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On Parametric Copositive Programming Problems

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Abstract: We consider a class of one-parametric copositive programming problems. The aim of the paper is to describe the behavior of solutions to these parametric problems under small parameter perturbations. To do this, we explore the properties of the unperturbed copositive programming problem and show that, under some assumptions, it is closely related to a special type of block semi-infinite programming problem. Taking into account this relationship and the known results on properties of the complementary set of the copositive cone, we obtain a system of a finite number of bi-linear and linear equations with respect to a finite number of variables and a parameter. Under the assumptions made, for any parameter value from a neighborhood of unperturbed parameter value, the system has a unique solution, which allows us to construct optimal primal and dual solutions to the corresponding parametric copositive programming problem.

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1. Introduction

A conic optimization problem is characterized by a constraint stipulating that the optimization variables must belong to a closed convex cone. Linear conic optimization deals with the optimization of a linear function over a feasible set given by the intersection of an affine space and a convex cone. Conic optimization is an important subfield of convex optimization, not only because its theoretical results are actively used in optimization, enabling one to model many non-linearities arising in practice with the help of the conic structure of the feasible set, but also because it has important applications, whose number is growing and the software is developing rapidly (see, *e.g.* [3, 20], and the references therein).

Copositive programming (CoP) problems are linear conic optimization problems over the cone $COP(p)$ of the so-called copositive $p \times p$ matrices (*i.e.*, matrices that are positive semidefinite on the non-negative orthant). Copositive models occur in many important applications. For the references on motivation and application of CoP see, *e.g.* [6, 9, 12], and the references given there.

In this paper, we explore parametric copositive programming problems. We study the behavior of solutions to these problems under small perturbations of the problem's data. Such a study is of great importance since it forms the basis of sensitivity analysis and allows for understanding the qualitative behavior of the optimal solution and the optimal value of the cost function as the parameters vary. Knowledge of the parameter's dependence on problem's parameters (data) is absolutely necessary for a complete understanding of the optimization problem. Nevertheless, so far, there are few publications devoted to the sensitivity analysis of conic optimization problems. There are some results related to parametric semidefinite and second-order programming problems [4, 8, 13, 21, 27]. As far as we know, at the moment there are no results on parametric conic optimization problems over the cone $COP(p)$.

It is known that copositive programming problems are particular cases of semi-infinite programming (SIP) problems, and there are some results on parametric SIP problems [15, 26]. Unfortunately, these results can not be applied to CoP problems since the studies in [15, 26] are performed under the assumption that, for an optimal solution, the set of active indices consists of a finite number of elements. However, a specific

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feature of CoP problems is that, for an optimal solution, the set of active indices contains a continuum of elements. In the paper, taking into account specific properties of CoP problems and the results obtained in [16], we managed to overcome this difficulty under some non-restrictive assumptions.

The remainder of the paper is structured as follows. In Section 2, we introduce the basic notation and formulate the problem addressed in the paper. In Section 3, we provide necessary definitions and some results on properties of copositive and completely positive matrices for later use. In Section 4, we impose assumptions on solutions to a pair of unperturbed dual CoP problems and analyze properties of solutions to these problems. Section 5 is devoted to exploring properties of a pair of parametric dual CoP problems. Under the assumptions made on solutions to the corresponding unperturbed dual CoP problems, we obtain a system of bi-linear and linear equations, which unambiguously determine optimal solutions to the parametric problems as functions of a parameter in a neighborhood of a given parameter value. An illustrative example is presented in Section 6. The paper ends with some conclusions presented in Section 7.

2. Notation and problem statement

Let p be a positive integer and $[p] := \{1, \dots, p\}$. Let $\mathbb{R}^p$ stay for the p-dimensional Euclidean vector space with the standard orthogonal basis $\{e_k, k \in [p]\}$ and denote by $\mathbb{R}^p_+$ the set of element-wise non-negative p-vectors. For $t = (t_k, k \in [p])^\top \in \mathbb{R}^p$, we denote its support by $\mathrm{supp}(t) := \{k \in [p] : t_k \neq 0\}$, and use the norm

$$\|t\|_1 = \sum_{k=1}^{p} |t_k|.$$

We denote by $\mathbb{S}(p)$ the real linear space of symmetric $p \times p$ matrices. For $U \in \mathbb{S}(p)$ and $W \in \mathbb{S}(p)$, the inner product is defined by $U \bullet W = \mathrm{trace}(UW)$. For a set $\mathcal{L} \subset \mathbb{S}(p)$, we denote by $\mathrm{int}\,\mathcal{L}$, $\mathrm{ri}\,\mathcal{L}$, $\mathrm{conv}\,\mathcal{L}$, and $\mathrm{cone}\,\mathcal{L}$ the interior, relative interior, convex hull, and conic hull of $\mathcal{L}$, respectively. We denote by $\mathrm{span}\,\mathcal{L}$ the space spanned by $\mathcal{L}$, and by $\mathcal{L}^\perp$ the orthogonal complement of its span. We use $\mathbb{O}(p)$, $E(p)$, and $\mathbf{0}$ to denote the $p \times p$ matrix of all zeroes, identity $p \times p$ matrix, and a null vector. The dimension of the vector will be clear from the context.

For a matrix $F \in \mathbb{S}(p)$, we denote by F_{kq}, $k \in [p]$, $q \in [p]$, its elements and define the vector $\mathrm{svec}(F) \in \mathbb{R}^{p(p+1)/2}$ by the following rule (see [2]):

$$\mathrm{svec}(F) = (F_{11}, \sqrt{2}F_{21}, \dots, \sqrt{2}F_{p1}, F_{22}, \sqrt{2}F_{32}, \dots, \sqrt{2}F_{p2}, \dots, F_{pp})^\top.$$

For a set of symmetric $p \times p$ matrices $F(i)$, $i \in I$, we define:

$$\mathrm{rank}(F(i),\ i \in I) := \mathrm{rank}(\mathrm{svec}(F(i)),\ i \in I).$$

For a cone $\mathcal{K} \subset \mathbb{S}(p)$, we denote by $\mathcal{K}^*$ its dual cone

$$\mathcal{K}^* := \{U \in \mathbb{S}(p) : U \bullet X \geq 0\ \forall X \in \mathcal{K}\}.$$

We use $\mathbb{S}_+(p)$ to denote the cone of symmetric positive semidefinite $p \times p$ matrices and $COP(p)$ to denote the cone of copositive $p \times p$ matrices

$$COP(p) := \{D \in \mathbb{S}(p) : t^\top D t \geq 0\ \forall t \in \mathbb{R}^p_+\}.$$

It is easy to see that the cone $COP(p)$ can be equivalently defined as follows:

$$COP(p) := \{D \in \mathbb{S}(p) : t^\top D t \geq 0\ \forall t \in T\} \text{ where } T := \{t \in \mathbb{R}^p_+ : \|t\|_1 = 1\}.$$

The cone $COP(p)$ is not self-dual, its dual cone is the cone of completely positive $p \times p$ matrices

$$CP(p) := (COP(p))^* = \mathrm{cone}\{t\, t^\top : t \in \mathbb{R}^p_+\}.$$

It follows from the above that any matrix $U \in CP(p)$ admits a representation $U = \sum\limits_{i \in I} t(i)(t(i))^\top$ with some finite index set I and vectors $t(i) \in \mathbb{R}^p_+ \setminus \{\mathbf{0}\}$ for all $i \in I$. In the literature (see, *e.g.* [5]), such a representation is called cp-factorization of a completely positive matrix U. Note that a matrix $U \in CP(p)$ may have several different cp-factorizations.

For a copositive matrix $X \in COP(p)$, a vector $t \in \mathbb{R}^p_+$, $t \neq \mathbf{0}$, is called a zero of X if $t^\top X t = 0$. Clearly, any positive scaling of a zero remains a zero; therefore, one can assume, without loss of generality, that $t \in T$,

in which case t is referred to as normalized. For simplicity, we will henceforth assume that all zeros are normalized and omit the term "normalized" in subsequent discussions.

In the paper, we consider a parametric linear copositive programming problem in the form

$$\mathbf{P}(\varepsilon): \qquad \min_{x\in\mathbb{R}^n} c^\top(\varepsilon)x \quad \text{s.t. } \mathcal{X}(\varepsilon,x) := \sum_{m=1}^{n} A_m x_m + A_0(\varepsilon) \in COP(p),$$

where $x = (x_1,\dots,x_n)^\top$ is the vector of decision variables, matrices $A_m \in \mathbb{S}(p)$, $m \in [n]$, are given, $c(\varepsilon) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $A_0(\varepsilon) \in \mathbb{S}(p)$, $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$, $\varepsilon_0 > 0$, are given sufficiently smooth vector and matrix functions.

For problem ($\mathbf{P}(\varepsilon)$), the Lagrange dual problem takes the form

$$\mathbf{D}(\varepsilon): \qquad \max -U\bullet A_0(\varepsilon),\ \text{s.t. } U\bullet A_m = c_m(\varepsilon)\ \forall m\in[n];\ U\in CP(p),$$

where the matrix $U \in \mathbb{S}(p)$ is the decision variable.

The following theorem is a well known duality result (see, for example, [22]).

