

Duality for Smooth Vector Continuous-Time Programming Problems

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Abstract: This paper considers smooth vector continuous-time programming problems (VCTP) with phase inequality constraints defined almost everywhere on a compact interval, and develops two duality models of Mond–Weir and Wolfe type. Since many earlier results in continuous-time programming relied on an infinite-dimensional version of Gordan’s transposition theorem [20], whose validity has been questioned in [1], we instead work with a new regularity condition, (generalized) concavity assumptions, and a constraint qualification (MFCQ) to obtain weak and strong duality by using alternative tools in infinite-dimensional spaces. For the primal problem, with efficiency understood in the integral sense, we formulate a Mond–Weir continuous-time dual (MWCTD) using a primal-like trajectory, a constant weight vector, and time-dependent nonnegative multipliers. Under concavity/generalized concavity of the phase inequality constraints and pseudoconcavity (or concavity) of a scalarization functional, we derive a weak duality theorem, and under the (MFCQ) together with a suitable regularity condition we obtain a strong duality theorem. An analogous construction yields a Wolfe-type continuous-time dual (WCTD), for which weak duality follows from joint concavity and strong duality from the (MFCQ) or from a continuous-time Slater condition. Several illustrative examples are provided to demonstrate how the theoretical results can be verified in practice.

Keywords: Continuous-time programming, vector problems, duality.

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1. Introduction

The study of optimization problems in continuous time goes back to the early work of Bellman [2] in the 1950s, where time-dependent decision processes were first formulated in an integral framework. Subsequent papers by Tindall [18] and other authors extended these ideas to linear continuous-time programming and established the first duality theorems. This line of research was then developed further [3, 4, 5] in the 1960s and 1970s, when duality theorems and Karush-Kuhn-Tucker (KKT) optimality conditions were proved for nonlinear continuous-time programming problems with time-dependent matrices and Lebesgue-integrable functions, typically under convexity/concavity assumptions and various regularity requirements. However, several of these early results relied on separation theorem in function spaces whose interior is empty, and later works pointed out gaps and inaccuracies in some of those duality proofs. This was one of the reasons why a more systematic functional-analytic approach to continuous-time programming started to appear in the 1980s.

A major step was made by Reiland [13, 14, 15], who considered convex and smooth nonlinear continuous-time problems and derived optimality conditions and duality theorems by using a generalized Farkas lemma for operators with finite-dimensional kernel and closed range. Although mathematically elegant, Reiland’s assumptions were rather restrictive and therefore not easy to verify in concrete optimization models. This motivated a long series of papers [19, 20, 21, 22, 23] by Zalmai in the mid-1980s, where smooth, convex, fractional and even homogeneous continuous-time programming problems were treated in a unified way, and where Fritz-John and KKT type necessary conditions, saddle-point theorems and duality theorems

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were obtained for scalar problems and later for multiobjective continuous-time programming problems. The key technical tool behind almost all of these works was the generalized Gordan transposition theorem [20] in infinite-dimensional spaces, which allowed one to pass from feasibility of a primal convex system to the existence of nontrivial multiplier functions.

Later, vector (multiobjective) continuous-time programming problems were considered in several papers [8, 10, 11, 12, 16] in which efficiency in the Pareto sense was taken as the optimality notion, and duality models of Mond–Weir and Wolfe type were proposed for nonsmooth case, typically under invex, preinvex or generalized invex assumptions. See, for instance, the works where efficient solutions were studied together with saddle-point characterizations and weak/strong duality theorems. In those papers, a slightly restricted definition of efficiency was sometimes adopted in order to exclude anomalous efficient points. Yet, again, one of the fundamental tools for deriving both optimality conditions and duality results was precisely the above-mentioned Gordan’s transposition theorem or some of its direct consequences.

However, the validity of that generalized Gordan transposition theorem has been seriously questioned in more recent years. Arutyunov, Zhukovsky, and Marinković [1] pointed out, by explicit counterexamples, that the aforementioned theorem and its consequences used in much of the continuous-time programming literature are not actually true in the stated form. As a consequence, the correctness of many duality and optimality results that depended on that theorem became doubtful. This critique affects not only the early scalar continuous-time papers, but also several later works on nonsmooth vector continuous-time problems, on preinvex and pseudoinvex models, and on rational continuous-time optimization, because all of them repeated the same device under slightly different assumptions.

In parallel with this development, a newer line of research has emerged, in which duality and KKT-type conditions for continuous-time problems are derived by using the new alternative theorem [1] with the necessary regularity condition and appropriate constraint qualifications. For example, after 2019, in the papers [6, 7], necessary first and zero-order conditions for smooth vector and non-smooth convex vector continuous-time programming problems with phase inequality constraints under appropriate constraint qualifications were obtained. In [17], the authors studied the interval-valued multiobjective continuous-time programming problem. First, both Karush-Kuhn-Tucker necessary and sufficient optimality conditions for some types of efficient solutions were obtained. Then, they proposed the Mond–Weir and Wolfe dual problems and investigated the weak and strong duality relations under convexity hypotheses.

The present paper follows precisely this latter direction. We consider a smooth vector continuous-time programming problem (VCTP) with phase inequality constraints, where all integrals are understood in the Lebesgue sense. Our goal is to develop duality results for this problem under the assumptions of concavity and generalized concavity, but using new results from this field. We work with a new alternative theorem, a regularity condition, and a continuous-time version of the constraint qualification (MFCQ). We also use an auxiliary problem that connects scalar and multiobjective continuous-time programming problems and construct two dual models – the Mond–Weir continuous-time dual (MWCTD) and the Wolff continuous-time dual (WCTD) – and prove weak and strong duality results between the primal problem and this one. In particular, we show that, under the concavity/pseudoconcavity assumptions of the involved functions and under appropriate regularity conditions for the auxiliary problem, every efficient solution of (VCTP) generates a solution of (MWCTD) or (WCTD). Here, to prove the strong duality theorem, we use the necessary optimality conditions [6, 9] obtained by applying the new theorem [1].

The paper is organized in the following way. Some preliminaries about problem are given in Section 2, where some definitions and auxiliary results are stated. In Section 3 and 4, we formulate and consider two duality-type problems for a smooth vector continuous-time problem and prove duality theorems.

2. Definitions and Preliminaries

In this work, we consider the following vector continuous-time problem:

$$\begin{aligned}
 \text{(VCTP)} \quad & \max \int_0^T f(t, x(t)) dt = \left(\int_0^T f_1(t, x(t)) dt, \dots, \int_0^T f_k(t, x(t)) dt \right) \\
 & \text{s.t. } g_i(t, x(t)) \geq 0, \quad i \in I = \{1, \dots, m\} \text{ a.e. in } [0, T], \\
 & \quad x(\cdot) \in L_\infty([0, T]; \mathbb{R}^n),
 \end{aligned}$$

where $f_j : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in J = \{1, \dots, k\}$ and $g_i : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I$, are given functions and $f_j(t, x(t))$ denotes the j -th component of $f(t, x(t)) \in \mathbb{R}^k$. Here, $L_\infty([0, T]; \mathbb{R}^n)$ denotes the Banach space of

all n -dimensional Lebesgue measurable essentially bounded functions defined on the compact interval $[0, T]$ of the real line $\mathbb{R}$ with the corresponding norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \leq i \leq n} \operatorname{ess\,sup}_{t \in [0, T]} |x_i(t)|.$$

For each $t \in [0, T]$, $x_k(t)$ is the k th component of $x(t) \in \mathbb{R}^n$. All integrals are given in the Lebesgue sense. All equalities and inequalities between vector-valued functions are understood componentwise, a.e. on $[0, T]$. By a constant vector function we mean $\lambda(\cdot) \in L_\infty([0, T]; \mathbb{R}^k)$ such that $\lambda(t) = \lambda \in \mathbb{R}^k$ for a.e. t . B denotes the open unit ball with centre at the origin, independently of the space or dimension. Ω denotes the feasible set of (VCTP), and throughout the paper we assume that $\Omega \neq \emptyset$. Let $\varepsilon > 0$ and $\hat{x}(\cdot) \in \Omega$. Suppose the following assumptions are valid:

1. For each $j \in J$, the function $f_j(t, \cdot)$ is continuously differentiable on $\hat{x}(t) + \varepsilon \bar{B}$ a.e. in $[0, T]$. For each $j \in J$, the function $f_j(\cdot, x)$ is Lebesgue measurable for each $x(\cdot)$, $f_j(\cdot, x(\cdot))$ is essentially bounded in $[0, T]$ for all $x(\cdot) \in L_\infty([0, T], \mathbb{R}^n)$ and there exists a number $K_f > 0$ such that

$$\|\nabla f_j(t, \hat{x}(t))\| \leq K_f \text{ a.e. in } [0, T];$$

2. For each $i \in I$, the function $g_i(t, \cdot)$ is continuously differentiable on $\hat{x}(t) + \varepsilon \bar{B}$ a.e. in $[0, T]$. For each $i \in I$, the function $g_i(\cdot, x)$ is Lebesgue measurable for each $x(\cdot)$, $g_i(\cdot, x(\cdot))$ is essentially bounded in $[0, T]$ for all $x(\cdot) \in L_\infty([0, T], \mathbb{R}^n)$ and there exists a number $K_g > 0$ such that

$$\|\nabla g_i(t, \hat{x}(t))\| \leq K_g, \quad i \in I, \text{ a.e. in } [0, T].$$

Definition 2.1. A feasible solution $\hat{x}(\cdot)$ for (VCTP) is said to be an *efficient solution* for (VCTP) if there is no other feasible solution $x(\cdot)$ for (VCTP) such that

$$\int_0^T f_j(t, x(t)) dt \geq \int_0^T f_j(t, \hat{x}(t)) dt, \quad j = 1, \dots, k,$$

with at least one strict inequality.

