

A Snowpiercer Model: Dynamic Beam Solution for Highly Nonlinear Reaction, Energy Inequality, and Granular Foundation

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Abstract: The dynamic contact problem for Euler-Bernoulli beam equation with highly nonlinear reaction term is formulated to model deflection of a rail road on a granular foundation under motion of a high-speed vehicle (snowpiercer). Based on the energy inequality and Galerkin approximation, variational solution to the evolutionary semi-linear differential equation is proved. The reaction of the granular foundation is presented for periodic beam vibrations, which illustrate plastic, ageing, and hardening phenomena.

Keywords: Dynamic Euler-Bernoulli beam, rail road, granular foundation, semi-linear hyperbolic equation, energy inequality.

2020 Mathematics Subject Classification: 35L76, 37K58, 74E20.

1. Introduction

Complex modeling of railway applications yield challenging systems constituted by dynamic equations and inequalities, see a survey [40]. We refer to the mathematical models of wear [35], frictional and viscous dampers [16, 17], granular media [24, 25], and to the experimental testing of foundations composed from granular materials [10] including Danube sand [34]. Typically, the rail way is modeled by the Euler-Bernoulli beam lying on a rigid or elastic foundation [1, 11], for modeling with Timoshenko beams see [4, 26, 42]. Time periodic solutions of semi-linear dynamic beam and wave equations were justified in [13, 38, 41], and its stabilization using damping was treated in [15, 46, 47]. For numerical methods used in the field of dynamic problems we cite [8, 9, 27].

We refer the interesting readers to [45] for treatment of vibration problems in engineering, in particular, to [12] for bi-Laplacian equations appearing in the mathematical biology. The hysteresis theory for hyperbolic equations and rate-independent systems was developed in [6, 28, 30]. We cite the related results on differential inclusions [33], on sweeping processes for non-convex systems [5, 29], and history-dependent sweeping processes [31]. See collection of works on dynamic control and optimization problems [7, 44], specifically dynamics of a spinning disc in a rarefied medium [32, 37]. Mechanical systems of impact with finite degrees of freedom as well as suitable numerical schemes can be found in [36].

There is a number of books on the variational theory of static and dynamic problems for contact and dissipation phenomena [14, 20, 22]. The variational inequalities describing contact and friction mechanics of solids were treated e.g. in [3, 19, 21] by using primal as well as dual optimization techniques. For the contact and dynamics of inelastic bodies see [2, 43]. We cite [18, 23, 39] for iterative algorithms solving unilaterally constrained problems for thin structures, beams and plates.

In Section 2 we formulate the problem of a dynamic beam under moving load and highly nonlinear reaction of inelastic foundation. In Section 3 its variational solution is provided by using the energy inequality. In Section 4 we present the example of history-dependent reaction for granular media.

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2. Problem formulation

We consider the dynamic contact problem in a time interval $t \in (0, T)$, $T > 0$, for moving in x -direction a high-speed vehicle (snowpiercer) over a rail road for $x \in (0, L)$, $L > 0$. The action of rail vehicle is represented by $q(t, x)$, which summarizes the gravity and dynamic load, see Figure 1.

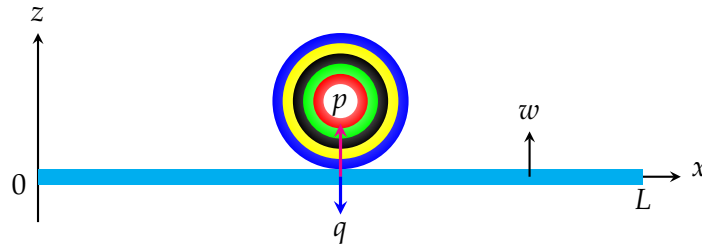


Figure 1. Dynamic contact problem.

Motion of the rail road in z -direction is modeled by the Euler-Bernoulli beam equation for the vertical deflection $w(t, x)$ in the mid-line $z = 0$:

$$\mu w_{tt} + Dw_{xxxx} = q - p(w) \quad \text{in } Q := (0, T) \times (0, L), \quad (1)$$

where the linear density $\mu > 0$ and flexural rigidity $D > 0$ are prescribed. Here p stands for reaction of a foundation, which is generally a nonlinear function of w . For prescribed initial deflection $w^0(x)$ and initial velocity $w^1(x)$, equation (1) is supported by initial conditions:

$$w = w^0, \quad w_t = w^1 \quad \text{at } t = 0, \quad (2)$$

and by clamped boundary conditions:

$$w = w_x = 0 \quad \text{at } x = 0, L. \quad (3)$$

Without loss of generality, (3) can be changed to hinge boundary conditions, free end, or their combination.

To describe the admissible functions $p(w)$ in (1) from a mathematical point of view is the main task of this paper. The standard, linear force-deflection relationship, known as a Winkler foundation model:

$$p = k_1 w \quad (k_1 > 0), \quad (4)$$

implies a continuous distribution of linear elastic springs with stiffness k .