Theorem 2.1. *For a fixed $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$, a vector $x^*(\varepsilon)$ and a matrix $U^*(\varepsilon)$ are optimal solutions to problems ($\mathbf{P}(\varepsilon)$) and ($\mathbf{D}(\varepsilon)$) if the following relations hold true:*

$$\mathcal{X}(\varepsilon,x^*(\varepsilon)) \in COP(p),\ U^*(\varepsilon)\in CP(p),\ \mathcal{X}(\varepsilon,x^*(\varepsilon))\bullet U^*(\varepsilon) = 0,$$
$$U^*(\varepsilon)\bullet A_m = c_m(\varepsilon)\ \forall\, m\in[n]. \tag{1}$$

Without loss of generality, we will assume that the matrices A_m, $m \in [n]$, are linearly independent.

Note that under this assumption, for any feasible (or optimal) solution $x(\varepsilon)\in\mathbb{R}^n$ ($x^*(\varepsilon)\in\mathbb{R}^n$) to problem ($\mathbf{P}(\varepsilon)$), the matrix $X(\varepsilon):=\mathcal{X}(\varepsilon,x(\varepsilon))$ ($X^*(\varepsilon):=\mathcal{X}(\varepsilon,x^*(\varepsilon))$) is uniquely determined by this vector $x(\varepsilon)$ ($x^*(\varepsilon)$). Thus, we will refer to both $X(\varepsilon)$ and $x(\varepsilon)$ ($X^*(\varepsilon)$ and $x^*(\varepsilon)$) as a feasible (or optimal) solution to problem ($\mathbf{P}(\varepsilon)$).

The aim of this paper is to study properties of the solutions $x^*(\varepsilon)$ and $U^*(\varepsilon)$ to problems ($\mathbf{P}(\varepsilon)$) and ($\mathbf{D}(\varepsilon)$) in a neighborhood of the unperturbed parameter value $\varepsilon = 0$.

At $\varepsilon = 0$, the corresponding problems ($\mathbf{P}(\varepsilon)$) and ($\mathbf{D}(\varepsilon)$) are referred to as the unperturbed problems and are denoted by ($\mathbf{P}$) and ($\mathbf{D}$), respectively.

To achieve our main goal, we begin with preliminaries and an analysis of the properties of the solutions to unperturbed problems ($\mathbf{P}$) and ($\mathbf{D}$).

3. Preliminaries

In this section, we present some properties that hold true for arbitrary matrices $X^* \in COP(p)$ and $U^* \in CP(p)$. All these properties will be used below.

For $X^* \in COP(p)$, denote by T_0 the corresponding set of (normalized) zeros

$$T_0 = T_0(X^*) := \{t\in T : t^\top X^* t = 0\},\ \text{where } T := \{t\in\mathbb{R}^p_+ : \|t\|_1 = 1\}.$$

Suppose that $T_0 \neq \emptyset$. A zero $t \in T_0$ is called a minimal zero if there does not exist a zero $\tau \in T_0$ such that strict inclusion $\operatorname{supp}(\tau) \subset \operatorname{supp}(t)$ holds true. It is known (see [14]) that for a given $X^* \in COP(p)$ there exists a finite number of minimal zeros.

Denote by

$$\mathcal{T} = \{\tau(j), j\in J\},\ \text{where } J\subset\mathbb{N},\ |J|<\infty,$$

the set of all minimal zeros of X^* and define

$$M(j) = M(j,X^*) := \{k\in[p] : e_k^\top X^*\tau(j) = 0\},\ j\in J. \tag{2}$$

For a given $X^* \in COP(p)$, using $\tau(j),\ M(j), j \in J$, define the set of pairs of indices

$$\mathcal{V} := \{(i,j) : i \in J,\ j \in J,\ i < j,\ \text{supp}(\tau(i)) \subseteq M(j)\}$$

and consider an undirected graph $G(X^*) := (J, \mathcal{V})$ with the set of vertices J and the set of edges $\mathcal{V}$.

Let $\{J(s), s \in S\}$ be the set of all maximal cliques of the graph $G(X^*)$. It was shown in [19] that the set T_0 can be represented in the form

$$T_0 = \bigcup_{s \in S} T_0(s),\ \text{where } T_0(s) := \text{conv}\{\tau(j), j \in J(s)\},\ \forall s \in S. \tag{3}$$

Denote

$$P_*(s) := \bigcup_{i \in J(s)} \text{supp}(\tau(i)) \subseteq [p],\ s \in S. \tag{4}$$

It is easy to see that

$$P_*(s) \subseteq M(j), j \in J(s), s \in S. \tag{5}$$

Note that, for a given $X^* \in COP(p)$, applying the algorithm from [19], one can find the corresponding set of minimal zeros and the set of maximal cliques of the graph $G(X^*)$. Hence one can easily construct the sets $P_*(s)$, $s \in S$.

Proposition 3.1. *Let $X^* \in COP(p)$ and $P_*(s), s \in S$, be the corresponding sets defined in (4). Then, for any $s \in S$, the sub-matrix $\begin{pmatrix} X^*_{kq}, q \in P_*(s) \\ k \in P_*(s) \end{pmatrix}$ of X^* is positive semi-definite.*

Proof. Let us consider a fixed $s \in S$. By construction we have $T_0(s) = \text{conv}\{\tau(j),\ j \in J(s)\} \subseteq T_0$. Hence $\tau^c(s) := \frac{1}{|J(s)|} \sum\limits_{j \in J(s)} \tau(j) \in T_0$ and $\text{supp}(\tau^c(s)) = P_*(s)$. Then the statement of the proposition follows from Lemma 2.4 in [11]. □

Let us recall some definitions commonly used in conic optimization (see [23]).

Definition 3.2. For $X^* \in COP(p)$ and $U^* \in CP(p)$, we say that these matrices are *complementary* if $X^* \bullet U^* = 0$. We say that X^* is strictly complementary to U^* if

$$X^* \in \text{ri}(COP(p) \cap U^{*\perp}) \tag{6}$$

and U^* is strictly complementary to X^* if

$$U^* \in \text{ri}(CP(p) \cap X^{*\perp}). \tag{7}$$

Let us introduce some notation. Denote

$$V(I) := \{(i,j) : i \in I,\ j \in I,\ i \le j\} \text{ for a finite index set } I \subset \mathbb{N},$$
$$\xi(i,j) := (\tau(i) + \tau(j))\ \forall (i,j) \in V(J(s)),\ s \in S. \tag{8}$$

Here $\{\tau(j), j \in J\}$ is the set of minimal zeros of $X^* \in COP(p)$ and $\{J(s), s \in S\}$ is the set of maximal cliques of $G(X^*)$.

For $s \in S$, let $J_b(s) \subseteq J(s)$ be a subset of $J(s)$ such that

$$\text{rank}(\tau(j), j \in J(s)) = \text{rank}(\tau(j), j \in J_b(s)) = |J_b(s)|. \tag{9}$$

Notice that it follows from Lemma 6.2 in [10], that given $s \in S$ and the vectors $\xi(i,j), (i,j) \in V(J(s))$ defined in (8),

$$\text{the matrices } \xi(i,j)\xi(i,j)^\top,\ (i,j) \in V(J_b(s)),\ \text{are linearly independent.} \tag{10}$$

Using (9) and (10), it is not difficult to show that

$$\operatorname{rank}(t\,t^\top, t \in T_0(s)) = \operatorname{rank}\big(\xi(i,j)\xi(i,j)^\top,\ (i,j) \in V(J_b(s))\big)\ \forall s \in S. \tag{11}$$

In what follows, we will need the following proposition proved in [16].

Proposition 3.3. *Let $U^* \in CP(p)$, $X^* \in COP(p)$, and suppose that U^* is strictly complementary to X^*. Then the matrix U^* admits a representation of the form*

$$U^* = \sum_{s\in S} \mathcal{U}^*(s),\ \text{where}\ \mathcal{U}^*(s) = \sum_{(i,j)\in V(J(s))} \alpha_{ij}\xi(i,j)\xi(i,j)^\top + \sum_{i\in I(s)} \alpha_i t(i)t(i)^\top, \tag{12}$$
$$\text{with some } \alpha_{ij} > 0, (i,j) \in V(J(s));\ \alpha_i > 0,\ t(i) \in T_0(s),\ i \in I(s);\ s \in S.$$

Here the sets $T_0(s)$ and vectors $\xi(i,j)$, $(i,j) \in V(J_b(s))$, $s \in S$, are defined in (3) and (8) using minimal zeros of the matrix X^*.

Notice that for some $s \in S$, the set $I(s)$ may be empty.

4. Unperturbed CoP problem: assumptions and solution properties

For parametric CoP problems (**P**(ε)), (**D**(ε)), let us consider a pair of the corresponding unperturbed dual CoP problems:

$$\mathbf{P}: \quad \min_{x\in\mathbb{R}^n} c^\top x \quad \text{s.t. } \mathcal{X}(x) := \sum_{m=1}^n A_m x_m + A_0 \in COP(p),$$
$$\mathbf{D}: \quad \max -U\bullet A_0,\ \text{s.t. } U\bullet A_m = c_m\ \forall m \in [n];\ U \in CP(p),$$

where $x = (x_1, \dots, x_n)^\top$ is the vector of primal decision variables, and the matrix $U \in \mathbb{S}(p)$ is the dual decision variable. The matrices $A_m \in \mathbb{S}(p)$ for $m \in [n]$, $A_0 = A_0(0) \in \mathbb{S}(p)$, and the vector $c = c(0) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, are defined as in problem (**P**(ε)) with $\varepsilon = 0$.