3. Mond–Weir continuous-time dual (MWCTD)

The Mond–Weir-type dual problem of (VCTP) is formulated as follows (MWCTD):

$$\begin{aligned} \min \quad & \int_0^T f(t, u(t)) dt = \left(\int_0^T f_1(t, u(t)) dt, \dots, \int_0^T f_k(t, u(t)) dt \right) \\ \text{s.t.} \quad & \sum_{j \in J} \lambda_j(t) \nabla f_j(t, u(t)) + \sum_{i \in I} w_i(t) \nabla g_i(t, u(t)) = 0, \quad \text{a.e. in } [0, T], \end{aligned} \quad (1)$$

$$w_i(t) g_i(t, u(t)) \leq 0, \quad w_i(t) \geq 0, \quad i \in I, \text{ a.e. in } [0, T], \quad (2)$$

$$\sum_{j \in J} \lambda_j(t) = 1, \quad \lambda_j(t) \geq 0, \quad \text{a.e. in } [0, T], \quad j \in J, \quad (3)$$

$$\lambda(\cdot) \in L_\infty([0, T]; \mathbb{R}^k), \quad u(\cdot) \in L_\infty([0, T]; \mathbb{R}^n), \quad w(\cdot) \in L_\infty([0, T]; \mathbb{R}^m).$$

Definition 3.1. A feasible solution $(u^0(\cdot), \lambda^0(\cdot), w^0(\cdot))$ for (MWCTD) is said to be an *efficient solution* for (MWCTD) if there is no other feasible solution $(u(\cdot), \lambda(\cdot), w(\cdot))$ for (MWCTD) such that

$$\int_0^T f_j(t, u(t)) dt \leq \int_0^T f_j(t, u^0(t)) dt, \quad j = 1, \dots, k,$$

with at least one strict inequality. □

In this section we present weak and strong duality relations between programs (VCTP) and (MWCTD). The weak duality results are given under (strictly) pseudoconcavity and quasiconcavity assumptions. Define, for each $\lambda \in \mathbb{R}^k$, the functional $\Phi_\lambda : L_\infty([0, T]; \mathbb{R}^n) \rightarrow \mathbb{R}$ by $\Phi_\lambda(u(\cdot)) = \sum_{j \in J} \int_0^T \lambda_j f_j(t, u(t)) dt$.

Theorem 3.2. (Weak Duality) Assume that for all feasible $x(\cdot)$ for (VCTP) and all feasible $(u(\cdot), \lambda(\cdot), w(\cdot))$ for (MWCTD) (where $\lambda(\cdot)$ is a constant vector function), $\sum_{i \in I} w_i(t) g_i(t, \cdot)$ is quasiconcave at $u(\cdot)$ throughout $[0, T]$. If also any of the following holds:

- (i) $\lambda_j > 0$, $\forall j \in J$ and $\Phi_\lambda(u(\cdot))$ is pseudoconcave on the feasible set,
- (ii) $\Phi_\lambda(u(\cdot))$ is strictly pseudoconcave on the feasible set,
- (iii) $f_j(t, \cdot)$, $j \in J$ are concave functions almost everywhere in $[0, T]$,

then the following cannot hold:

$$\int_0^T f_j(t, x(t)) dt \geq \int_0^T f_j(t, u(t)) dt \quad \forall j \in J, \quad (4)$$

and

$$\int_0^T f_k(t, x(t)) dt > \int_0^T f_k(t, u(t)) dt \quad \text{for some } k \in J. \quad (5)$$

Proof. For each $x(\cdot) \in \Omega$ for (VCTP) and each feasible $(u(\cdot), \lambda, w(\cdot))$ for (MWCTD) we have

$$w_i(t) g_i(t, x(t)) \geq 0 \geq w_i(t) g_i(t, u(t)), \quad i \in I, \quad \text{a.e. in } [0, T],$$

i.e.

$$\sum_{i \in I} w_i(t) g_i(t, x(t)) \geq \sum_{i \in I} w_i(t) g_i(t, u(t)) \quad \text{a.e. in } [0, T].$$

Since $\sum_{i \in I} w_i(t) g_i(t, \cdot)$ is quasiconcave at $u(\cdot)$, this implies

$$\sum_{i \in I} w_i(t) \nabla g_i(t, u(t)) (x(t) - u(t)) \geq 0, \quad \text{a.e. in } [0, T]. \quad (6)$$

Applying (6) to (1), yields

$$\sum_{j \in J} \lambda_j \nabla f_j(t, u(t)) (x(t) - u(t)) \leq 0, \quad \text{a.e. in } [0, T]. \quad (7)$$

Now suppose contrary to the result of the theorem that (4) and (5) hold. If $\lambda_j > 0$ for all $j \in J$, then (4) and (5) imply

$$\int_0^T \lambda_j f_j(t, x(t)) dt \geq \int_0^T \lambda_j f_j(t, u(t)) dt \quad \forall j \in J, \quad (8)$$

and

$$\int_0^T \lambda_k f_k(t, x(t)) dt > \int_0^T \lambda_k f_k(t, u(t)) dt \quad \text{for some } k \in J. \quad (9)$$

respectively. Inequalities (8) and (9) also imply

$$\sum_{j \in J} \int_0^T \lambda_j f_j(t, x(t)) dt > \sum_{j \in J} \int_0^T \lambda_j f_j(t, u(t)) dt. \quad (10)$$

By hypothesis (i), i.e., $\Phi_\lambda(u(\cdot))$ is pseudoconcave, (10) implies

$$\sum_{j \in J} \int_0^T \lambda_j \nabla f_j(t, u(t))(x(t) - u(t)) dt > 0, \quad (11)$$

contradicting (7). Now, by $\lambda_j \geq 0, j \in J$, we have

$$\sum_{j \in J} \int_0^T \lambda_j f_j(t, x(t)) dt \geq \sum_{j \in J} \int_0^T \lambda_j f_j(t, u(t)) dt, \quad (12)$$

and by (ii) strictly pseudoconcavity $\Phi_\lambda(u(\cdot))$ at $u(\cdot)$, (12) implies (11), again contradicting (7).

The proof that the Theorem is also valid under the stronger assumption (iii) follows directly from the implication:

$$\begin{aligned} f_j(t, \cdot), j \in J, \text{ is concave a.e. in } [0, T] &\implies \sum_{j \in J} \lambda_j f_j(t, \cdot) \text{ is concave a.e. in } [0, T] \\ \implies \Phi_\lambda(u(\cdot)) = \int_0^T \sum_{j \in J} \lambda_j f_j(t, u(t)) dt &\text{ is concave} \\ \implies \Phi_\lambda(u(\cdot)) \text{ is pseudoconcave.} &\quad \square \end{aligned}$$

Corollary 3.3. Assume weak duality Theorem 3.2 holds between (VCTP) and (MWCTD). If $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible for (MWCTD) such that $u^0(\cdot)$ is feasible for (VCTP), then $u^0(\cdot)$ is an efficient solution for (VCTP) and $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is an efficient solution for (MWCTD).

Proof. First we show that $u^0(\cdot)$ is efficient solution for (VCTP). Let us suppose this proposition is false. Then there exists a feasible $x(\cdot)$ for (VCTP) such that (4) and (5) hold. But $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible for (MWCTD). It follows the results of weak duality Theorem 3.2 is contradicted. Consequently $u^0(\cdot)$ must be efficient solution for (VCTP). Now, suppose that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is not efficient solution for (MWCTD). Similarly, then there exists a feasible $(\bar{u}(\cdot), \bar{\lambda}, \bar{w}(\cdot))$ for (MWCTD) such that

$$\int_0^T f_j(t, \bar{u}(t)) dt \leq \int_0^T f_j(t, u^0(t)) dt \quad \forall j \in J, \quad (13)$$

and

$$\int_0^T f_k(t, \bar{u}(t)) dt < \int_0^T f_k(t, u^0(t)) dt \quad \text{for some } k \in J. \quad (14)$$

This contradict the weak duality theorem. Hence $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is efficient for (MWCTD). $\square$

Let $r \in J$. Let $\hat{x}(\cdot)$ be an efficient solution of (VCTP). To prove the strong duality theorem, a certain regularity condition for auxiliary problem

$$\begin{aligned} &\max \int_0^T f_r(t, x(t)) dt \\ P_r(\hat{x}(\cdot)) \quad &\bar{g}_j(t, x(t)) = f_j(t, x(t)) - f_j(t, \hat{x}(t)) \geq 0, j \in J \setminus \{r\} \text{ a.e. in } [0, T], \\ &\text{s.t. } g_i(t, x(t)) \geq 0, i \in I \text{ a.e. in } [0, T], \\ &x(\cdot) \in L_\infty([0, T]; \mathbb{R}^n) \end{aligned}$$

must be satisfied (see [6]).

Given $b > 0$ and $\hat{x}(\cdot) \in \Omega$, we will denote by $I_b(t)$ and $J_b(t)$ the index sets of b -active constraints at instant t , that is,

$$I_b(t) = \{i \in I : 0 \leq g_i(t, \hat{x}(t)) \leq b\}, \quad J_b(t) = \{j \in J \setminus \{r\} : 0 \leq \bar{g}_j(t, \hat{x}(t)) \leq b\},$$

for each $t \in [0, T]$. For all $i \in I$, let us define the functions $\delta_i^b, \bar{\delta}_j^b : [0, T] \rightarrow \mathbb{R}$ as

$$\delta_i^b(t) = \begin{cases} 1, & i \in I_b(t) \\ 0, & \text{otherwise,} \end{cases} \quad \bar{\delta}_j^b(t) = \begin{cases} 1, & j \in J_b(t) \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $J_b(t) = J \setminus \{r\}$ and $\bar{\delta}_j^b(t) = 1$ for all $t \in [0, T]$. We have that for almost every $t \in [0, T]$

$$\begin{aligned}\phi_j(t, h) &= \bar{g}_j(t, x(t))|_{x(t)=\hat{x}(t)} - \bar{\delta}_j^b(t) \nabla \bar{g}_j(t, x(t))^T|_{x(t)=\hat{x}(t)} h \\ &= (f_j(t, x(t)) - f_j(t, \hat{x}(t)))|_{x(t)=\hat{x}(t)} - \bar{\delta}_j^b(t) \nabla (f_j(t, x(t)) - f_j(t, \hat{x}(t)))^T|_{x(t)=\hat{x}(t)} h \\ &= -\bar{\delta}_j^b(t) \nabla f_j(t, \hat{x}(t))^T h = -\nabla f_j(t, \hat{x}(t))^T h\end{aligned}$$

for all $j \in J \setminus \{r\}$ and $h \in \mathbb{R}^n$. Let

$$\begin{aligned}\phi_r(t, h) &= -\int_0^T \nabla f_r(t, \hat{x}(t))^T h dt < 0, \\ \phi_j(t, h) &= -\nabla f_j(t, \hat{x}(t))^T h \leq 0, \quad j \in J \setminus \{r\}, \\ \phi_i(t, h) &= -g_i(t, \hat{x}(t)) - \delta_i^b(t) \nabla g_i(t, \hat{x}(t))^T h \leq 0, \quad i \in I, \quad h \in \mathbb{R}^n,\end{aligned}\tag{15}$$

be a system corresponding to the problem (VCTP) and for $t \in [0, T]$

$$\mathcal{I}(t, h) = \left\{ p : \phi_p(t, h) = \max \left\{ \phi_r(t, h), \{ \phi_i(t, h) \}_{i \in I}, \{ \phi_j(t, h) \}_{j \in J \setminus \{r\}} \right\} \right\}, \quad h \in \mathbb{R}^n.$$

Definition 3.4. [1] System (15) is said to be *regular* when there exist a function $\bar{x}(\cdot) \in L_\infty([0, T]; \mathbb{R}^n)$, real numbers $R \geq 0$ and $\alpha > 0$ such that for a.e. $t \in [0, 1]$ and for all $h \in \mathbb{R}^n$ with $\|h - \bar{x}(t)\| \geq R$, there exists a unit vector $e = e(t, h) \in \mathbb{R}^n$, satisfying $\langle \partial_h \phi_p(t, h), e \rangle \geq \alpha$ for all $p \in \mathcal{I}(t, h)$. $\square$

Definition 3.5. We say that the constraint qualification (MFCQ) holds at $\hat{x}(\cdot)$, if there exists $\bar{\gamma}(\cdot) \in L_\infty([0, T]; \mathbb{R}^n)$ and $b > 0$ such that, for almost every $t \in [0, T]$,

$$\nabla \bar{g}_j^T(t, \hat{x}(t)) \bar{\gamma}(t) \geq \beta, \quad j \in J \setminus \{r\}, \quad \text{and} \quad \nabla g_i^T(t, \hat{x}(t)) \bar{\gamma}(t) \geq \beta, \quad i \in I_b(t),$$

for some $\beta > 0$. $\square$

For $u(\cdot) \in \Omega$, define the active index set

$$I_0(t) := \{ i \in I : g_i(t, u(t)) = 0 \}, \quad t \in [0, T].$$

(CC_{act}) (Active-cone compensation condition).

