficiently large n and (38) is clear. Condition (39) is a direct consequence of the equality

$$s(n)\epsilon(n) = h_n = \frac{1}{n} + \frac{(-1)^n}{n \ln n}.$$

To show that (57) fails, we consider

$$\begin{aligned} |(s\epsilon)^{\Delta}(n)| &\geq \left| (-1)^{n+1} \left(\frac{1}{n \ln n} + \frac{1}{(n+1) \ln(n+1)} \right) \right| - \left| \frac{1}{n+1} - \frac{1}{n} \right| \\ &\geq \frac{2}{(n+1) \ln(n+1)} - \frac{1}{n(n+1)}, \end{aligned}$$

which gives $\sum_{n=0}^{\infty} (|(s\epsilon)^{\Delta}(n)| + |s^{\Delta}(n)|) = \infty$.

To demonstrate the validity of (56), we analyze two limits. The first one considers the following

$$\frac{(s\epsilon)^{\Delta}(n)}{s(n)\epsilon(n)} = \frac{\frac{1}{n+1} \left(1 + \frac{(-1)^{n+1}}{\ln(n+1)} \right)}{\frac{1}{n} \left(1 + \frac{(-1)^n}{\ln n} \right)} - 1 \rightarrow 0.$$

The second limit is

$$\begin{aligned} |n(s(n+1) - s(n))| &\leq n \left| \frac{h_{n+1} - h_n}{\epsilon(n+1)} \right| + n h_n \left| \frac{1}{\epsilon(n+1)} - \frac{1}{\epsilon(n)} \right| \\ &\leq n \cdot \frac{\frac{3}{n \ln n}}{\epsilon(n+1)} + n h_n \left(\sqrt{\ln(n+1)} - \sqrt{\ln n} \right) \\ &= \frac{3\sqrt{\ln(n+1)}}{\ln n} + \left(1 + \frac{(-1)^n}{\ln n} \right) \cdot \frac{\ln(1 + \frac{1}{n})}{\sqrt{\ln(n+1)} + \sqrt{\ln n}} \\ &\leq \frac{3\sqrt{\ln(n+1)}}{\ln n} + \left(1 + \frac{(-1)^n}{\ln n} \right) \cdot \frac{\frac{1}{n}}{\sqrt{\ln(n+1)} + \sqrt{\ln n}}, \end{aligned}$$

which gives $0 = \lim_{n \rightarrow \infty} n(s(n+1) - s(n)) = \lim_{n \rightarrow \infty} \frac{s^{\Delta}(n)}{s(n)\epsilon(n)}$.

Consequently, the limit (56) of the sum is shown to be the sum of the individual limits and it is equal to 0. $\square$

Example 4.8. (Meets (57) but not (56).) Let

$$n_k = 4^k, \quad s(n) = \begin{cases} \frac{1}{\sqrt[4]{(n+1)^3}} & \text{if } n \notin \{n_k : k \in \mathbb{Z}, k \geq 0\}, \\ \frac{2}{\sqrt[4]{(n+1)^3}} & \text{if } n \in \{n_k : k \in \mathbb{Z}, k \geq 0\}, \end{cases} \quad \epsilon(n) = \frac{1}{(n+1)s(n)}.$$

Note that condition (37) holds for sufficiently large n and (38)–(39) are clear. Condition (56) fails to hold, since

$$\begin{aligned} \frac{|s^\Delta(n_k)|}{s(n_k)\epsilon(n_k)} &= \frac{\left| \frac{1}{\sqrt[4]{(n_k+2)^3}} - \frac{2}{\sqrt[4]{(n_k+1)^3}} \right|}{\frac{1}{n_k+1}} \\ &= \frac{2\sqrt[4]{(n_k+2)^3} - \sqrt[4]{(n_k+1)^3}}{\sqrt[4]{(n_k+2)^3}} \cdot \sqrt[4]{n_k+1} \\ &\geq \frac{\sqrt[4]{(n_k+1)^3}}{\sqrt[4]{(n_k+2)^3}} \cdot \sqrt[4]{n_k+1} \rightarrow \infty. \end{aligned}$$

To establish the validity of (57), we can divide it into two convergent series. The first one considers

$$\sum_{n=0}^{\infty} |(s\epsilon)^\Delta(n)| = \sum_{n=0}^{\infty} \left| \frac{1}{n+2} - \frac{1}{n+1} \right| = 1.$$