Since the matrices A_m, $m \in [n]$, are linearly independent, for a feasible (an optimal) solution $x \in \mathbb{R}^n$ ($x^* \in \mathbb{R}^n$) to problem (**P**), the matrix $\mathcal{X}(x)$ ($\mathcal{X}(x^*)$) is uniquely determined by the vector x (or x^*), and vice versa. Thus, we will refer to both $\mathcal{X}(x)$ and x (or $\mathcal{X}(x^*)$ and x^*) as a feasible (or optimal) solution to problem (**P**).

4.1. Assumptions on solutions to unperturbed problems (P) and (D)

Let x^* and U^* be optimal solutions to problems (**P**) and (**D**), respectively, and matrices $X^* = \mathcal{X}(x^*)$ and U^* be complementary. Hence, the following conditions hold true:

$$X^* \in COP(p),\ U^* \in CP(p),\ X^*\bullet U^* = 0,\ U^*\bullet A_m = c_m\ \forall m \in [n]. \tag{13}$$

These relations and the assumption $c \neq \mathbf{0}$ imply that the set $T_0 = T_0(X^*)$ of all zeros of X^* is non-empty.

Since $X^* \in COP(p)$ and $U^* \in CP(p)$, all constructions and results from Section 3 can be applied to them. In particular, one can find the set $\{\tau(j),\ j \in J\}$ of all minimal zeros of X^*, define the sets $M(j)$, $j \in J$, by the rules (2), form a graph $G(X^*)$, find the set $\{J(s),\ s \in S\}$ of its maximal cliques, describe the polytopes $T_0(s) = \operatorname{conv}\{\tau(j),\ j \in J(s)\}$, $s \in S$, forming the set T_0 (see (3)), and determine the set $P_*(s)$, $s \in S$, by the rules (4).

Note that it follows from (13) that the matrix U^* admits a representation

$$U^* = \sum_{s\in S} \mathcal{U}^*(s),\ \mathcal{U}^*(s) \in \mathcal{F}(s) := \operatorname{cone}\{t\,t^\top, t \in T_0(s)\} \subset CP(p)\ \forall s \in S. \tag{14}$$

Let us impose some assumptions on solutions X^* and U^* to unperturbed problems (**P**) and (**D**).

As a first step, we strengthen the relations given in (5) by introducing the following assumption.

Assumption 4.1. *Suppose that* $P_*(s) = M(j)\ \forall j \in J(s),\ \forall s \in S.$

It is not difficult to show that Assumption 1 implies that

$$T_0(s) \cap T_0(\bar{s}) = \emptyset\ , \forall s \in S,\ \forall \bar{s} \in S,\ s \neq \bar{s},$$

where $T_0(s),\ s \in S$, are the polytopes forming the set T_0, see (3).

Now, let us strengthen conditions (10) by making the following assumption.

Assumption 4.2. *The matrices* $\xi(i,j)\xi(i,j)^\top,\ (i,j) \in V(J_b(s)),\ s \in S$, *are linearly independent.*

It follows from Proposition 4.5 in [17] that under Assumption 4.2, for a given optimal dual solution U^*, there is a unique representation in the form (14), *i.e.*, there is no other set of matrices $\{\bar{\mathcal{U}}(s),\ s \in S\}$ such that $U^* = \sum_{s\in S} \bar{\mathcal{U}}(s)$, $\bar{\mathcal{U}}(s) \in \mathcal{F}(s)$ for all $s \in S$. Note that this assumption does not exclude the possibility that, for some $s \in S$, the matrix $\mathcal{U}^*(s)$ in a unique representation (14) may admit multiple cp-factorizations.

4.2. Properties of optimal solutions to problems (P), (D)

In this subsection, we establish some properties of optimal solutions to the pair of problems (**P**) and (**D**) that will be used in what follows.

Let X^* and U^* be optimal solutions to problems (**P**) and (**D**), respectively. Let $\{\tau(j), j \in J\}$ be the set of minimal zeros of X^*, and let $\{J(s), s \in S\}$ be the set of all maximal cliques of the graph $G(X^*)$. Furthermore, let the sets $P_*(s) \subseteq [p]$, for $s \in S$, be defined as in (4).

For each $s \in S$, using the given set $P_*(s)$ and the number $p(s) := |P_*(s)|$, we introduce the following matrix mappings:

$$\mathcal{A}(X,s) : \mathbb{S}(p) \to \mathbb{S}(p(s)) \text{ and } \mathcal{B}(W,s) : \mathbb{S}(p(s)) \to \mathbb{S}(p),$$

defined by the rules

$$\mathcal{A}(X,s) =: X(s) = \begin{pmatrix} X_{kq}, q \in P_*(s) \\ k \in P_*(s) \end{pmatrix},\ \mathcal{B}(W,s) =: \mathcal{U}(s) = \begin{pmatrix} \mathcal{U}_{kq}, q \in [p] \\ k \in [p] \end{pmatrix},$$
$$\mathcal{U}_{kq} = W_{kq}\ \forall k \in P_*(s),\ \forall q \in P_*(s),\ \mathcal{U}_{kq} = 0\ \forall k \in [p] \setminus P_*(s),\ \forall q \in [p], \tag{15}$$

where X_{kq} and W_{kq} are the (k,q)-th elements of matrices X and W, respectively.

For the minimal zeros $\tau(j) = (\tau_k(j),\ k \in [p])^\top,\ j \in J$, and the vectors $\xi(i,j) = (\xi_k(i,j),\ k \in [p])^\top,\ (i,j) \in V(J(s)),\ s \in S$, (see (8)), let us define sub-vectors

$$\tau(s,j) := (\tau_k(j),\ k \in P_*(s))^\top, j \in J(s),\ s \in S, \tag{16}$$
$$\xi(s,i,j) := (\xi_k(i,j),\ k \in P_*(s))^\top,\ (i,j) \in V(J(s)),\ \ s \in S. \tag{17}$$

Proposition 4.3. *Let* $X^* = \mathcal{X}(x^*)$ *be an optimal solution to problem* (**P**) *and* $Y^*(s) := \mathcal{A}(\mathcal{X}(x^*), s)$, $s \in S$. *Then, for* $s \in S$, *any matrix* $W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp$ *can be represented in the form*

$$W(s) = \sum_{(i,j)\in V(J_b(s))} \gamma_{ij}\xi(s,i,j)\xi(s,i,j)^\top \text{ with some } \gamma_{ij} \in \mathbb{R},\ (i,j) \in V(J_b(s)), \tag{18}$$

where $J_b(s)$ *is a subset of* $J(s)$ *satisfying* (9).

Proof. Notice that it follows from Proposition 3.1 that $Y^*(s) \in \mathbb{S}_+(p(s))$ for all $s \in S$. Fix an index $s \in S$ and consider a sub-space

$$H(Y^*(s)) := \{\eta \in \mathbb{R}^{p(s)} : Y^*(s)\eta = \mathbf{0}\},$$

and a vector $\tau^*(s) := \sum\limits_{j\in J(s)} \tau(s,j)$, where $\tau(s,j),\ j \in J(s)$, are defined in (16). Since $Y^*(s) \in \mathbb{S}_+(p(s))$, $\tau^*(s) > \mathbf{0}$ and $H(Y^*(s))\tau^*(s) = 0$, it follows from Proposition A.3 in [16] that $H(Y^*(s)) = \text{span}\{\tau(s,j), j \in J(s)\}$.

Taking into account that, by construction, $\tau_k(j) = 0\ \forall k \in [p] \setminus P_*(s), \forall j \in J(s)$, we conclude that condition (9) can be equivalently rewritten as

$$\text{rank}\big(\tau(s,j), j \in J(s)\big) = \text{rank}\big(\tau(s,j), j \in J_b(s)\big) = |J_b(s)|.$$

It follows from these equalities and the equality $H(Y^*(s)) = \mathrm{span}\{\tau(s,j), j \in J(s)\}$, that for any $\eta \in H(Y^*(s))$, there exist numbers $\alpha_i = \alpha_i(\eta)$, $i \in J_b(s)$, such that $\eta = \sum_{i\in J_b(s)} \alpha_i \tau(s,i)$. Hence,

$$\eta\eta^\top = \Big(\sum_{i\in J_b(s)} \alpha_i \tau(s,i)\Big)^\top \Big(\sum_{j\in J_b(s)} \alpha_j \tau(s,j)\Big)$$
$$= \sum_{\substack{(i,j)\in V(J_b(s))\\ i\neq j}} \alpha_i\alpha_j\big(\tau(s,i)\tau(s,j)^\top + \tau(s,j)\tau(s,i)^\top\big) + \sum_{i\in J_b(s)} \alpha_i^2\, \tau(s,i)\tau(s,i)^\top. \tag{19}$$