Let $u^0(\cdot)$ be a feasible solution of (VCTP) and $(\tilde{\lambda}^0(\cdot), \tilde{w}^0(\cdot)) \in L_\infty([0, T]; \mathbb{R}^k \times \mathbb{R}^m)$ be a multiplier pair satisfying, for almost every $t \in [0, T]$,

$$\sum_{j \in J} \tilde{\lambda}_j^0(t) \nabla f_j(t, u^0(t)) + \sum_{i \in I} \tilde{w}_i^0(t) \nabla g_i(t, u^0(t)) = 0, \tag{16}$$

$$\tilde{w}_i^0(t) g_i(t, u^0(t)) = 0, \quad \tilde{w}_i^0(t) \geq 0, \quad i \in I, \tag{17}$$

$$\sum_{j \in J} \tilde{\lambda}_j^0(t) = 1, \quad \tilde{\lambda}_j^0(t) \geq 0, \quad j \in J. \tag{18}$$

Define the constant vector $\lambda^0 = (\lambda_1^0, \dots, \lambda_k^0) \in \mathbb{R}^k$, where $\lambda_j^0 := \frac{1}{T} \int_0^T \tilde{\lambda}_j^0(t) dt$, $j \in J$. We say that (CC_{act}) holds for the multiplier representation $(\tilde{\lambda}^0(\cdot), \tilde{w}^0(\cdot))$ if for almost every $t \in [0, T]$,

$$\sum_{j \in J} (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) \in K := \text{cone} \{ \nabla g_i(t, u^0(t)) : i \in I_0(t) \}, \tag{19}$$

that is, if there exists a function $\eta(\cdot) \in L_\infty([0, T]; \mathbb{R}^m)$ such that

$$\eta_i(t) \geq 0 \quad \text{for } i \in I_0(t), \quad \eta_i(t) = 0 \quad \text{for } i \notin I_0(t),$$

and

$$\sum_{j \in J} (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) = \sum_{i \in I_0(t)} \eta_i(t) \nabla g_i(t, u^0(t))$$

for almost every $t \in [0, T]$.

Condition (CC_{act}) is trivial whenever the selected multiplier function $\tilde{\lambda}^0(\cdot)$ is already constant. Indeed, if $\tilde{\lambda}^0(\cdot) \equiv \lambda^0 \in \mathbb{R}^k$, then, by the definition of λ^0 , $\tilde{\lambda}_j^0(t) - \lambda_j^0 = 0$ for a.e. $t \in [0, T]$, $j \in J$. Hence the left-hand side in (CC_{act}) vanishes identically, and one may simply take $\eta(\cdot) \equiv 0$. Therefore, (CC_{act}) becomes relevant only in the genuinely time-dependent case, where $\tilde{\lambda}^0(\cdot)$ is not constant and one needs a nontrivial compensation through active constraint gradients.

In fact, (CC_{act}) should be viewed as a bridge between the time-dependent multiplier representation delivered by the Theorem 6 [5] (which we will use in the proof of the following Strong duality theorem) and the constant-weight framework required by Weak duality theorem. It should be emphasized that if $u^0(\cdot)$ is an efficient solution of (VCTP) and certain regularity conditions are satisfied, Theorem [5] (necessary optimality conditions) guarantees the existence of a multiplier representation that satisfies (16)–(18).

Theorem 3.6. (Strong Duality) *Assume that $u^0(\cdot)$ is an efficient solution of (VCTP), $P_r(u^0(\cdot))$ satisfies (MFCQ) at $u^0(\cdot)$ for some $r \in J$, the corresponding system is regular and (CC_{act}) holds for at least one multiplier representation. Then there exists $(\lambda^0, w^0(\cdot)) \in \mathbb{R}^k \times L_\infty([0, T]; \mathbb{R}^m)$ such that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible solution for (MWCTD). If also weak duality Theorem 3.2 holds between (VCTP) and (MWCTD) then $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is an efficient solution for (MWCTD).*

Proof. Since $u^0(\cdot)$ is an efficient solution of (VCTP), $P_r(u^0(\cdot))$ satisfies (MFCQ) for some $r \in J$ and the corresponding system is regular, hence, from Theorem 6 in [6] that there exists $(\tilde{\lambda}^0(\cdot), \tilde{w}^0(\cdot)) \in L_\infty([0, T]; \mathbb{R}^k \times \mathbb{R}^m)$ satisfying (16)–(18) for almost every $t \in [0, T]$. Therefore, by definition of (MWCTD), $(u^0(\cdot), \tilde{\lambda}^0(\cdot), \tilde{w}^0(\cdot))$ is feasible for (MWCTD).

Also, (CC_{act}) holds. Since $\tilde{\lambda}^0(t) \geq 0$ and $\sum_{j \in J} \tilde{\lambda}_j^0(t) = 1$ for a.e. t in $[0, T]$, we have $\lambda^0 \geq 0$ and $\sum_{j \in J} \lambda_j^0 = 1$. Thus, the normalization and nonnegativity condition (18) holds with the constant vector function $\lambda^0(\cdot) \equiv \lambda^0$ and there exists $\eta(\cdot) \in L_\infty([0, T]; \mathbb{R}_+^m)$ such that, for almost every $t \in [0, T]$, $\eta_i(t) = 0$ for all $i \notin I_0(t)$ and

$$\sum_{i \in I_0(t)} \eta_i(t) \nabla g_i(t, u^0(t)) = \sum_{j \in J} (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) \quad (20)$$

with an essentially bounded measurable choice of conic coefficients $\eta(\cdot)$ satisfying (20). Let $\eta(\cdot) \in L_\infty([0, T]; \mathbb{R}_+^m)$ be as in (20) and define

$$w^0(t) := \tilde{w}^0(t) + \eta(t) \quad \text{for a.e. } t \in [0, T]. \quad (21)$$

Then $w^0(t) \geq 0$ for a.e. t . We next verify the complementarity condition (17). Fix $i \in I$ and t such that (17) and (20) hold. If $i \notin I_0(t)$, then $g_i(t, u^0(t)) > 0$ and (20) gives $\eta_i(t) = 0$, hence $w_i^0(t) = \tilde{w}_i^0(t)$. Therefore

$$w_i^0(t) g_i(t, u^0(t)) = \tilde{w}_i^0(t) g_i(t, u^0(t)) = 0$$

by (17). If $i \in I_0(t)$, then $g_i(t, u^0(t)) = 0$ and thus $w_i^0(t) g_i(t, u^0(t)) = 0$ trivially. Consequently, (17) holds with $w^0(\cdot)$ for almost every t . Finally, we verify the stationarity equality (16) for the pair $(\lambda^0, w^0(\cdot))$. Using (20) and (21), we obtain

$$\begin{aligned} & \sum_{j \in J} \lambda_j^0 \nabla f_j(t, u^0(t)) + \sum_{i \in I} w_i^0(t) \nabla g_i(t, u^0(t)) \\ &= \sum_{j \in J} \lambda_j^0 \nabla f_j(t, u^0(t)) + \sum_{i \in I} \tilde{w}_i^0(t) \nabla g_i(t, u^0(t)) + \sum_{j \in J} (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) \\ &= \sum_{j \in J} \tilde{\lambda}_j^0(t) \nabla f_j(t, u^0(t)) + \sum_{i \in I} \tilde{w}_i^0(t) \nabla g_i(t, u^0(t)). \end{aligned}$$

The right-hand side equals 0 by (16). Therefore,

$$\sum_{j \in J} \lambda_j^0 \nabla f_j(t, u^0(t)) + \sum_{i \in I} w_i^0(t) \nabla g_i(t, u^0(t)) = 0 \quad \text{for a.e. } t \in [0, T],$$

which also proves (16) for $(\lambda^0, w^0(\cdot))$. Thus, $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is also feasible for (MWCTD) with a constant vector function λ^0 . Also, we have that weak duality Theorem 3.2 holds between (VCTP) and (MWCTD), when $\lambda^0(\cdot) = \lambda^0$ is a constant vector function. Therefore, the efficiency of $(u^0(\cdot), \lambda^0, w^0(\cdot))$ for (MWCTD) now follows from Corollary 3.3. $\square$

Remark 3.7. Condition (CC_{act}) guarantees that the time-variation of the weights $\lambda(\cdot)$ can be absorbed through active constraint gradients, yielding a dual feasible triple with a constant weight vector λ , which is necessary to apply weak duality Theorem 3.2. $\square$

As an illustration, we consider the following example.