To prove the convergence of the second series, we introduce

$$c_n := s(n) - \frac{1}{\sqrt[4]{(n+1)^3}} = \begin{cases} 0 & \text{if } n \notin \{n_k : k \in \mathbb{Z}, k \geq 0\}, \\ \frac{1}{\sqrt[4]{(n+1)^3}} & \text{if } n \in \{n_k : k \in \mathbb{Z}, k \geq 0\}. \end{cases}$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} |s^\Delta(n)| &\leq \sum_{n=0}^{\infty} |c_{n+1} - c_n| + \sum_{n=0}^{\infty} \left| \frac{1}{\sqrt[4]{(n+2)^3}} - \frac{1}{\sqrt[4]{(n+1)^3}} \right| \\ &= \sum_{n=0}^{\infty} |c_{n+1} - c_n| + 1, \end{aligned}$$

where we observe

$$|c_{n+1} - c_n| = \begin{cases} \frac{1}{\sqrt[4]{(n_k+1)^3}} & \text{if } n \in \{n_k : k \in \mathbb{Z}, k \geq 0\}, \\ \frac{1}{\sqrt[4]{(n_k+2)^3}} & \text{if } n \in \{n_k - 1 : k \in \mathbb{Z}, k \geq 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we get

$$\sum_{n=1}^{\infty} |s^\Delta(n)| \leq \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt[4]{n_k^3}} + \frac{1}{\sqrt[4]{n_k^3}} \right) + 1 < \infty.$$

5. Examples

In this section, the theoretical results are empirically validated through illustrative examples, which specifically confirm the observations made in Remark 3.13. These examples indicate that the convergence rate of our method, in comparison to the existing ones, depends on particular problems; it can be either faster or slower.

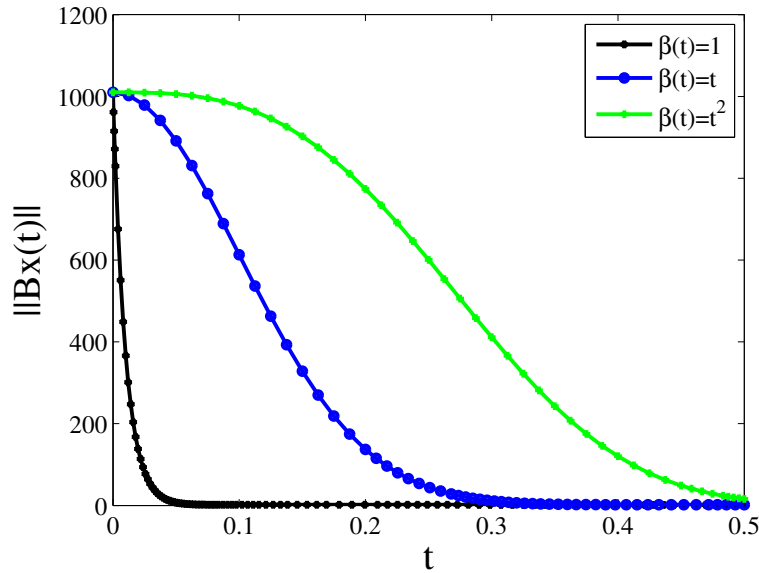


Figure 1. Performance of the dynamical system (6) with different stepsizes $\beta(t)$ in Example 5.1

Example 5.1. (The case when $A \neq I$) Let $\mathcal{H} = \mathbb{R}^n$, $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$A(x) = \begin{bmatrix} 2x_1 + x_2 + x_3 + \dots + x_n \\ x_1 + 2x_2 + x_3 + \dots + x_n \\ x_1 + x_2 + 2x_3 + \dots + x_n \\ \dots \\ x_1 + x_2 + x_3 + \dots + 2x_n \end{bmatrix}, \quad B(x) = \begin{bmatrix} -x_1 - 2x_2 - x_3 - \dots - x_n \\ 2x_1 + x_2 + x_3 + \dots + x_n \\ x_1 + x_2 + 2x_3 + \dots + x_n \\ \dots \\ x_1 + x_2 + x_3 + \dots + 2x_n \end{bmatrix}.$$

As demonstrated in Example 2.4, the operators A, B satisfy all conditions of Theorem 3.5. Let $n = 100$ and $x^0 = [1, 1, \dots, 1]^T$. First, we test the dynamical system (6) with $F = I$, $\varepsilon(t) = \frac{1}{t+1}$, using different selections of $\beta(t)$. The results are presented in Figure 1.