Notice that

$$\tau(s,i)\tau(s,j)^\top + \tau(s,j)\tau(s,i)^\top$$
$$= \xi(s,i,j)\xi(s,i,j)^\top - \frac{1}{4}\Big(\xi(s,j,j)\xi(s,j,j)^\top + \xi(s,i,i)\xi(s,i,i)^\top\Big), \quad \forall (i,j) \in V(J(s)).$$

It follows from these equalities and (19), that for any $\eta \in H(Y^*(s))$, the matrix $\eta\eta^\top$ can be represented in the form

$$\eta\eta^\top = \sum_{(i,j)\in V(J_b(s))} \alpha_{ij}\xi(s,i,j)\xi(s,i,j)^\top \text{ with some } \alpha_{ij} \in \mathbb{R},\ (i,j) \in V(J_b(s)).$$

This implies that

$$\mathrm{rank}\big(\eta\eta^\top, \eta \in H(Y^*(s))\big) = \mathrm{rank}\big(\xi(s,i,j)\xi(s,i,j)^\top, (i,j) \in V(J_b(s))\big). \tag{20}$$

By assumption, $W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp$. Hence, the matrix $W(s)$ admits a representation

$$W(s) = \sum_{i\in\tilde{I}} \eta(i)\eta(i)^\top \text{ with some index set } \tilde{I} \text{ and } \eta(i) \in H(Y^*(s)),\ i \in \tilde{I}.$$

It follows from this representation and equality (20), that (18) holds true. □

Define the matrices $B_m(s) := \mathcal{A}(A_m, s),\ m = 0, 1, \ldots, n,\ s \in S$.

Proposition 4.4. *Let the dual problems* (**P**) *and* (**D**) *have unique optimal solutions* $X^* = \mathcal{X}(x^*)$ *and* U^*, *and let* U^* *be strictly complementary to* X^*. *Suppose that the Assumptions* 4.1 *and* 4.2 *hold true. Then the following properties,* (C1)–(C5), *are satisfied for*

$$Y^*(s) := \mathcal{A}(\mathcal{X}(x^*), s) = \sum_{m=1}^n B_m(s)x_m^* + B_0(s),\ W^*(s) := \mathcal{A}(\mathcal{U}^*(s), s),\ s \in S, \tag{21}$$

where the matrices $\mathcal{U}^*(s)$, $s \in S$, *are defined in* (14).

(C1) *The following relations hold true:*

$$Y^*(s) \in \mathbb{S}_+(p(s)),\ W^*(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp\ \ \forall s \in S;$$
$$\sum_{s\in S} B_m(s)\bullet W^*(s) = c_m\ \ \forall m \in [n]. \tag{22}$$

(C2) *The following inclusions hold true:*

$$W^*(s) \in \mathrm{ri}(\mathbb{S}_+(p(s)) \cap Y^*(s)^\perp)\ \forall s \in S, \tag{23}$$
$$Y^*(s) \in \mathrm{ri}(\mathbb{S}_+(p(s)) \cap W^*(s)^\perp)\ \forall s \in S. \tag{24}$$

(C3) *There is a unique set of matrices* $\{W^*(s), s \in S\}$ *satisfying properties* (C1).

(C4) *There is a unique vector* $x^* \in \mathbb{R}^n$ *satisfying properties* (C1).

(C5) *The equalities* $\sum_{m=1}^n B_m(s)y_m = \mathbb{O}(p(s)),\ s \in S$, *imply the equalities* $y_m = 0\ \forall m \in [n]$.

Proof. Let us prove property (C1). In fact, it was shown above that the matrix U^* admits representation (14): $U^* = \sum\limits_{s\in S} \mathcal{U}^*(s)$, $\mathcal{U}^*(s) \in \mathcal{F}(s)\ \forall s \in S$. It follows from the latter inclusions that for $s \in S$, we have

$$e_k^\top \mathcal{U}^*(s) e_q = 0 \text{ if } k \in [p] \setminus P_*(s) \text{ or } q \in [p] \setminus P_*(s).$$

By assumption, the matrices $W^*(s), s \in S$, are defined by the rules (21) using the matrices $\mathcal{U}^*(s)$, $s \in S$. Consequently, $A_m \bullet \mathcal{U}^*(s) = B_m(s) \bullet W^*(s)$, $m \in [n]$, from which it follows that

$$A_m \bullet U^* = \sum_{s\in S} B_m(s) \bullet W^*(s) = c_m,\ m \in [n]. \tag{25}$$

Since U^* and $\mathcal{X}(x^*)$ are complementary, we have $0 = \mathcal{X}(x^*) \bullet U^* = \sum\limits_{s\in S} \mathcal{X}(x^*) \bullet \mathcal{U}^*(s)$.

Then, taking into account the inclusions $\mathcal{X}(x^*) \in COP(p)$, $\mathcal{U}^*(s) \in CP(p)\ \forall s \in S$, we obtain

$$\mathcal{X}(x^*) \bullet \mathcal{U}^*(s) = 0 \implies Y^*(s) \bullet W^*(s) = 0\ \forall s \in S. \tag{26}$$

By construction, $W^*(s) \in \mathcal{CP}(p(s)) \subset \mathbb{S}_+(p(s))$, and it follows from Proposition 3.1 that $Y^*(s) := \mathcal{A}(\mathcal{X}(x^*), s) \in \mathbb{S}_+(p(s))$, $s \in S$. These inclusions and equalities (25), (26) imply the relations (22).

Now, let us prove property (C2). By assumption, U^* is strictly complementary to X^*, hence according to Proposition 3.3, the matrix U^* admits representation (12). This implies that

$$W^*(s) := \mathcal{A}(\mathcal{U}^*(s), s) = \sum_{(i,j)\in V(J(s))} \alpha_{ij} \xi(s,i,j) \xi(s,i,j)^\top + \sum_{i\in I(s)} \alpha_i t(s,i) t(s,i)^\top, \tag{27}$$

where

$$\alpha_{ij} > 0,\ (i,j) \in V(J(s));\ \alpha_i > 0,\ i \in I(s),$$
$$t(s,i) := (t_k(i), k \in P_*(s))^\top \geq \mathbf{0},\ i \in I(s),$$
$$\xi(s,i,j) := (\xi_k(i,j), k \in P_*(s))^\top,\ (i,j) \in V(J(s)),\ s \in S.$$

Moreover, it was shown (see property (C1)) that $W^*(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp\ \ \forall s \in S$.
Notice that, for $s \in S$, condition (23) is equivalent to the following one:

$$\forall W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp\ \ \exists \gamma > 0 \text{ such that } W^*(s) - \gamma W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp.$$

Let us fix $s \in S$ and consider any matrix $W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp$. It follows from Proposition 4.3 that matrix $W(s)$ can be represented in the form (18). Using this representation and (27), and settting $\gamma_{ij} = 0$ for $(i,j) \in V(J(s)) \setminus V(J_b(s))$, $s \in S$, we obtain

$$W^*(s) - \gamma W(s) = \sum_{i\in I(s)} \alpha_i t(s,i) t(s,i)^\top + \sum_{(i,j)\in V(J(s))} (\alpha_{ij} - \gamma\gamma_{ij}) \xi(s,i,j) \xi(s,i,j)^\top.$$

For a sufficiently small $\gamma > 0$, the inequalities $\alpha_{ij} - \gamma\gamma_{ij} \geq 0$, $(i,j) \in V(J(s))$, are satisfied. Consequently, we have shown that, for a sufficiently small $\gamma > 0$, it holds $W^*(s) - \gamma W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp$. Hence, we have proved the inclusions (23).

Notice, that it is not difficult to show that for $W \in \mathbb{S}_+(k)$ and $X \in \mathbb{S}_+(k)$, the following equivalences hold:

$$W \in \mathrm{ri}\,(\mathbb{S}_+(k) \cap X^\perp) \iff X \in \mathrm{ri}\,(\mathbb{S}_+(k) \cap U^\perp) \iff (W + X) \in \mathrm{int}(\mathbb{S}_+(k)).$$

Hence, it follows from the inclusions $Y^*(s) \in \mathbb{S}_+(p(s))$, $W^*(s) \in \mathbb{S}_+(p(s))$, and (23), that the inclusions (24) hold true as well.

Let us prove property (C3). Suppose the contrary: there exists a set of matrices $\{W(s), s \in S\}$ such that $\{W(s), s \in S\} \neq \{W^*(s), s \in S\}$ and

$$W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp\ \forall s \in S;\ \sum_{s\in S} B_m(s) \bullet W(s) = c_m\ \forall m \in [n]. \tag{28}$$

As established in the proof of property (C2), for $s \in S$, matrix $W^*(s)$ can be represented in form (27). Since

$$W^*(s) \in \mathcal{CP}(p(s)) \cap Y^*(s)^\perp \subset \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp,$$

Proposition 4.3 ensures the existence of scalars γ^*_{ij}, $(i,j) \in V(J_b(s))$, such that matrix $W^*(s)$ can be represented in another form:

$$W^*(s) = \sum_{(i,j)\in V(J_b(s))} \gamma^*_{ij}\xi(s,i,j)\xi(s,i,j)^\top. \tag{29}$$

By assumption, $W(s) \in \mathbb{S}_+(p(s)) \cap Y^*(s)^\perp$, $s \in S$, hence, due to Proposition 4.3, we conclude that, for $s \in S$, the matrix $W(s)$ admits representation (18). Then, it follows from (18) and (27) that there exists $\lambda \in (0,1]$ such that

$$\bar{W}(s) := (1-\lambda)W^*(s) + \lambda W(s) \in \mathcal{CP}(p(s)),$$

and due to (22) and (28) we have

$$Y^*(s)\bullet\bar{W}(s) = 0 \ \forall s \in S; \ \sum_{s\in S} B_m(s)\bullet\bar{W}(s) = c_m \ \forall m \in [n].$$

It follows from these relations that the matrix $\bar{U} := \sum\limits_{s\in S} \mathcal{B}(\bar{W}(s), s)$ is an optimal solution to the dual problem (**D**). By assumption, the matrix U^* is a unique optimal solution to the dual problem (**D**) and, by construction, $U^* = \sum\limits_{s\in S} \mathcal{B}(W^*(s), s)$.