Example 3.8.

$$\begin{aligned} & \max \left(\int_0^1 \left(1 - \frac{1}{4}x_1^2(t) - \frac{1}{8}x_2(t) \right) dt, \int_0^1 (1 + 4t - x_1(t) - 2x_2(t)) dt \right) \\ & \text{s.t.} \quad 4x_1(t) - x_1^2(t) - 2x_2(t) - 3 + 4t \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad 4x_1(t) - x_1^2(t) + 2x_2(t) - 3 - 4t \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad x_1^2(t) - \frac{5}{2}x_1(t) - 2x_2(t) + \frac{3}{2} + 4t \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad x(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2). \end{aligned} \quad (\text{VCTP})$$

It is obvious that $f_1(t, x(t)) = 1 - \frac{1}{4}x_1^2(t) - \frac{1}{8}x_2(t)$, $f_2(t, x(t)) = 1 + 4t - x_1(t) - 2x_2(t)$ and for almost every $t \in [0, 1]$, $u^0(t) = \hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t)) = (1, 2t)$ is an efficient solution of preceding problem. The Mond–Weir-type dual problem of (VCTP) is formulated as follows (MWCTD):

$$\begin{aligned} & \min \left(\int_0^1 \left(1 - \frac{1}{4}u_1^2(t) - \frac{1}{8}u_2(t) \right) dt, \int_0^1 (1 + 4t - u_1(t) - 2u_2(t)) dt \right) \\ & -\frac{1}{2}\lambda_1(t)u_1(t) - \lambda_2(t) + (4 - 2u_1(t))(w_1(t) + w_2(t)) + (2u_1(t) - \frac{5}{2})w_3(t) = 0 \quad \text{a.e. in } [0, 1], \\ & -\frac{1}{8}\lambda_1(t) - 2\lambda_2(t) - 2w_1(t) + 2w_2(t) - 2w_3(t) = 0 \quad \text{a.e. in } [0, 1], \\ & w_1(t)(4u_1(t) - u_1^2(t) - 2u_2(t) - 3 + 4t) \leq 0 \quad \text{a.e. in } [0, 1], \\ & w_2(t)(4u_1(t) - u_1^2(t) + 2u_2(t) - 3 - 4t) \leq 0 \quad \text{a.e. in } [0, 1], \\ & w_3(t)(u_1^2(t) - \frac{5}{2}u_1(t) - 2u_2(t) + \frac{3}{2} + 4t) \leq 0 \quad \text{a.e. in } [0, 1], \\ & \lambda_1(t) + \lambda_2(t) = 1, \quad \lambda_1(t), \lambda_2(t) \geq 0, \quad w_1(t) \geq 0, \quad w_2(t) \geq 0, \quad w_3(t) \geq 0 \quad \text{a.e. in } [0, 1], \\ & u(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2), \quad \lambda(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2), \quad w(\cdot) \in L_\infty([0, 1]; \mathbb{R}^3). \end{aligned}$$

The problem $P_1(u^0(\cdot))$ is given by

$$\begin{aligned} & \max \int_0^1 \left(1 - \frac{1}{4}x_1^2(t) - \frac{1}{8}x_2(t) \right) dt \\ & \text{s.t.} \quad \bar{g}_1(t, x(t)) = 1 + 4t - x_1(t) - 2x_2(t) \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad g_1(t, x(t)) = 4x_1(t) - x_1^2(t) - 2x_2(t) - 3 + 4t \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad g_2(t, x(t)) = 4x_1(t) - x_1^2(t) + 2x_2(t) - 3 - 4t \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad g_3(t, x(t)) = x_1^2(t) - \frac{5}{2}x_1(t) - 2x_2(t) + \frac{3}{2} + 4t \geq 0 \quad \text{a.e. in } [0, 1], \\ & \quad \quad x(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2). \end{aligned} \quad P_1(u^0(\cdot))$$

We will verify that corresponding system is regular and (MFCQ) is satisfied for $P_1(u^0(\cdot))$. It is obvious

that $J_b(t) = \{1\}$, $I_b(t) = \{1, 2, 3\}$ for $b = \frac{1}{2}$. It can be easily verified that for almost every $t \in [0, 1]$,

$$\begin{aligned} \nabla f_1(t, \hat{x}(t)) &= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{8} \end{pmatrix}, \quad \nabla \bar{g}_1(t, \hat{x}(t)) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad \nabla g_1(t, \hat{x}(t)) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \\ \nabla g_2(t, \hat{x}(t)) &= \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \nabla g_3(t, \hat{x}(t)) = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix}. \end{aligned}$$

Take $\bar{\gamma}(t) = (\frac{4}{5}, -\frac{3}{5})^T$ a.e. in $[0, 1]$ and $\beta = \frac{1}{5}$. It follows that $\nabla g_i(t, \hat{x}(t))^T \bar{\gamma}(t) \geq \beta$, $i = 1, 2, 3$, $\nabla \bar{g}_1(t, \hat{x}(t))^T \bar{\gamma}(t) \geq \beta$ a.e. in $[0, 1]$. Thus, constraint qualification (MFCQ) is satisfied. Let $\delta(t) = \tilde{\lambda}_1^0(t) - \lambda_1^0$. Obviously $-\delta(t) = \tilde{\lambda}_2^0(t) - \lambda_2^0$. Since $|J| = 2$ we get

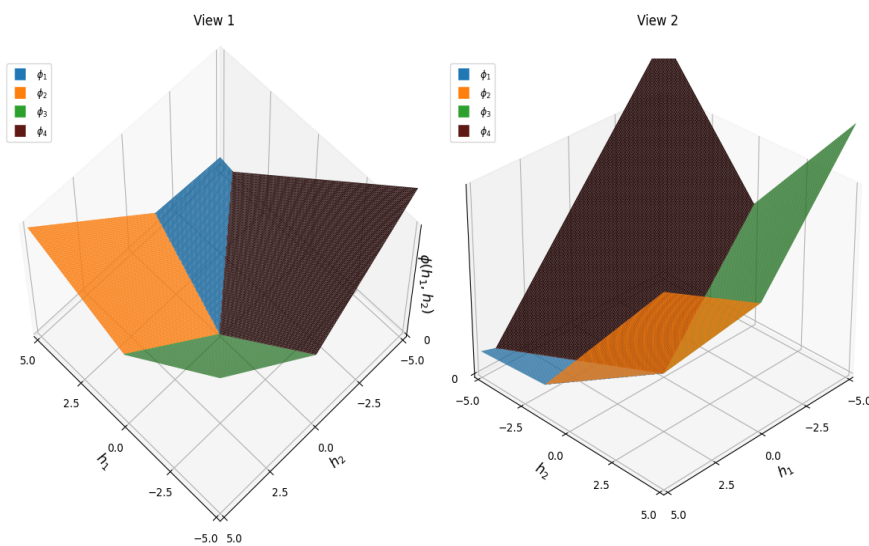
$$\sum_{j \in J} (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) = (\tilde{\lambda}_1^0(t) - \lambda_1^0) v = \delta(t) v,$$

where $v := (\nabla f_1 - \nabla f_2)(t, u^0(t)) = (\frac{1}{2}, \frac{15}{8})$.

Moreover $I_0(t) = \{1, 2, 3\}$ and $K := \text{cone}\{(2, -2), (2, 2), (-\frac{1}{2}, -2)\}$. It is clear that $v \notin K$ and $-v \notin K$. If we assume that condition (CC_{act}) holds nontrivially for some non-constant multipliers, then, based on the above discussion and the definition of (CC_{act}), the inclusion $\delta(t)v \in K$ can hold for a.e. $t \in [0, 1]$ only if $\delta(t) = 0$ for a.e. $t \in [0, 1]$, i.e. $\tilde{\lambda}_1^0(t) = \lambda_1^0$ and also $\tilde{\lambda}_2^0(t) = \lambda_2^0$ a.e. $t \in [0, 1]$. This shows that, in this example, condition (CC_{act}) is satisfied only in the trivial case, namely when the multiplier function $\tilde{\lambda}^0(\cdot)$ is constant.

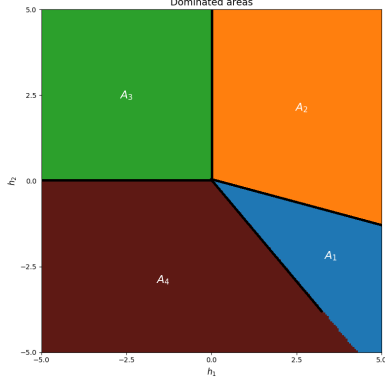
Now, we show that the corresponding system to $P_1(u^0(\cdot))$ is regular. Let $(h_1, h_2) \in \mathbb{R}^2$ and for almost everywhere in $[0, 1]$, let $\phi_1(t, h) = \frac{1}{2}h_1 + \frac{1}{8}h_2$, $\phi_2(t, h) = h_1 + 2h_2$, $\phi_3(t, h) = -2h_1 + 2h_2$, $\phi_4(t, h) = -2h_1 - 2h_2$, and $\phi_5(t, h) = \frac{1}{2}h_1 + 2h_2$. Define

$$\begin{aligned} A_1 &= \left\{ (h_1, h_2) \in \mathbb{R}^2 : \frac{20}{17}h_1 + h_2 \geq 0, \frac{4}{15}h_1 + h_2 \leq 0 \right\}, \\ A_2 &= \left\{ (h_1, h_2) \in \mathbb{R}^2 : \frac{4}{15}h_1 + h_2 \geq 0, h_1 \geq 0 \right\}, \\ A_3 &= \left\{ (h_1, h_2) \in \mathbb{R}^2 : h_1 \leq 0, h_2 \geq 0 \right\}, \\ A_4 &= \left\{ (h_1, h_2) \in \mathbb{R}^2 : \frac{20}{17}h_1 + h_2 \leq 0, h_2 \leq 0 \right\}. \end{aligned}$$



It can be easily verified that for $(h_1, h_2) \in \text{int} A_i$ we have $\mathcal{I}(t, h) = \{i\}$ a.e. in $[0, 1]$, $i = 1, 2, 3, 4$, and $\bigcup_{i=1}^4 A_i = \mathbb{R}^2$. For $(h_1, h_2) \in A_1 \cap A_2$, $\mathcal{I}(t, h) = \{1, 2\}$ a.e. in $[0, 1]$. For $(h_1, h_2) \in A_2 \cap A_3$, $\mathcal{I}(t, h) = \{2, 3, 5\}$ a.e. in $[0, 1]$. For $(h_1, h_2) \in A_3 \cap A_4$, $\mathcal{I}(t, h) = \{3, 4\}$ a.e. in $[0, 1]$. For $(h_1, h_2) \in A_1 \cap A_4$, $\mathcal{I}(t, h) = \{1, 4\}$ a.e. in $[0, 1]$.

The regularity of the system

$$\begin{aligned}
 \phi_1(t, h) &= \frac{1}{2}h_1 + \frac{1}{8}h_2 < 0, \\
 \phi_2(t, h) &= h_1 + 2h_2 \leq 0, \\
 \phi_3(t, h) &= -2h_1 + 2h_2 \leq 0, \\
 \phi_4(t, h) &= -2h_1 - 2h_2 \leq 0, \\
 \phi_5(t, h) &= \frac{1}{2}h_1 + 2h_2 \leq 0, \\
 h &\in \mathbb{R}^2,
 \end{aligned}
 \tag{22}$$


is verified with $\bar{x} \equiv (0, 0)$, $R \geq 0$, $\alpha = \frac{1}{60}$ and for almost every $t \in [0, 1]$,

1. $e = e(t, h) = (-\frac{4}{5}, \frac{3}{5})$ for $(h_1, h_2) \in \text{int}(A_2 \cup A_3 \cup A_4)$,
2. $e = e(t, h) = (\frac{7}{25}, -\frac{24}{25})$ for $(h_1, h_2) \in A_1 \cap A_4$ or $(h_1, h_2) \in \text{int} A_1$, and
3. $e = e(t, h) = (\frac{4}{5}, \frac{3}{5})$ for $(h_1, h_2) \in (A_1 \cap A_2)$.