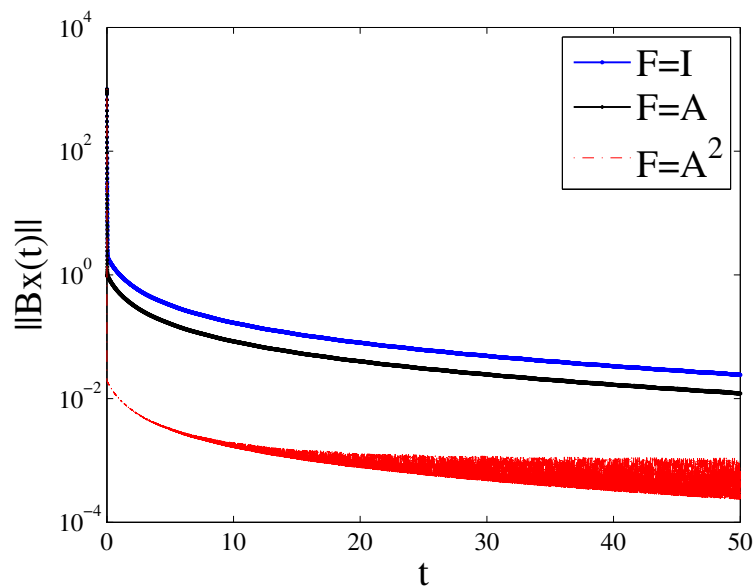


Figure 2. Performance of the dynamical system (6) with different regularization operators F in Example 5.1

To study the effect of the regularization operator F , we test the dynamical system (6) in cases: $F = I$, $F = A$ and $F = A^2$. Observation of Figure 2 suggests that setting $F = A^2$ yields the best accuracy, aligning with the theoretical prediction in Remark 3.13, but causes oscillations. The choice $F = A$ introduces stability at the cost of moderate accuracy.

Example 5.2 (*The case when $A = I$*). We compare the system (6) with [8, the system (2.1)] by Boţ and Nguyen. Let $\mathcal{H} = \mathbb{R}^n$ and $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $B(x) = \mathcal{B}x$, where $\mathcal{B} = MM^T$, M is a randomly generated $n \times n$ matrix. In (6), we choose $\beta(t) = 1$ for all t . Both methods use $\varepsilon(t) = \frac{1}{t+1}$ and $x^0 = [1, 1, \dots, 1]^T$. Unlike the Boţ's method that keeps the regularization operator F as the identity, our method leverages F as a tunable parameter to potentially enhance performance. Let K be an $n \times n$ matrix with all elements equal to 1. We test the methods with $F = K + I$, $F = K + 2I$ and $F = K + 3I$. For $n = 10$, the result is presented in Figure 3.

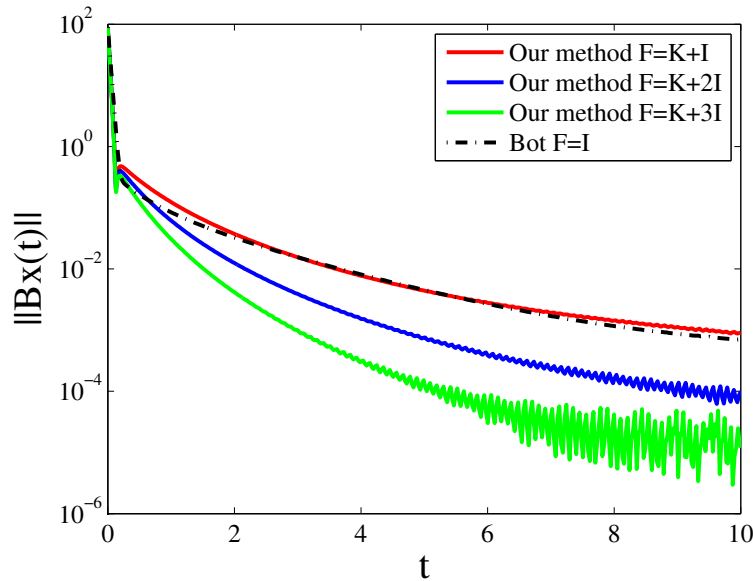


Figure 3. Comparison of our method and that of Boţ in Example 5.2

6. Conclusion

This paper introduces regularized dynamical systems designed to identify the zeros of generalized monotone operators. Our contributions are summarized as follows: First, we consider a broad analytical framework by addressing operators that satisfy generalized monotonicity conditions. This approach expands the scope of existing methods to a wider set of operators. Second, we propose regularized dynamical systems in both continuous- and discrete-time settings, employing a broader regularization framework than those found in the existing literature. Finally, for the continuous-time case, we demonstrate that the convergence rate is highly dependent on the choice of regularization operators; this provides a mechanism for accelerating convergence, which we validate through illustrative examples. The extension of these rate estimates to discrete time, along with the corresponding numerical experiments, is identified as a significant area for our future research.

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