Consequently,

$$\sum_{s\in S} \mathcal{B}(W^*(s), s) - \sum_{s\in S} \mathcal{B}(\bar{W}(s), s) = \mathbb{O}(p). \tag{30}$$

Taking into account (18), (29), and the equality $\bar{W}(s) := (1-\lambda)W^*(s) + \lambda W(s)$, we conclude that condition (30) is equivalent to the following one:

$$\sum_{s\in S} \sum_{(i,j)\in V(J_b(s))} (\gamma^*_{ij} - \gamma_{ij})\xi(i,j)\xi(i,j)^\top = \mathbb{O}(p). \tag{31}$$

It follows from the condition $\{W(s), s \in S\} \neq \{W^*(s), s \in S\}$ and the representations (18) and (29), that $(\gamma^*_{ij} - \gamma_{ij}, (i,j) \in V(J_b(s)), s \in S)^\top \neq \mathbf{0}$. But this inequality and equality (31) contradict Assumption 4.2. Property (C3) is proved.

Let us prove property (C4). Suppose the contrary: there is a vector $x^0 \neq x^*$ with

$$Y^0(s) := \mathcal{A}(\mathcal{X}(x^0), s) \in \mathbb{S}_+(p(s)), \ \ Y^0(s)\bullet W^*(s) = 0 \ \forall s \in S. \tag{32}$$

Consider a vector function $x(\lambda) = (1-\lambda)x^* + \lambda x^0$, $\lambda \in [0,1]$.

Since U^* is strict complementary to X^*, then it follows from Proposition 3.3 that for $s \in S$, matrix $W^*(s)$ admits representation (27). It follows from (27), (32) that

$$\xi(s,i,j)^\top Y^0(s)\xi(s,i,j) = 0, \ (i,j) \in V(J(s)), \implies Y^0(s)\tau(s,j) = \mathbf{0}, \ j \in J(s).$$

The latter equalities can be rewritten in the form

$$e_k^\top \mathcal{X}(x^0)\tau(j) = 0 \ \forall k \in P_*(s), \ \forall j \in J(s).$$

Taking into account these equalities and Assumption 4.1, we can conclude that there exists $0 < \bar{\lambda} \leq 1$ such that for $\lambda \in [0, \bar{\lambda}]$ we have

$$\begin{aligned} &e_k^\top \mathcal{X}(x(\lambda))\tau(j) = 0 \ \forall k \in P_*(s), \\ &e_k^\top \mathcal{X}(x(\lambda))\tau(j) > 0 \ \forall k \in [p] \setminus P_*(s), \ \forall j \in J(s), \ \forall s \in S. \end{aligned} \tag{33}$$

It follows from these relations and Theorem 2.1 in [18], that there exists a sufficiently small $\lambda^0 > 0$ such that the vector $x(\lambda^0)$ is a feasible solution to problem (**P**). Based on equalities (33), one can show that $\mathcal{X}(x(\lambda^0))\bullet U^* = 0$. Hence, $x(\lambda^0)$ is an optimal solution to problem (**P**) and $x(\lambda^0) \neq x^*$. But this contradicts the assumption that x^* is a unique optimal solution to problem (**P**).

Let us prove property (C5). Suppose the contrary: there exists a vector

$$y = (y_m, m \in [n])^\top \neq \mathbf{0} \ \text{ such that } \ \sum_{m=1}^n B_m(s)y_m = \mathbb{O}(p(s)) \ \forall s \in S.$$

Then it is easy to see that relations (22) hold true with x^* replaced by $x^* + y \neq x^*$. But this contradicts property (C4) proved above. □

5. Perturbed CoP problems

Let us consider the pair of dual perturbed CoP problems (**P**(ε)) and (**D**(ε)) for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, and the corresponding unperturbed problems (**P**) and (**D**) obtained by setting $\varepsilon = 0$.

Let X^* and U^* be optimal solutions to problems (**P**) and (**D**), respectively, and let $P_*(s) \subseteq [p]$, $s \in S$, denote the sets defined in (4) based on X^*.

For $s \in S$, using the set $P_*(s)$ and the number $p(s) := |P_*(s)|$, define the mappings $\mathcal{A}(X, s) : \mathbb{S}(p) \to \mathbb{S}(p(s))$ and $\mathcal{B}(W, s) : \mathbb{S}(p(s)) \to \mathbb{S}(p)$ by the rules (15).

The main contribution of this paper is the following theorem.

Theorem 5.1. *Let x^* and U^* be unique optimal solutions to problems* (**P**) *and* (**D**)*, respectively. Suppose that Assumptions 4.1, 4.2 hold true, and the matrix U^* is strictly complementary to $X^* = \mathcal{X}(x^*)$. Then there exist $\varepsilon_0 > 0$ and unique continuously differentiable functions $x(\varepsilon) \in \mathbb{R}^n$, $\mathcal{Y}(\varepsilon, s) \in \mathbb{S}(p(s))$, $\mathcal{W}(\varepsilon, s) \in \mathbb{S}(p(s))$, $s \in S$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, such that*

$$\begin{aligned} \mathcal{Y}(\varepsilon, s) - \mathcal{A}(\mathcal{X}(\varepsilon, x(\varepsilon)), s) &\equiv \mathbb{O}(p(s)), \\ \mathcal{Y}(\varepsilon, s)\mathcal{W}(\varepsilon, s) + \mathcal{W}(\varepsilon, s)\mathcal{Y}(\varepsilon, s) &\equiv \mathbb{O}(p(s)),\ s \in S, \\ \sum_{s \in S} \mathcal{W}(\varepsilon, s) \bullet B_m(s) - c_m(\varepsilon) &\equiv 0,\ m \in [n],\ \varepsilon \in [-\varepsilon_0, \varepsilon_0], \end{aligned} \tag{34}$$

$$x(0) = x^*,\ \mathcal{Y}(0, s) = Y^*(s),\ \mathcal{W}(0, s) = W^*(s)\ \forall s \in S, \tag{35}$$

and $x(\varepsilon)$ and $U(\varepsilon) := \sum\limits_{s \in S} \mathcal{B}(\mathcal{W}(\varepsilon, s), s)$ are optimal solutions to the pair of dual problems (**P**(ε)) *and* (**D**(ε))*. Here, the matrices $Y^*(s)$, $W^*(s)$, $s \in S$, are defined in (21).*

Proof. First of all, note that it follows from Proposition 4.4 and the assumptions of this theorem, that the properties (C1)–(C5) hold true. Also, notice that system (34) can be equivalently rewritten in the vector form

$$\begin{aligned} \operatorname{svec}(\mathcal{Y}(s)) - \operatorname{svec}\Big(\sum_{m=1}^{n} B_m(s) x_m + B_0(\varepsilon, s)\Big) &= \mathbf{0}, \\ \mathcal{K}(\mathcal{W}(s))\operatorname{svec}(\mathcal{Y}(s)) &= \mathbf{0},\ s \in S, \\ \sum_{s \in S} \operatorname{svec}(\mathcal{W}(s))^\top \operatorname{svec}(B_m(s)) - c_m(\varepsilon) &= 0,\ m \in [n], \end{aligned} \tag{36}$$

w.r.t. the variables

$$x \in \mathbb{R}^n,\ \operatorname{svec}(\mathcal{Y}(s)) \in \mathbb{R}^{p_*(s)},\ \operatorname{svec}(\mathcal{W}(s)) \in \mathbb{R}^{p_*(s)},\ s \in S, \tag{37}$$

and the parameter ε. Here $p_*(s) := p(s)(p(s)+1)/2$, $B_0(\varepsilon, s) := \mathcal{A}(A_0(\varepsilon), s)$, $s \in S$; $\mathcal{K}(U) := U \otimes_s E$ is the *symmetric Kronecker product* of $U \in \mathbb{S}(k)$ and the $k \times k$ identity matrix $E = E(k)$. A more precise definition and the properties of the symmetric Kronecker product are described in [1]. Due to these properties, we know that $\mathcal{K}(\mathcal{W}(s)) \in \mathbb{R}^{p_*(s) \times p_*(s)}$.