In consequence of Theorem 3.6 there exists $(\lambda^0, w^0(\cdot)) \in \mathbb{R}^2 \times L_\infty([0, T]; \mathbb{R}^3)$ such that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible solution for (MWCTD) (i.e. (1)–(3) hold), that is, for almost every $t \in [0, 1]$,

$$\lambda_1^0 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{8} \end{pmatrix} + \lambda_2^0 \begin{pmatrix} -1 \\ -2 \end{pmatrix} + w_1^0(t) \begin{pmatrix} 2 \\ -2 \end{pmatrix} + w_2^0(t) \begin{pmatrix} 2 \\ 2 \end{pmatrix} + w_3^0(t) \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\lambda_1^0 = \frac{4}{5}$, $\lambda_2^0 = \frac{1}{5}$, $w_1^0(t) \in [0, \frac{1}{40}]$, $w_2^0(t) = \frac{19}{60} - \frac{5}{3}w_1^0(t)$ and $w_3^0(t) = \frac{1}{15} - \frac{8}{3}w_1^0(t)$ where $w_1^0 : [0, 1] \rightarrow [0, \frac{1}{40}]$ is a measurable function. Moreover, Theorem 3.2 holds between (VCTP) and (MWCTD). For $w_1^0(t) \in [0, \frac{1}{40}]$, we have that almost every t in $[0, 1]$, $(u^0(\cdot), \lambda^0, w^0(\cdot)) = ((1, 2t), (\frac{4}{5}, \frac{1}{5}), (w_1^0(t), \frac{19}{60} - \frac{5}{3}w_1^0(t), \frac{1}{15} - \frac{8}{3}w_1^0(t)))$ is an efficient solution for (MWCTD).

Example 3.9. for $n = 2$, $k = 3$.

We now construct a new example with $n = 2$ and $k = 3$ for which all assumptions of Theorem 3.6 are satisfied and where the multipliers $\tilde{\lambda}^0(\cdot)$ are genuinely non-constant. Consider the vector continuous-time problem

$$\begin{aligned}
 &\max \left(\int_0^1 f_1(t, x(t)) dt, \int_0^1 f_2(t, x(t)) dt, \int_0^1 f_3(t, x(t)) dt \right) \\
 &\text{s.t. } g_1(x(t)) := x_1(t) \geq 0, \quad g_2(x(t)) := x_2(t) \geq 0 \quad \text{a.e. on } [0, 1], \\
 &\quad x(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2),
 \end{aligned}$$

where $f_1(t, x(t)) = x_1(t) - x_1^2(t) - x_2(t)$, $f_2(t, x(t)) = 2x_1(t) - x_1^2(t) - x_2(t)$, and $f_3(t, x(t)) = -x_1(t) - x_1^2(t) - x_2(t)$. It is obvious that $u^0(t) = \hat{x}(t) = (0, 0)$ is an efficient solution for almost every $t \in [0, 1]$. The corresponding Mond–Weir dual is

$$\begin{aligned}
 &\min \left(\int_0^1 f_1(t, u(t)) dt, \int_0^1 f_2(t, u(t)) dt, \int_0^1 f_3(t, u(t)) dt \right) \\
 &\lambda_1(t)(1 - 2u_1(t)) + \lambda_2(t)(2 - 2u_1(t)) - \lambda_3(t)(1 + 2u_1(t)) + w_1(t) = 0, \quad \text{a.e. in } [0, 1], \\
 &\quad -\lambda_1(t) - \lambda_2(t) - \lambda_3(t) + w_2(t) = 0, \quad \text{a.e. in } [0, 1], \\
 &\quad w_1(t)u_1(t) \leq 0, \quad \text{a.e. in } [0, 1], \\
 &\quad w_2(t)u_2(t) \leq 0, \quad \text{a.e. in } [0, 1], \\
 &\lambda_1(t) + \lambda_2(t) + \lambda_3(t) = 1, \quad \lambda_1(t), \lambda_2(t), \lambda_3(t) \geq 0, \quad \text{a.e. in } [0, 1], \\
 &\quad w_1(t) \geq 0, \quad w_2(t) \geq 0, \quad \text{a.e. in } [0, 1], \\
 &u(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2), \quad \lambda(\cdot) \in L_\infty([0, 1]; \mathbb{R}^3), \quad w(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2),
 \end{aligned}$$

where $f_1(t, u(t)) = u_1(t) - u_1^2(t) - u_2(t)$, $f_2(t, u(t)) = 2u_1(t) - u_1^2(t) - u_2(t)$, and $f_3(t, u(t)) = -u_1(t) - u_1^2(t) - u_2(t)$. We have that for almost every $t \in [0, 1]$,

$$\begin{aligned}\nabla f_1(t, u^0(t)) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}, & \nabla f_2(t, u^0(t)) &= \begin{pmatrix} 2 \\ -1 \end{pmatrix}, & \nabla f_3(t, u^0(t)) &= \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \\ \nabla g_1(t, u^0(t)) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \nabla g_2(t, u^0(t)) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\end{aligned}$$

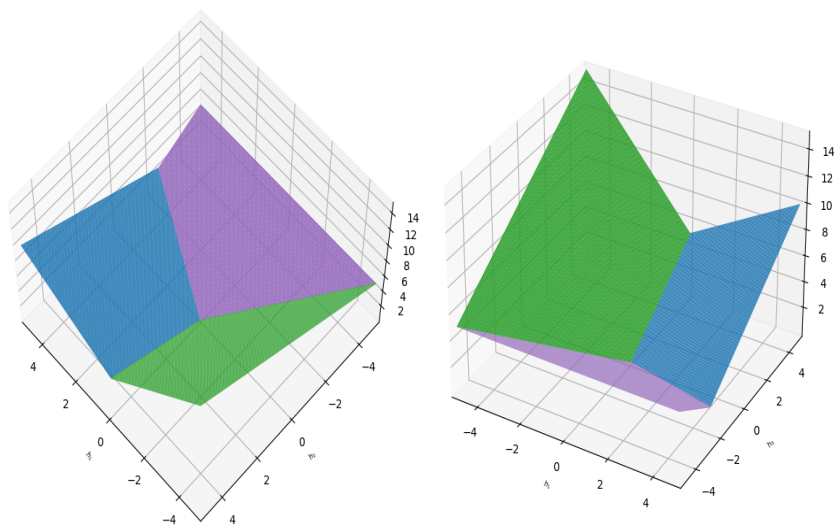
Choose $r = 3$. The problem $P_3(u^0(\cdot))$ is given by

$$\begin{aligned}P_3(u^0(\cdot)) \quad & \max \int_0^1 (-x_1(t) - x_1^2(t) - x_2(t)) dt \\ & \text{s.t. } \bar{g}_1(t, x(t)) = x_1(t) - x_1^2(t) - x_2(t) \geq 0 \text{ a.e. in } [0, 1], \\ & \bar{g}_2(t, x(t)) = 2x_1(t) - x_1^2(t) - x_2(t) \geq 0 \text{ a.e. in } [0, 1], \\ & g_1(t, x(t)) = x_1(t) \geq 0 \text{ a.e. in } [0, 1], \\ & g_2(t, x(t)) = x_2(t) \geq 0 \text{ a.e. in } [0, 1], \\ & x(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2).\end{aligned}$$

We will verify that corresponding system is regular and (MFCQ) is satisfied for $P_3(u^0(\cdot))$. It is obvious that $J_b(t) = \{1, 2\}$, $I_b(t) = \{1, 2\}$ for $b = 1$. It can be easily verified that for almost every $t \in [0, 1]$, we have $\nabla \bar{g}_1(t, u^0(t)) = (1, -1)^T$ and $\nabla \bar{g}_2(t, u^0(t)) = (2, -1)^T$. Let $\bar{\gamma}(t) \equiv (2, 1)^T$. Thus MFCQ holds with $\beta = 0.5$.

Now, we show that the corresponding system to $P_3(u^0(\cdot))$ is regular. Let $(h_1, h_2) \in \mathbb{R}^2$ and consider for almost everywhere in $[0, 1]$, $\phi_1(t, h) = -h_1 + h_2$, $\phi_2(t, h) = -2h_1 + h_2$, $\phi_3(t, h) = h_1 + h_2$, $\phi_4(t, h) = -h_1$, $\phi_5(t, h) = -h_2$. Define

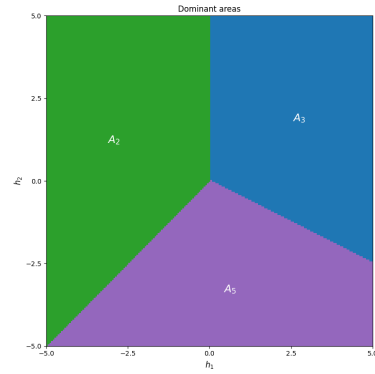
$$\begin{aligned}A_2 &= \{(h_1, h_2) \in \mathbb{R}^2 : h_2 - h_1 \geq 0, h_1 \leq 0\}, \\ A_3 &= \{(h_1, h_2) \in \mathbb{R}^2 : \frac{1}{2}h_1 + h_2 \geq 0, h_1 \geq 0\}, \\ A_5 &= \{(h_1, h_2) \in \mathbb{R}^2 : h_2 - h_1 \leq 0, \frac{1}{2}h_1 + h_2 \leq 0\}.\end{aligned}$$



It can be easily verified that for $(h_1, h_2) \in \text{int } A_i$, we have $\mathcal{I}(t, h) = \{i\}$ a.e. in $[0, 1]$, $i = 2, 3, 5$, and $A_2 \cup A_3 \cup A_5 = \mathbb{R}^2$. For $(h_1, h_2) \in A_2 \cap A_3$, $\mathcal{I}(t, h) = \{2, 3\}$ a.e. in $[0, 1]$. For $(h_1, h_2) \in A_2 \cap A_5$, $\mathcal{I}(t, h) = \{2, 4, 5\}$ a.e. in $[0, 1]$. For $(h_1, h_2) \in A_3 \cap A_5$, $\mathcal{I}(t, h) = \{3, 5\}$ a.e. in $[0, 1]$. The regularity of the

system

$$\begin{aligned}\phi_3(t, h) &= h_1 + h_2 < 0, \\ \phi_1(t, h) &= -h_1 + h_2 \leq 0, \\ \phi_2(t, h) &= -2h_1 + h_2 \leq 0, \\ \phi_4(t, h) &= -h_1 \leq 0, \\ \phi_5(t, h) &= -h_2 \leq 0, \\ h &\in \mathbb{R}^2,\end{aligned}$$



is verified with $\bar{x} \equiv (0, 0)$, $R \geq 0$, $\alpha = 0.05$ and for almost every $t \in [0, 1]$,

1. $e = e(t, h) = (0, 1)$ for $(h_1, h_2) \in \text{int}(A_2 \cup A_3)$,
2. $e = e(t, h) = (-\frac{4}{5}, -\frac{3}{5})$ for $(h_1, h_2) \in \text{int} A_5$ or $(h_1, h_2) \in A_2 \cap A_5$, and
3. $e = e(t, h) = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ for $(h_1, h_2) \in (A_3 \cap A_5)$.

Nonconstant multipliers $\tilde{\lambda}^0(\cdot)$ and verification of (CC_{act}) .

For $t \in [0, 1]$, define

$$\tau(t) := \frac{1}{32} \sin(2\pi t), \quad \tilde{\lambda}_1^0(t) = \frac{1}{8} + \tau(t), \quad \tilde{\lambda}_2^0(t) = \frac{1}{12} - \frac{2}{3}\tau(t), \quad \tilde{\lambda}_3^0(t) = \frac{19}{24} - \frac{1}{3}\tau(t).$$

Then $\tilde{\lambda}_1^0(t) + \tilde{\lambda}_2^0(t) + \tilde{\lambda}_3^0(t) = 1$ for all $t \in [0, 1]$ and, since $|\tau(t)| \leq \frac{1}{32}$, we have

$$\tilde{\lambda}_1^0(t) \geq \frac{3}{32}, \quad \tilde{\lambda}_2^0(t) \geq \frac{1}{16}, \quad \tilde{\lambda}_3^0(t) \geq \frac{25}{32}.$$

Hence all three multipliers are nonnegative and non-constant. Let $\tilde{w}_1^0(t) \equiv \frac{1}{2}$ and $\tilde{w}_2^0(t) \equiv 1$.

Then the stationarity condition is satisfied, because it is obvious that

$$\begin{aligned}& \underbrace{\left(\frac{1}{8} + \tau(t)\right)}_{\tilde{\lambda}_1^0(t)} \cdot \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\nabla f_1(t, u^0(t))} + \underbrace{\left(\frac{1}{12} - \frac{2}{3}\tau(t)\right)}_{\tilde{\lambda}_2^0(t)} \cdot \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\nabla f_2(t, u^0(t))} + \underbrace{\left(\frac{19}{24} - \frac{1}{3}\tau(t)\right)}_{\tilde{\lambda}_3^0(t)} \cdot \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{\nabla f_3(t, u^0(t))} \\& + \underbrace{\frac{1}{2}}_{\tilde{w}_1^0(t)} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\nabla g_1(t, u^0(t))} + \underbrace{1}_{\tilde{w}_2^0(t)} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\nabla g_2(t, u^0(t))} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned}$$

for a.e. $t \in [0, 1]$. Since $g_1(t, u^0(t)) = g_2(t, u^0(t)) = 0$, complementary slackness also holds. The averaged multiplier vector is

$$\lambda^0 = \left(\int_0^1 \tilde{\lambda}_1^0(t) dt, \int_0^1 \tilde{\lambda}_2^0(t) dt, \int_0^1 \tilde{\lambda}_3^0(t) dt \right) = \left(\frac{1}{8}, \frac{1}{12}, \frac{19}{24} \right).$$

Moreover, $I_0(t) = \{1, 2\}$ for all $t \in [0, 1]$. Hence $K = \text{cone}\{(1, 0), (0, 1)\}$ and

$$\sum_{j=1}^3 (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) = \tau(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{2}{3}\tau(t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \frac{1}{3}\tau(t) \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for all $t \in [0, 1]$. Therefore

$$\sum_{j=1}^3 (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) \in K$$

for all $t \in [0, 1]$, so (CC_{act}) holds.

By the Theorem 3.6, $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is also feasible solution for (MWCTD), that is, for almost every $t \in [0, 1]$,

$$\lambda_1^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2^0 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda_3^0(t) \begin{pmatrix} -1 \\ -1 \end{pmatrix} + w_1^0(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w_2^0(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $(\lambda_1^0, \lambda_2^0, \lambda_3^0) = (\frac{1}{8}, \frac{1}{12}, \frac{19}{24})$ and $(w_1^0(t), w_2^0(t)) = (\frac{1}{2}, 1)$. Finally, Theorem 3.2 (Weak duality) holds between (VCTP) and (MWCTD). Therefore $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is an efficient solution of the dual problem.

4. Wolf continuous-time dual (WCTD)

The Wolfe-type dual problem of (VCTP) is formulated as follows (WCTD):

$$\begin{aligned} \min & \left(\int_0^T (f_1(t, u(t)) + w(t)^T g(t, u(t))) dt, \dots, \int_0^T (f_k(t, u(t)) + w(t)^T g(t, u(t))) dt \right) \\ \text{s.t.} & \sum_{j \in J} \lambda_j(t) \nabla f_j(t, u(t)) + \sum_{i \in I} w_i(t) \nabla g_i(t, u(t)) = 0, \quad \text{a.e. in } [0, T], \end{aligned} \quad (23)$$

$$w_i(t) \geq 0, \quad i \in I, \quad \text{a.e. in } [0, T], \quad (24)$$

$$\sum_{j \in J} \lambda_j(t) = 1, \quad \lambda_j(t) \geq 0, \quad \text{a.e. in } [0, T], \quad j \in J, \quad (25)$$

$$\lambda(\cdot) \in L_\infty([0, T]; \mathbb{R}^k), \quad u(\cdot) \in L_\infty([0, T]; \mathbb{R}^n), \quad w(\cdot) \in L_\infty([0, T]; \mathbb{R}^m).$$

In this section we present weak and strong duality relations between programs (VCTP) and (WCTD). The weak duality results are given under concavity assumptions.

Theorem 4.1. (Weak Duality) Assume that for all feasible $x(\cdot)$ for (VCTP) and all feasible $(u(\cdot), \lambda(\cdot), w(\cdot))$ for (WCTD) (where $\lambda(\cdot)$ is a constant vector function), $f_j(t, \cdot)$, $j \in J$, and $g_i(t, \cdot)$, $i \in I$ are concave functions almost everywhere in $[0, T]$. If also either

- (i) $\lambda_j > 0$, $\forall j \in J$, a.e. in $[0, T]$ or
- (ii) $\sum_{j \in J} \lambda_j f_j(t, \cdot) + \sum_{i \in I} w_i(t) g_i(t, \cdot)$ is strictly concave for almost everywhere in $[0, T]$,

then the following cannot hold:

$$\int_0^T f_j(t, x(t)) dt \geq \int_0^T (f_j(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t))) dt \quad \forall j \in J, \quad (26)$$

and

$$\int_0^T f_k(t, x(t)) dt > \int_0^T (f_k(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t))) dt \quad \text{for some } k \in J. \quad (27)$$

Proof. Define the functional $J : L_\infty([0, T]; \mathbb{R}^n) \rightarrow \mathbb{R}$ by

$$J(u(\cdot)) = \int_0^T \left(\sum_{j \in J} \lambda_j f_j(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t)) \right) dt.$$

Suppose contrary to the result that (26) and (27) hold. Then since $x(\cdot)$ is feasible for (VCTP) and $w_i(t) \geq 0$, $i \in I$, a.e. in $[0, T]$, i.e.,

$$\sum_{i \in I} w_i(t) g_i(t, x(t)) \geq 0, \quad \text{a.e. in } [0, T],$$

(26) and (27) imply

$$\begin{aligned} & \int_0^T \left(f_j(t, x(t)) + \sum_{i \in I} w_i(t) g_i(t, x(t)) \right) dt \\ & \geq \int_0^T \left(f_j(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t)) \right) dt, \quad \forall j \in J \end{aligned} \quad (28)$$

and

$$\begin{aligned} & \int_0^T \left(f_k(t, x(t)) + \sum_{i \in I} w_i(t) g_i(t, x(t)) \right) dt \\ & > \int_0^T \left(f_k(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t)) \right) dt, \end{aligned} \quad (29)$$

for some $k \in J$, respectively.

Hypothesis (i) and $\sum_{j \in J} \lambda_j = 1$ imply

$$\begin{aligned} J(x(\cdot)) &= \int_0^T \left(\sum_{j \in J} \lambda_j f_j(t, x(t)) + \sum_{i \in I} w_i(t) g_i(t, x(t)) \right) dt \\ &> \int_0^T \left(\sum_{j \in J} \lambda_j f_j(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t)) \right) dt. \end{aligned} \quad (30)$$

Since $f_j(t, \cdot)$, $j \in J$, and $g_i(t, \cdot)$, $i \in I$, are concave functions at $u(\cdot)$ throughout $[0, T]$ and $\lambda(t) > 0$, $w(t) \geq 0$ a.e. in $[0, T]$, it follows

$$f_j(t, x(t)) - f_j(t, u(t)) \leq \nabla f_j(t, u(t))(x(t) - u(t)) \quad (31)$$

$$g_i(t, x(t)) - g_i(t, u(t)) \leq \nabla g_i(t, u(t))(x(t) - u(t)). \quad (32)$$

By multiplying equations (31) and (32) by λ_j and $w_i(t)$ respectively and summing, we obtain

$$\begin{aligned} & \sum_{j \in J} \lambda_j (f_j(t, x(t)) - f_j(t, u(t))) + \sum_{i \in I} w_i(t) (g_i(t, x(t)) - g_i(t, u(t))) \\ &= \left(\sum_{j \in J} \lambda_j f_j(t, x(t)) + \sum_{i \in I} w_i(t) g_i(t, x(t)) \right) - \left(\sum_{j \in J} \lambda_j f_j(t, u(t)) + \sum_{i \in I} w_i(t) g_i(t, u(t)) \right) \\ &\leq \left(\sum_{j \in J} \lambda_j \nabla f_j(t, u(t)) + \sum_{i \in I} w_i(t) \nabla g_i(t, u(t)) \right) (x(t) - u(t)), \text{ a.e. in } [0, T]. \end{aligned}$$

Integrating the previous inequality from 0 to T , it follows from (30) that

$$\begin{aligned} 0 &< J(x(\cdot)) - J(u(\cdot)) \\ &\leq \int_0^T \left(\sum_{j \in J} \lambda_j \nabla f_j(t, u(t)) + \sum_{i \in I} w_i(t) \nabla g_i(t, u(t)) \right) (x(t) - u(t)) dt. \end{aligned} \quad (33)$$

This is a contradiction to (23), i.e. $(u(\cdot), \lambda, w(\cdot))$ is not feasible for (WCTD).