Using matrices $B_m(s) := \mathcal{A}(A_m, s)$, $m = 0, 1, \ldots, n$, $s \in S$, let us form a pair of dual block semi-definite programming problems

$$\textbf{PSDP}: \quad \min c^\top x,\ \text{s.t.} \sum_{m=1}^{n} B_m(s) x_m + B_0(s) \in \mathbb{S}_+(p(s))\ \forall\, s \in S,$$

$$\begin{aligned} \textbf{DSDP}: \quad & \max -\sum_{s \in S} B_0(s) \bullet W(s), \\ & \text{s.t.} \sum_{s \in S} B_m(s) \bullet W(s) = c_m,\ m = 1, \ldots, n;\ W(s) \in \mathbb{S}_+(p(s)), s \in S, \end{aligned}$$

where $x^\top = (x_m, m \in [n])$ and $\{W(s) \in \mathbb{S}(p(s)),\ s \in S\}$ are primal and dual decision variables. Problems of this type, involving arbitrary finite numbers $p(s) \in \mathbb{N}$ and arbitrary matrices $B_m(s) \in \mathbb{S}(p(s))$, $m = 0, 1, \ldots, n$, $s \in S$, were studied in [2].

Notice that if the data of the CoP problem (**P**) is given, then the block semi-definite programming problems above are uniquely determined by a given set $\{P_*(s),\ s \in S\}$ of subsets of the set $[p]$.

It follows from the results of [2] and the properties (C1), (C3), and (C4) that x^* and $\{W^*(s) := \mathcal{A}(\mathcal{U}^*(s), s),\ s \in S\}$ are unique optimal solutions to problems (**PSDP**) and (**DSDP**), respectively, and it follows from property (C2) that these solutions are strictly complementary. Here we take into account that according to [23, 25] property (C2) is equivalent to the definition of strict complementarity in [2]. Furthermore, property (C5) implies that Assumption 1 in [2] is fulfilled.

Since the optimal solutions x^* and $\{W^*(s), s),\ s \in S\}$ to problems (**PSDP**) and (**DSDP**), respectively, are unique and strictly complementary, it follows from Theorem 3.3.3 in [23] that these solutions are non-degenerate (see the definition in [2, 23]). Hence, applying Theorem 9 from [2], we conclude that the Jacobian of system (36) w.r.t. variables (37) is nonsingular at $(x^*, \mathrm{svec}(Y^*(s)), \mathrm{svec}(W^*(s)), s \in S)$.

Then, it follows from the implicit function theorem, that under the assumptions of Theorem 5.1, there exist $\bar{\varepsilon} > 0$ and unique continuously differentiable functions $x(\varepsilon)$, $\mathcal{Y}(\varepsilon, s)$, $\mathcal{W}(\varepsilon, s)$, $s \in S$, $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, satisfying equalities (34) and initial conditions (35).

Hence, for $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, the matrices $\mathcal{X}(\varepsilon, x(\varepsilon))$ and

$$U(\varepsilon) = \sum_{s \in S} \mathcal{B}(\mathcal{W}(\varepsilon, s), s)$$

satisfy the equalities

$$\mathcal{X}(\varepsilon, x(\varepsilon)) \bullet U(\varepsilon) = 0, \quad A_m \bullet U(\varepsilon) = c_m(\varepsilon)\ \forall m \in [n].$$

It follows from the assumptions of this theorem, relations (34), and Theorem 4.2 in [16], that there exists $\varepsilon_0 \in (0, \bar{\varepsilon})$ such that $\mathcal{X}(\varepsilon, x(\varepsilon)) \in COP(p)$ and $U(\varepsilon) \in CP(p)$ for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Hence, it follows from Theorem 2.1 that $x(\varepsilon)$ and $U(\varepsilon)$ are optimal solutions to (**P**(ε)) and (**D**(ε)) for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. □

It is easy to see that the variables $\mathrm{svec}(\mathcal{Y}(s))$, $s \in S$, can be excluded from system (36). As a result, we obtain the equivalent system in the form

$$\mathcal{K}(\mathcal{W}(s))\,\mathrm{svec}\Big(\sum_{m=1}^{n} B_m(s) x_m + B_0(\varepsilon, s)\Big) = \mathbf{0}\ \forall s \in S, \tag{38}$$

$$\sum_{s \in S} \mathrm{svec}(\mathcal{W}(s))^\top \mathrm{svec}(B_m(s)) - c_m(\varepsilon) = 0\ \forall m \in [n], \tag{39}$$

w.r.t. parameter ε and variables

$$x \in \mathbb{R}^n,\ \mathrm{svec}(\mathcal{W}(s)) \in \mathbb{R}^{p^*(s)},\ s \in S. \tag{40}$$

Thus, we have shown that for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, under the assumptions of Theorem 5.1, the solutions $x(\varepsilon)$, $U(\varepsilon)$ to problems (**P**(ε)), (**D**(ε)) are uniquely determined by the system of bi-linear (38) and linear (39) equations and initial condition (35). System (38), (39) is a system of $n + \sum\limits_{s \in S} p_*(s)$ equations w.r.t. to $n + \sum\limits_{s \in S} p_*(s)$ variables (40).

Theorem 5.1 permits us to explore the behavior of solutions $x(\varepsilon)$, $U(\varepsilon)$ to perturbed problems (**P**(ε)), (**D**(ε)) in a neighborhood of the unperturbed parameter value $\varepsilon = 0$. It also allows us to easily find solutions $x(\varepsilon_*)$, $U(\varepsilon_*)$ to the perturbed problems (**P**(ε_*)), (**D**(ε_*)) with a given $\varepsilon_* \in [-\varepsilon_0, \varepsilon_0]$. In fact, suppose that for the unperturbed problems (**P**), (**D**), we have the optimal solutions x^* and U^* that are unique, satisfy Assumptions 4.1, 4.2, and U^* is strictly complementary to $X^* = \mathcal{X}(x^*)$. Then, for any fixed $\varepsilon_* \in [-\varepsilon_0, \varepsilon_0]$, we can find solutions $x(\varepsilon_*)$, $U(\varepsilon_*)$ to the perturbed problems (**P**(ε_*)), (**D**(ε_*)) by solving, with respect to variables (40), the (finite) system of equations (38), (39), applying, for example, the Newton method starting from a known initial approximation (35). To illustrate this opportunity, a small example is presented in the next section.

Theorem 5.1 allows one to calculate the derivatives $\frac{dx(\varepsilon)}{d\varepsilon}|_{\varepsilon=0}$, $\frac{dU(\varepsilon)}{d\varepsilon}|_{\varepsilon=0}$, and analyze other properties of solutions to problems (**P**(ε)), (**D**(ε)) under small perturbations of the parameter ε.

5.1. Particular cases

Case 1. Let the set $T_0 = T_0(X^*)$ be presented in the form (3). Suppose that $S_* := \{s \in S : |T_0(s)| = 1\} \neq \emptyset$. In this case, for all $s \in S_*$, the polytope $T_0(s)$ is an isolated point $\tau(s)$. Consequently, for $s \in S_*$, we have that $J(s) = \{j(s)\}$ with some $j(s) \in J$ and $P_*(s) = \mathrm{supp}(\tau(j(s)))$. Without loss of generality, we may consider that $j(s) = s$ for $s \in S_*$.

We can simplify system (38), (39) replacing it by the following system:

$$\begin{gathered} e_k^\top(s)\Big(\sum_{m=1}^{n} B_m(s)x_m + B_0(\varepsilon, s)\Big)\bar{\tau}(s) = 0 \ \forall k \in P_*(s), \ \forall s \in S_*, \\ \mathcal{K}(\mathcal{W}(s))\,\mathrm{svec}\Big(\sum_{m=1}^{n} B_m(s)x_m + B_0(\varepsilon, s)\Big) = \mathbf{0}, \ s \in S \setminus S_*, \\ \sum_{s\in S\setminus S_*} \mathrm{svec}(\mathcal{W}(s))^\top \mathrm{svec}(B_m(s)) + \sum_{s\in S_*} \bar{\tau}(s)^\top B_m(s)\bar{\tau}(s) - c_m(\varepsilon) = 0, \ m \in [n], \end{gathered} \tag{41}$$

with the parameter ε and variables

$$x = (x_m, m \in [n])^\top, \ \bar{\tau}(s) := (\bar{\tau}_k(s), \ k \in P_*(s))^\top, \ s \in S_*, \ \mathcal{W}(s) \in \mathbb{S}(p(s)), \ s \in S \setminus S_*.$$

Here, for $s \in S_*$, $e_k(s) = (e_{kq}(s), \ k \in P_*(s))^\top$, $e_{kq}(s) = 0$ for $q \in P_*(s) \setminus \{k\}$, and $e_{kk}(s) = 1$.