Similarly, since $\lambda_j \geq 0$, $j \in J$ and $\sum_{j \in J} \lambda_j = 1$, (28) and (29) imply

$$J(x(\cdot)) \geq J(u(\cdot)). \quad (34)$$

Hypothesis (ii) and (34) also imply

$$\int_0^T \left(\sum_{j \in J} \lambda_j \nabla f_j(t, u(t)) + \sum_{i \in I} w_i(t) \nabla g_i(t, u(t)) \right) (x(t) - u(t)) dt > 0,$$

again contradicting (23). $\square$

Corollary 4.2. Assume that weak duality Theorem 4.1 holds between (VCTP) and (WCTD). If $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible for (WCTD) with

$$w_i^0(t) g_i(t, u^0(t)) = 0, \quad i \in I \text{ a.e. in } [0, T], \quad (35)$$

and $u^0(\cdot)$ is feasible for (VCTP), then $u^0(\cdot)$ is an efficient solution for (VCTP) and $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is an efficient solution for (WCTD).

Proof. First we show that $u^0(\cdot)$ is efficient solution for (VCTP). Suppose this proposition is false. Then there exists a feasible $x(\cdot)$ for (VCTP) such that

$$\int_0^T f_j(t, x(t))dt \geq \int_0^T f_j(t, u^0(t))dt \quad \forall j \in J, \quad (36)$$

and

$$\int_0^T f_k(t, x(t))dt > \int_0^T f_k(t, u^0(t))dt \quad \text{for some } k \in J. \quad (37)$$

By hypothesis (35) formulas (36) and (37) can be written as

$$\int_0^T f_j(t, x(t))dt \geq \int_0^T \left(f_j(t, u^0(t)) + \sum_{i \in I} w_i^0(t) g_i(t, u^0(t)) \right) dt \quad \forall j \in J,$$

and

$$\int_0^T f_k(t, x(t))dt > \int_0^T \left(f_k(t, u^0(t)) + \sum_{i \in I} w_i^0(t) g_i(t, u^0(t)) \right) dt \quad \text{for some } k \in J.$$

These inequalities contradict Theorem 4.1. Therefore, $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is efficient for (WCTD). Similarly, suppose that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is not efficient for (WCTD), i.e. there exists a feasible $(\bar{u}(\cdot), \bar{\lambda}, \bar{w}(\cdot))$ for (WCTD) such that

$$\begin{aligned} & \int_0^T \left(f_j(t, \bar{u}(t)) + \sum_{i \in I} \bar{w}_i(t) g_i(t, \bar{u}(t)) \right) dt \\ & \leq \int_0^T \left(f_j(t, u^0(t)) + \sum_{i \in I} w_i^0(t) g_i(t, u^0(t)) \right) dt, \quad \forall j \in J \end{aligned} \quad (38)$$

and

$$\begin{aligned} & \int_0^T \left(f_k(t, \bar{u}(t)) + \sum_{i \in I} \bar{w}_i(t) g_i(t, \bar{u}(t)) \right) dt \\ & < \int_0^T \left(f_k(t, u^0(t)) + \sum_{i \in I} w_i^0(t) g_i(t, u^0(t)) \right) dt, \quad \text{for some } k \in J. \end{aligned} \quad (39)$$

By hypothesis (35), (38) and (39) reduce to

$$\int_0^T \left(f_j(t, \bar{u}(t)) + \sum_{i \in I} \bar{w}_i(t) g_i(t, \bar{u}(t)) \right) dt \leq \int_0^T f_j(t, u^0(t))dt \quad \forall j \in J,$$

and

$$\int_0^T \left(f_k(t, \bar{u}(t)) + \sum_{i \in I} \bar{w}_i(t) g_i(t, \bar{u}(t)) \right) dt < \int_0^T f_k(t, u^0(t))dt \quad \text{for some } k \in J.$$

This inequalities also contradict Theorem 4.1. It follows $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is efficient for (WCTD). $\square$

Theorem 4.3. (Strong Duality) Assume that $u^0(\cdot)$ is an efficient solution of (VCTP), $P_r(u^0(\cdot))$ satisfies (MFCQ) at $u^0(\cdot)$ for some $r \in J$, the corresponding system is regular and (CC_{act}) holds for at least one multiplier representation. Then there exists $(\lambda^0, w^0(\cdot)) \in \mathbb{R}^k \times L_\infty([0, T]; \mathbb{R}^m)$ such that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible solution for (WCTD) and for almost every $t \in [0, T]$,

$$w_i^0(t) g_i(t, u^0(t)) = 0, \quad i \in I. \quad (40)$$

If also the weak duality Theorem 4.1 holds between (VCTP) and (WCTD) then $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is an efficient solution for (WCTD).

Proof. Since $u^0(\cdot)$ is an efficient solution of (VCTP), and satisfies constraint qualification (MFCQ) for at least one r , hence, from Theorem 6 in [6] that there exist $(\tilde{\lambda}^0(\cdot), \tilde{w}^0(\cdot)) \in L_\infty([0, T]; \mathbb{R}^k \times \mathbb{R}^m)$ such that, for almost every $t \in [0, T]$,

$$\sum_{j \in J} \tilde{\lambda}_j^0(t) \nabla f_j(t, u^0(t)) + \sum_{i \in I} \tilde{w}_i^0(t) \nabla g_i(t, u^0(t)) = 0, \quad (41)$$

$$\tilde{w}_i^0(t) g_i(t, u^0(t)) = 0, \quad \tilde{w}_i^0(t) \geq 0, \quad i \in I, \quad (42)$$

$$\sum_{j \in J} \tilde{\lambda}_j^0(t) = 1, \quad \tilde{\lambda}_j^0(t) \geq 0, \quad j \in J. \quad (43)$$

Therefore, we conclude that $(u^0(\cdot), \tilde{\lambda}^0(\cdot), \tilde{w}^0(\cdot))$ is feasible solution for (WCTD). Also, (CC_{act}) holds. As in the proof of Theorem 3.6, we conclude that there exists $(\lambda^0, w^0(\cdot)) \in \mathbb{R}^k \times L_\infty([0, T]; \mathbb{R}_+^m)$ such that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible solution for (WCTD), where $\lambda^0 \geq 0$ and $\sum_{j \in J} \lambda_j^0 = 1$. Also, $w_i^0(t) g_i(t, u^0(t)) = 0$, a.e. in $[0, T]$, $i \in I$. If weak duality Theorem 4.1 holds between (VCTP) and (WCTD) where $\lambda^0(\cdot)$ is a constant vector function, the efficiency of $(u^0(\cdot), \lambda^0, w^0(\cdot))$ for (WCTD) follows from Corollary 4.2. $\square$

Definition 4.4. Let $u^0(\cdot) \in \Omega$. We say that $P_r(u^0(\cdot))$ satisfies the constraint qualification (SCQ), if there exist $s(\cdot) \in \Omega$ and $b > 0$ such that, for almost every $t \in [0, T]$,

$$\bar{g}_j(t, s(t)) \geq \alpha, \quad j \in J \setminus \{r\}, \quad \text{and} \quad g_i(t, s(t)) \geq \alpha, \quad i \in I_b(t),$$

for some $\alpha > 0$ and $\alpha > b$.

Corollary 4.5. Assume that $f_j(t, \cdot)$ and $g_i(t, \cdot)$, $j \in J$, $i \in I$ are concave function almost everywhere in $[0, T]$. Then $(SCQ) \implies (MFCQ)$.

Proof. Let $u^0(\cdot) \in \Omega$. The concavity of the function $\bar{g}_j(t, \cdot)$ follows from the concavity of the function $f_j(t, \cdot)$, $j \in J$. We have

$$\nabla \bar{g}_j(t, u^0(t))^T (s(t) - u^0(t)) \geq \bar{g}_j(t, s(t)) - \bar{g}_j(t, u^0(t)) \geq \alpha > \alpha - b, \quad j \in J \setminus \{r\},$$

$$\nabla g_i(t, u^0(t))^T (s(t) - u^0(t)) \geq g_i(t, s(t)) - g_i(t, u^0(t)) \geq \alpha - b, \quad i \in I_b(t).$$

Hence, (MFCQ) is satisfied for $\beta = \alpha - b$ and $s(t) - u^0(t) = \bar{\gamma}(t)$. $\square$

Based on the previous Definition, we see that Theorem 4.3 still holds with other constraint qualification (SCQ).

Theorem 4.6. (Strong Duality) Assume that $u^0(\cdot)$ is an efficient solution of (VCTP), $P_r(u^0(\cdot))$ satisfies the constraint qualification (SCQ) at $u^0(\cdot)$ for some $r \in J$, corresponding system is regular and (CC_{act}) holds for at least one multiplier representation. Then there exists $(\lambda^0, w^0(\cdot)) \in \mathbb{R}^k \times L_\infty([0, T]; \mathbb{R}^m)$ such that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is feasible solution for (WCTD) and for almost every $t \in [0, T]$,

$$w_i^0(t) g_i(t, u^0(t)) = 0, \quad i \in I. \quad (44)$$

If also the weak duality Theorem 4.1 holds between (VCTP) and (WCTD) then $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is an efficient solution for (WCTD).

Example 4.7.

$$\begin{aligned} \max \quad & \left(\int_0^1 (-x(t) - t - \exp(x(t))) dt, \int_0^1 (4x(t) - 5 - x^2(t)) dt \right) \\ \text{s.t.} \quad & x(t) \geq 0 \text{ a.e. in } [0, 1], \quad x(\cdot) \in L_\infty([0, 1]; \mathbb{R}). \end{aligned}$$

The Wolf-type dual problem of (VCTP) is formulated as follows (WCTD):

$$\begin{aligned} \min \quad & \left(\int_0^1 (-u(t) - t - \exp(u(t)) + w(t)u(t)) dt, \int_0^1 (4u(t) - 5 - u^2(t) + w(t)u(t)) dt \right) \\ \text{s.t.} \quad & \lambda_1(t)(-1 - \exp(u(t))) + \lambda_2(t)(4 - 2u(t)) + w(t) = 0 \text{ a.e. in } [0, 1], \\ & \lambda_1(t) + \lambda_2(t) = 1, \quad \lambda_1(t), \lambda_2(t) \geq 0, \quad w(t) \geq 0 \text{ a.e. in } [0, 1], \\ & u(\cdot) \in L_\infty([0, 1]; \mathbb{R}), \quad \lambda(\cdot) \in L_\infty([0, 1]; \mathbb{R}^2), \quad w(\cdot) \in L_\infty([0, 1]; \mathbb{R}). \end{aligned}$$

It is obvious that $f_1(t, x(t)) = -x(t) - t - \exp(x(t))$, $f_2(t, x(t)) = 4x(t) - 5 - x^2(t)$, $g_1(t, x(t)) = x(t)$ and for almost all $t \in [0, 1]$, $u^0(t) = \hat{x}(t) = 0$ and $u^0(t) = \hat{x}(t) = 1$ are efficient solutions of the preceding problem.