One can show that under the assumptions of Theorem 5.1, there exist $\varepsilon_0 > 0$ and unique continuously differentiable functions

$$x(\varepsilon), \ \bar{\tau}(s, \varepsilon), \ s \in S_*, \ \mathcal{W}(\varepsilon, s), \ s \in S \setminus S_*, \ \varepsilon \in [-\varepsilon_0, \varepsilon_0],$$

which satisfy equalities (41) and the initial conditions

$$x^*(0) = x^*, \ \bar{\tau}(s, 0) = (\tau_k(s), \ k \in P_*(s))^\top, \ s \in S_*, \ \mathcal{W}(0, s) = W^*(s) \ \forall s \in S \setminus S_*.$$

For $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the vector $x(\varepsilon)$ and the matrix

$$U(\varepsilon) = \sum_{s\in S_*} \mathcal{B}(\bar{\tau}(s, \varepsilon)\bar{\tau}(s, \varepsilon)^\top, s) + \sum_{s\in S\setminus S_*} \mathcal{B}(\mathcal{W}(\varepsilon, s), s)$$

are solutions to the pair of dual problems $(\mathbf{P}(\varepsilon))$ and $(\mathbf{D}(\varepsilon))$.

Under the additional condition that $S_* = S$, the situation becomes significantly simpler, as $S \setminus S_* = \emptyset$. In this case, to explore the behavior of solutions in a neighborhood of a given parameter value, we can directly apply the results from parametric SIP theory (see, for example [15]), since one of the key assumptions required for the application of this theory is satisfied: the constraints matrix $\mathcal{X}(x^*)$ has *a finite number* of zeros,

$$T_0 = \{t \in T : t^\top \mathcal{X}(x^*)t = 0\} = \{\tau(j), j \in J\}, \ |J| < \infty.$$

Case 2. Suppose that the set T_0 consists of one polytope, *i.e.*, we consider that $S = \{1\}$ and $T_0 = T_0(1) = \mathrm{conv}\{\tau(j), j \in J\}$, $P_*(1) = \bigcup\limits_{j\in J} \mathrm{supp}(\tau(j))$, $p(1) = |P_*(1)|$.

In this case, we do not need Assumption 4.2, since it is always fulfilled. It is evident that in this case, system (34) and the initial conditions (35) become simpler, they are the same as for the following parametric semi-definite programming problem

$$\min c(\varepsilon)^\top x, \ \text{s.t.} \ \sum_{m=1}^{n} B_m(1)x_m + B_0(1, \varepsilon) \in \mathbb{S}_+(p(1)),$$

where $B_0(1, \varepsilon) := \mathcal{A}(A_0(\varepsilon), 1)$. Such parametric semi-definite programming problems were studied in [4]. Hence, in this case, one can use these results to analyze the solution's behavior in a neighborhood of a given parameter value.

6. Example

Let us consider a parametric CoP problem with the following data: $p = 5$, $n = 4$, $A_0(\varepsilon) = A_0 + \varepsilon\Delta A_0$, $c(\varepsilon) = (c_m(\varepsilon), m = 1, \dots, 4)^\top$, where

$$c_1(\varepsilon) = 10 + \varepsilon,\ c_2(\varepsilon) = 3 - \varepsilon,\ c_3(\varepsilon) = -11 - 3\varepsilon,\ c_4(\varepsilon) = 1.5 + 4\varepsilon^2,$$

$$A_0 = \begin{pmatrix} 2 & -1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 1 & 2 \\ -1 & -1 & 2 & -0.5 & 2 \\ 2 & 1 & -0.5 & 2 & -2 \\ -2 & 2 & 2 & -2 & 2 \end{pmatrix}, \qquad \Delta A_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -3 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 1 \\ -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_4 = \frac{1}{2}\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

Let us show that the unperturbed primal and dual problems with this data have the following optimal solutions: $x^* = (0, 0, 0, 0)^\top$, $X^* := \mathcal{X}(x^*) = A_0$,

$$U^* = \mathcal{U}^*(1) + \mathcal{U}^*(2) = \begin{pmatrix} 2.5 & 1.0 & 1.0 & 0.5 & 2.0 \\ 1.0 & 1.0 & 1.0 & 0 & 0 \\ 1.0 & 1.0 & 1.0 & 0 & 0 \\ 0.5 & 0 & 0 & 1.5 & 2.0 \\ 2.0 & 0 & 0 & 2.0 & 4.0 \end{pmatrix},$$

where $\mathcal{U}^*(1) = 9\tau(1)\tau(1)^\top$,

$$\mathcal{U}^*(2) = 4\tau(2)\tau(2)^\top + 2(\tau(3) + \tau(2))(\tau(3) + \tau(2))^\top + 4\tau(3)\tau(3)^\top,$$

$$\tau(1) = \frac{1}{3}(1, 1, 1, 0, 0)^\top,\ \tau(2) = \frac{1}{2}(0, 0, 0, 1, 1)^\top,\ \tau(3) = \frac{1}{2}(1, 0, 0, 0, 1)^\top. \tag{42}$$

First of all, let us show that $A_0 \in \mathcal{COP}(5)$. To this end, we will use the following proposition which is a direct consequence of Theorem 2 in [24].

Proposition 6.1. *Consider a $(m + 1) \times (m + 1)$ matrix in the form*

$$A = \begin{pmatrix} C & b \\ b^\top & a \end{pmatrix} \text{ with } C \in \mathbb{S}(m),\ b \in \mathbb{R}^m,\ a \in \mathbb{R}$$

and a polyhedral cone $K := \{y \in \mathbb{R}^m_+ : y^\top b \le 0\}$. Let $Y \in \mathbb{R}^{m\times q}$ be a representative matrix of the cone K. Then $A \in \mathcal{COP}(m + 1)$ if and only if $C \in \mathcal{COP}(m)$, $a \ge 0$, $Y^\top(aC - b\,b^\top)Y \in \mathcal{COP}(q)$.

For $k = 3, 4$, denote $A_0(k) := \begin{pmatrix} A_{0ij},\ j = 1, \dots, k \\ i = 1, \dots, k \end{pmatrix}$ where A_{0ij} is the (i, j)-th element of the matrix A_0. It is easy to see that $A_0(3) \in \mathbb{S}_+(3)$ and hence $A_0(3) \in \mathcal{COP}(3)$.

Consider matrix $A_0(4)$ and represent it in the form

$$A_0(4) = \begin{pmatrix} A_0(3) & b(3) \\ b^\top(3) & 2 \end{pmatrix} \text{ with } b(3) = (2, 1, -0.5)^\top.$$

For cone $K_3 := \{y \in \mathbb{R}^3_+ :\ y^\top b(3) \le 0\}$, a representative matrix is as follows

$$Y_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 2 \end{pmatrix}.$$

Then $Y_3^\top(2A_0(3)-b(3)\,b(3)^\top)Y_3 \in \mathcal{N}(3)$. Here and in what follows $\mathcal{N}(p)$ is the cone of symmetric $p\times p$ matrices with non-negative elements. Since $\mathcal{N}(3) \subset \mathcal{COP}(3)$, it follows from Proposition 6.1 that $A_0(4) \in \mathcal{COP}(4)$.

Now consider matrix A_0 and represent it in the form

$$A_0 = \begin{pmatrix} A_0(4) & b(4) \\ b^\top(4) & 2 \end{pmatrix} \text{ with } b(4) = (-2,\ 2,\ 2,\ -2)^\top.$$

Consider cone $K_4 := \{y \in \mathbb{R}^4_+ :\ y^\top b(4) \le 0\}$ and its a representative matrix

$$Y_4 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

and a matrix $D := Y_4^\top(2A_0(4) - b(4)\,b(4)^\top)Y_4$. Denote by D_{ij} (i,j)-th element of matrix D. By direct calculations it is easy to check that

$$D_* := \begin{pmatrix} D_{ij},\ j = 3,4,5 \\ i = 3,4,5 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 8 \\ -2 & 4 & -1 \\ 8 & -1 & 12 \end{pmatrix}$$

and all other elements of D are non-negative.

Note that the matrix D_* admits a representation $D_* = D_0 + D_+$ where

$$D_0 = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \in \mathbb{S}_+(3),\ D_+ = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 1 \\ 6 & 1 & 10 \end{pmatrix} \in \mathcal{N}(3).$$

This implies that $D_* \in \mathcal{COP}(3)$ and, hence, $D \in \mathcal{COP}(6)$. Then it follows from Proposition 6.1 that $A_0 \in \mathcal{COP}(5)$. Hence, $A_0 = \mathcal{X}(x^*) \in \mathcal{COP}(5)$ and we conclude that x^* is a primal feasible solution.

By construction we have $U^* \in \mathcal{CP}(5)$ and by direct calculations one can check that $U^* \bullet A_m = c_m$, $m = 1, \ldots, 5$. Hence, U^* is a dual feasible solution. The equality $\mathcal{X}(x^*) \bullet U^* = 0$ holds true. Applying Theorem 2.1 one conclude that x^* and U^* are primal and dual optimal solutions to unperturbed problem.

Notice that vectors (42) are minimal zeros of $X^* = \mathcal{X}(x^*) \in COP(p)$ and

$$T_0(X^*) = T_0(1) \cup T_0(2),\ T_0(1) = \{\tau(1)\},\ T_0(2) = \{t = \alpha\tau(2) + (1-\alpha)\tau(3),\ \alpha \in [0,1]\}.$$

Hence, in this example we have $S = \{1,2\}$, $J(1) = \{1\}$, $J(2) = \{2,\ 3\}$, $P_*(1) = \{1,2,3\}$, $P_*(2) = \{1,4,5\}$. One can check that $P_*(s) = M(j)$ for all $j \in J(s)$, $s = 1,2$. Thus, we conclude that *Assumption 4.1 is fulfilled*.