Case 1: $u^0(t) = 0$ is an efficient solution.

The problem $P_1(u^0(\cdot))$ is given by

$$\begin{aligned} & \max \int_0^1 (-x(t) - t - \exp(x(t))) dt \\ & \text{s.t. } 4x(t) - x^2(t) \geq 0 \text{ a.e. in } [0, 1], \\ & x(t) \geq 0 \text{ a.e. in } [0, 1], \quad x(\cdot) \in L_\infty([0, 1]; \mathbb{R}). \end{aligned}$$

We will verify that corresponding system is regular and (MFCQ) is satisfied for $P_1(u^0(\cdot))$. It is obvious that $I_b(t) = J_b(t) = \{2\}$. It can be easily verified that for almost every $t \in [0, 1]$,

$$\nabla f_1(t, \hat{x}(t)) = \{-1 - \exp(\hat{x}(t))\}, \nabla f_2(t, \hat{x}(t)) = \{4 - 2\hat{x}(t)\}, \nabla g_1(t, \hat{x}(t)) = \{1\}.$$

Thus, constraint qualification (MFCQ) is satisfied for $\bar{\gamma}(t) = t + 1$ a.e. in $[0, 1]$ and $\beta = \frac{1}{2}$. Also, $I_0(t) = 1$ and $K := \text{cone}\{\nabla g_1(t, u^0(t))\} = \mathbb{R}_+$. The stationarity and normalization conditions become $-2\tilde{\lambda}_1^0(t) + 4\tilde{\lambda}_2^0(t) + \tilde{w}^0(t) = 0$, $\tilde{\lambda}_1^0(t) + \tilde{\lambda}_2^0(t) = 1$.

Then $\tilde{w}^0(t) = 6\tilde{\lambda}_1^0(t) - 4$. Therefore, conditions (16)–(18) are satisfied if and only if $\frac{2}{3} \leq \tilde{\lambda}_1^0(t) \leq 1$ a.e. in $[0, 1]$. This shows that the multipliers are not unique in this case. Let $\delta(t) = \tilde{\lambda}_1^0(t) - \lambda_1^0$. Obviously $-\delta(t) = \tilde{\lambda}_2^0(t) - \lambda_2^0$. Since $|J| = 2$ we get

$$\sum_{j \in J} (\tilde{\lambda}_j^0(t) - \lambda_j^0) \nabla f_j(t, u^0(t)) = (\tilde{\lambda}_1^0(t) - \lambda_1^0) v = \delta(t)v,$$

where $v := (\nabla f_1 - \nabla f_2)(t, u^0(t)) = -6$. If we assume that condition (CC_{act}) holds nontrivially for some non-constant multipliers, then the definition of (CC_{act}) implies that the inclusion $\delta(t)v \in K$ can hold for a.e. $t \in [0, 1]$ only if $\tilde{\lambda}_1^0(t) \leq \lambda_1^0$ for a.e. $t \in [0, 1]$. Since λ_1^0 is the average value of $\tilde{\lambda}_1^0(t)$, this is possible only if $\tilde{\lambda}_1^0(t) = \lambda_1^0$ and also $\tilde{\lambda}_2^0(t) = \lambda_2^0$ a.e. in $[0, 1]$. This shows that, in this example, condition (CC_{act}) is satisfied only in the trivial case, namely when the multiplier function $\tilde{\lambda}^0(\cdot)$ is constant.

Now, we show that corresponding system to $P_1(u^0(\cdot))$ is regular. Given $h \in \mathbb{R}$, for almost every t in $[0, 1]$, $\phi_1(t, h) = 2h$, $\phi_2(t, h) = -4h$, $\phi_3(t, h) = -h$. We have

$$I(t, h) = \begin{cases} \{1\}, & h > 0 \\ \{2\}, & h < 0. \end{cases}$$

The regularity of the system

$$\begin{aligned} \phi_1(t, h) &= 2h < 0, \\ \phi_2(t, h) &= -4h \leq 0, \\ \phi_3(t, h) &= -h \leq 0, \\ h &\in \mathbb{R}, \end{aligned} \tag{45}$$

is verified with $\bar{x}(\cdot) \equiv 1$, $R = 2$, $\alpha = \frac{1}{2}$ and, for almost all $t \in [0, 1]$, $e = e(t, h) = -1$ for $h \leq -1$ and $e = e(t, h) = 1$ for $h \geq 3$. Indeed, for $h \in \mathbb{R} \setminus (-1, 3)$, we have $\{3\} \notin I(t, h)$, $I(t, h) = \{1\}$, for $h \geq 3$, $I(t, h) = \{2\}$ for $h \leq -1$ and

1. $\langle \partial_h \phi_1, e \rangle \geq \alpha$, for $h \geq 3$ and $e = 1$,
2. $\langle \partial_h \phi_2, e \rangle \geq \alpha$, for $h \leq -1$ and $e = -1$.

In consequence of Theorem 4.3 there exist $(\lambda^0, w^0(\cdot)) \in \mathbb{R}^2 \times L_\infty([0, T]; \mathbb{R})$ such that $(u^0(\cdot), \lambda^0, w^0(\cdot))$ is a feasible solution for (WCTD) (i.e. (23)–(25) hold), that is, for almost every $t \in [0, 1]$,

$$\lambda_1^0(-1 - \exp(u^0(t))) + \lambda_2^0(4 - 2u^0(t)) + w^0(t) = 0.$$

The multipliers $\lambda_1^0 = 1$, $\lambda_2^0 = 0$ and $w^0(t) = 2$ satisfy the previous condition of stationarity. Moreover, Theorem 4.1 holds between (VCTP) and (WCTD), then for almost every t in $[0, 1]$, $(u^0(\cdot), \lambda^0, w^0(\cdot)) = (0, (1, 0), 2)$ is an efficient solution for (WCTD).

Case 2: $u^0(t) = 1$ is an efficient solution.

In this case, the problem $P_1(u^0(\cdot))$ is given by

$$\begin{aligned} & \max \int_0^1 (-x(t) - t - \exp(x(t))) dt \\ & \text{s.t. } 4x(t) - x^2(t) - 3 \geq 0 \text{ a.e. in } [0, 1], \\ & \quad x(t) \geq 0 \text{ a.e. in } [0, 1], \quad x(\cdot) \in L_\infty([0, 1]; \mathbb{R}). \end{aligned}$$

Now, show that corresponding system to $P_1(u^0(\cdot))$ is regular. Given $h \in \mathbb{R}$, for almost every t in $[0, 1]$, $\phi_1(t, h) = (1 + e)h$, $\phi_2(t, h) = -2h$, $\phi_3(t, h) = -1$. We have

$$\mathcal{I}(t, h) = \begin{cases} \{1\}, & h > 0 \\ \{2\}, & h < 0. \end{cases}$$

The regularity of the system

$$\begin{aligned} \phi_1(t, h) &= (1 + e)h < 0, \\ \phi_2(t, h) &= -2h \leq 0, \\ \phi_3(t, h) &= -1 \leq 0, \\ h &\in \mathbb{R}, \end{aligned} \tag{46}$$

is verified with $\bar{x}(\cdot) \equiv 0$, $R = 1$, $\alpha = \frac{1}{2}$ and, for almost all $t \in [0, 1]$, $e = e(t, h) = -1$ for $h \leq -1$ and $e = e(t, h) = 1$ for $h \geq 1$. Indeed, for $h \in \mathbb{R} \setminus (-1, 1)$, we have $\{3\} \notin \mathcal{I}(t, h)$, $\mathcal{I}(t, h) = \{1\}$, for $h \geq 1$, $\mathcal{I}(t, h) = \{2\}$ for $h \leq -1$ and

1. $\langle \partial_h \phi_1, e \rangle \geq \alpha$, for $h \geq 1$ and $e = 1$,
2. $\langle \partial_h \phi_2, e \rangle \geq \alpha$, for $h \leq -1$ and $e = -1$.

Also, $I_0(t) = \{\emptyset\}$ and $K = \{0\}$. It is clear that (CC_{act}) trivially holds. Indeed, by the complementary slackness (constraint g_1 is not active), we obtain that $\tilde{w}^0(t) = 0$. Also, by stationarity

$$\tilde{\lambda}_1^0(t) \overbrace{(-1 - \exp(u^0(t)))}^1 + \tilde{\lambda}_2^0(t) \overbrace{(4 - 2u^0(t))}^1 + \overbrace{\tilde{w}^0(t)}^0 = 0$$

and $\tilde{\lambda}_1^0(t) + \tilde{\lambda}_2^0(t) = 1$, we have

$$\tilde{\lambda}_1^0(t) \equiv \frac{2}{3 + \exp(1)}, \quad \tilde{\lambda}_2^0(t) \equiv \frac{1 + \exp(1)}{3 + \exp(1)}, \quad \tilde{w}^0(t) \equiv 0 \text{ a.e. in } [0, 1].$$

In this case, for a given efficient solution, the multipliers must be constants and (CC_{act}) trivially holds. Moreover, Theorem 4.1 holds between (VCTP) and (WCTD), then for almost every $t \in [0, 1]$, $(1, (\frac{2}{3 + \exp(1)}, \frac{1 + \exp(1)}{3 + \exp(1)}), 0)$ is an efficient solution for (WCTD).

5. Conclusion

In this paper we studied a smooth vector continuous-time programming problem with inequality constraints given almost everywhere and developed two parallel duality models of Mond–Weir and Wolfe type.

The main contributions can be summarized as follows:

- We formulated continuous-time versions of the Mond–Weir and Wolfe duals for the same primal (VCTP);
- We proved weak duality theorems under natural concavity/pseudoconcavity assumptions on the data;

- We established strong duality theorems under a continuous-time (MFCQ) (and, in the concave case, under a Slater-type condition);
- In the examples, we have shown the validity of the theoretical results.

These results place vector continuous-time duality on a firmer and more transparent basis, and they provide a unified framework that can be further adapted to nonsmooth data, generalized convexity settings and multiobjective optimal control. Future research may focus on extending the present approach to the original nonsmooth problem and on deriving optimality and duality results without relying on concavity / convexity assumptions.

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