Let us show that for this example, all other assumptions of Theorem 5.1 are fulfilled as well.

• In this example, $J_b(s) = J(s)$ for $s = 1,2$ and $V(J(1)) = \{(1,1)\}$, $V(J(2)) = \{(2,2),(3,3),(2,3)\}$. One can check that matrices

$$\xi(i,j)\xi(i,j)^\top,\ (i,j) \in V(J(1)) \cup V(J(2)),$$

are linearly independent. This implies that *Assumption 4.2 holds true*. Moreover, it follows from Corollary 4.2 in [17] that U^* *is strictly complementary to* X^*.

• Each optimal solution $x \in \mathbb{R}^4$ to the primal unperturbed problem should satisfy the following relations:

$$\Big(\sum_{m=1}^4 A_m x_m + A_0\Big) \in COP(5),\ \Big(\sum_{m=1}^4 A_m x_m + A_0\Big) \bullet U^* = 0.$$

These relations imply the equalities

$$e_k^\top \Big(\sum_{m=1}^4 A_m x_m + A_0\Big)\tau(j) = 0,\ k \in P_*(s),\ j \in J(s),\ s = 1,2. \tag{43}$$

One can check that, in this example, we have

$$\operatorname{rank} \begin{pmatrix} e_k^\top A_m \tau(j),\ m = 1, \ldots, 4 \\ k \in P_*(s),\ j \in J(s),\ s = 1,2 \end{pmatrix} = 4.$$

Hence, linear system (43) w.r.t. unknowns x_m, $m = 1, \ldots, 4$, can not have more than one solution. Consequently, we have shown that x^* *is a unique optimal solution* to the primal unperturbed problem.

• Using equalities (11), one can show that every dual optimal solution U to the unperturbed problem admits a representation

$$U = \sum_{s\in S} \sum_{(i,j)\in V(J_b(s))} \beta_{ij}\xi(i,j)\xi(i,j)^\top \text{ with some } \beta_{ij} \in \mathbb{R},\ (i,j) \in V(J_b(s)),\ s \in S,$$

and satisfies the equalities $A_m \bullet U = c_m$, $m = 1,\dots,4$. These relations imply the following linear system w.r.t. unknowns β_{ij}, $(i,j) \in V(J_b(s))$, $s \in S$:

$$\sum_{s\in S} \sum_{(i,j)\in V(J_b(s))} \beta_{ij}\xi(i,j)^\top A_m \xi(i,j) = c_m,\ m = 1,\dots,4. \tag{44}$$

In this example, $\operatorname{rank}\begin{pmatrix}\xi(i,j)^\top A_m\xi(i,j),\ m=1,\dots,4\\ (i,j)\in V(J_b(s)),\ s=1,2\end{pmatrix} = 4.$

Hence, system (44) can not have more than one solution. Consequently, we have shown that U^* *is a unique optimal solution* to the dual unperturbed problem.

Thus, all assumptions of Theorem 5.1 are fulfilled, and we can apply this theorem.

In this example, system (38), (39) is a system with $6 + 6 + 4 = 16$ equations w.r.t. to 16 variables x_m, $m = 1,\dots,4$; $\mathcal{W}_{ij}(s), (i,j) \in \{(k,q) : k \in P_*(s),\ q \in P_*(s),\ k \le q\}$, $s = 1,2$, and parameter $\varepsilon \in \mathbb{R}$. The initial conditions (35) are as follows:

$$x(0) = x^* = (0,\ 0,\ 0,\ 0)^\top,$$

$$\mathcal{W}(0,1) = \mathcal{A}(\mathcal{U}^*(1),1) = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}, \tag{45}$$

$$\mathcal{W}(0,2) = \mathcal{A}(\mathcal{U}^*(2),2) = \begin{pmatrix}1,5 & 0.5 & 2\\ 0.5 & 1.5 & 2\\ 2 & 2 & 4\end{pmatrix}.$$

For the parameter value $\varepsilon_* = 0.2$, we solved system (38), (39) by the Newton method starting with the initial approximation (45). As a result, we obtained the following solution to this system: $x(\varepsilon_*) = (0.0899,\ 0.3705,\ 0.0500,\ 0.2399)^\top$,

$$\mathcal{W}(\varepsilon_*,1) = \begin{pmatrix}0.6809 & 0.8719 & 0.4734\\ 0.8719 & 1.1166 & 0.6062\\ 0.4734 & 0.6062 & 0.3291\end{pmatrix},\ \mathcal{W}(\varepsilon_*,2) = \begin{pmatrix}2.3547 & 0.6277 & 3.0141\\ 0.6277 & 1.0051 & 1.6501\\ 3.0141 & 1.6501 & 4.7137\end{pmatrix}.$$

The corresponding matrices

$$X(\varepsilon_*) = \mathcal{X}(\varepsilon_*, x(\varepsilon_*)) \text{ and } U(\varepsilon_*) = \mathcal{B}(\mathcal{W}(\varepsilon_*,1),1) + \mathcal{B}(\mathcal{W}(\varepsilon_*,2),2)$$

take the form

$$X(\varepsilon_*) = \begin{pmatrix}1.9000 & -1.2505 & -0.4295 & 1.9000 & -1.8801\\ -1.2505 & 1.5194 & -1.0000 & 1.5705 & 2.1199\\ -0.4295 & -1.0000 & 2.4598 & -0.4500 & 1.9301\\ 1.9000 & 1.5705 & -0.4500 & 1.9000 & -1.8801\\ -1.8801 & 2.1199 & 1.9301 & -1.8801 & 1.8603\end{pmatrix},$$

and

$$U(\varepsilon_*) = \begin{pmatrix}3.0356 & 0.8719 & 0.4734 & 0.6277 & 3.0141\\ 0.8719 & 1.1166 & 0.6062 & 0 & 0\\ 0.4734 & 0.6062 & 0.3291 & 0 & 0\\ 0.6277 & 0 & 0 & 1.0051 & 1.6501\\ 3.0141 & 0 & 0 & 1.6501 & 4.7137\end{pmatrix}.$$

To complete the example, we have to show that $X(\varepsilon_*)$ and $U(\varepsilon_*)$ are primal and dual optimal solutions to perturbed problem $(\mathbf{P}(\varepsilon))$ with $\varepsilon = \varepsilon_*$. Notice that it follows from system (38), (39) that, by construction, the equalities

$$A_m \bullet U(\varepsilon_*) = c_m(\varepsilon_*) \; \forall \, m = 1, \ldots, 4; \; X(\varepsilon_*) \bullet U(\varepsilon_*) = 0$$

hold true. Hence, our aim is to test that $X(\varepsilon_*) \in COP(5)$ and $U(\varepsilon_*) \in CP(5)$.

Let us start with $U(\varepsilon_*)$. It follows from the construction above that to show that $U(\varepsilon_*) \in CP(5)$, it is enough to show that $\mathcal{W}(\varepsilon_*, s) \in CP(3)$ for $s = 1, 2$. Let us do this. In fact, it is easy to see that $\mathcal{W}(\varepsilon_*, s) \in \mathbb{S}_+(3) \cap \mathcal{N}(3)$ for $s = 1, 2$. Then, taking into account that $\mathbb{S}_+(3) \cap \mathcal{N}(3) = CP(3)$ (see [12]), we obtain that $\mathcal{W}(\varepsilon_*, s) \in CP(3)$ for $s = 1, 2$, and, consequently, $U(\varepsilon_*) \in CP(5)$.

To show that $X(\varepsilon_*) \in COP(5)$ we used an approach proposed in [7] via solving a quadratic programming problem: $\min t^\top X(\varepsilon_*) t$, s.t. $t \in T$.

Notice that in this example, we have that $T_0(1) = \{\tau(1)\}$. Hence, $S_* = \{1\} \subset S$ and instead of system (38), (39) with 16 variables $(x, \mathcal{W}_{ij}(s), (i, j) \in V(J(s)), s \in S)$ we can solve system (41) with 13 variables

$$(x, \; \bar{\tau}_k(s), k = 1, 2, 3; \; \mathcal{W}_{ij}(s), \; (i, j) \in V(J(2))).$$

Thus, we have illustrated how, for sufficiently small $|\varepsilon|$, we can find primal and dual optimal solutions to perturbed problem $(\mathrm{P}(\varepsilon))$ by solving the corresponding system (38), (39) w.r.t. $(x, \mathcal{W}_{ij}(s), (i, j) \in V(J(s)), s \in S)$ with fixed ε.

7. Conclusion

The main goal of our research was to study parametric linear conic problems $(\mathbf{P}(\varepsilon))$ and $(\mathbf{D}(\varepsilon))$ over the cones of copositive and completely positive matrices.

We derived relations in the form of bi-linear and linear equations which, under certain assumptions, unambiguously determine the optimal solutions to these parametric problems as functions of the parameter ε in a neighborhood of a given parameter value.

The paper's findings are significant because they can be applied in sensitivity analysis, providing a qualitative understanding of how optimal solutions and the optimal value of the cost function behave as parameters vary. To the best of our knowledge, no prior results exist for parametric conic optimization problems over the copositive cone and its dual.

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