The nonlinear elastic Winkler-Pasternak relationships rely on the power law

$$p = k_1 |w|^{n-1} w \quad (n > 0), \quad (5)$$

Coulomb friction dampers were introduced in [17]:

$$p = k_2 \text{sign}(w_t) \quad (k_2 > 0), \quad (6)$$

such that equation of motion was formulated in the form of a differential inclusion. In [16] the elasto-frictional foundation was improved by composing in parallel (that is the sum) with viscous dampers:

$$p = k_3 w_t \quad (k_3 > 0). \quad (7)$$

We are motivated by a foundation model catching inelastic evolutionary behavior of granular media [2]. Based on hypoplastic modeling [24, 25], we suggest the following highly nonlinear relationship to have in mind:

$$p(w)(t) = -\frac{k}{3} \left(1 + 2 \frac{e^{-k \int_0^t \xi(w)(s) ds} - 1}{\xi(w)(t)} w_t(t) \right) S(w)(t) \quad (k > 0), \quad (8)$$

where the dissipation function:

$$S(w)(t) := e^{-k \int_0^t \left(\frac{7w_t(s)}{3} + \xi(w)(s) + \frac{2}{3} \frac{e^{-k \int_0^s \xi(w)(\tau) d\tau} - 1}{\xi(w)(s)} w_t^2(s) \right) ds}, \quad (9)$$

and the plastic part distinguishing loading $w_t < 0$ from unloading $w_t > 0$:

$$\xi(w) := (1 + \kappa)|w_t| - w_t \quad (\kappa > 0). \quad (10)$$

In contrast to previous considerations, function (8) is constituted without any summation of terms like (4)–(7).

3. Variational solution

Let the load $q \in L^2(Q)$, and initial data $w^0 \in H_0^2(0, L)$, $w^1 \in H_0^1(0, L)$. We assume to be prescribed a continuous operator:

$$p : L^\infty(Q) \mapsto L^2(Q). \quad (11)$$

Integrating by parts over space, weak formulation to the problem (1)–(3) reads: find $w \in L^\infty(0, T; H_0^2(0, L)) =: V$ and $w_{tt} \in L^2(Q)$ justifying

$$\int_0^L (\mu w_{tt} \bar{w} + Dw_{xx} \bar{w}_{xx} + p(w) \bar{w}) dx = \int_0^L q \bar{w} dx \quad (12)$$

for all test functions $\bar{w} \in H_0^2(0, L)$.

For smooth $w \in W^{1,\infty}(0, T; H_0^2(0, L))$ we assume the free energy:

$$\mathcal{E}(w) := \int_0^L \left(\frac{1}{2} \mu w_t^2 + \frac{1}{2} D w_{xx}^2 + S(w) \right) dx, \quad (13)$$

with some operator $S : V \mapsto W^{1,\infty}(0, T; L^1(0, L))$ such that for any w :

$$S(w) \geq 0, \quad S(w)_t \leq p(w)w_t \quad \text{in } Q. \quad (14)$$

Theorem 3.1. *Under the assumption (14) on the nonlinear reaction (11), there exists solution to the evolutionary variational equation (12).*

Proof. Let us choose an orthogonal basis in $L^2(0, L)$ satisfying the clamped boundary conditions (3), that is

$$\phi_i(x) = \cos(\lambda_i x) - \cosh(\lambda_i x) + \frac{\cos(\lambda_i L) - \cosh(\lambda_i L)}{\sin(\lambda_i L) - \sinh(\lambda_i L)} (\sin(\lambda_i x) - \sinh(\lambda_i x)),$$

where λ_i solves transcendental equation $\cos(\lambda_i L) \cosh(\lambda_i L) = 1$ for $i \in \mathbb{N}$. We look for the Galerkin approximation $w^n(t, x) = \sum_{j=0}^n w_j(t) \phi_j(x)$ solving

$$\int_0^L (\mu w_{tt}^n \phi_i + Dw_{xx}^n (\phi_i)_{xx} + p(w^n) \phi_i) dx = \int_0^L q \phi_i dx \quad (15)$$

for $i = 0, \dots, n$. The summation over i of (15) multiplied by w'_i leads to

$$\int_0^L (\mu w_{tt}^n w_t^n + Dw_{xx}^n w_{xxt}^n + p(w^n) w_t^n) dx = \int_0^L q w_t^n dx. \quad (16)$$

Inserting here (13) and (14) follows the energy inequality:

$$\mathcal{E}(w^n)_t \leq \int_0^L q w_t^n dx \quad \text{a.e. } (0, T). \quad (17)$$

From (16) and (17) we obtain the uniform estimate:

$$\sup_{t \in (0, T)} (\|w_t^n(t)\|_{L^2(0, L)} + \|w_{xx}^n(t)\|_{L^2(0, L)} + \int_0^L S(w^n)(t) dx) \leq c_1, \quad (18)$$

where $c_1 > 0$. Since $S \geq 0$ according to the assumption (14), then

$$\|w_t^n\|_{L^\infty(0, T; L^2(0, L))} + \|w_{xx}^n\|_{L^\infty(0, T; L^2(0, L))} \leq c_1, \quad (19)$$

After integration of the first term in (15) by parts yields

$$\begin{aligned} \int_0^t \int_0^L p(w^n) \phi_i \, dx \, dt &= \int_0^t \int_0^L (q \phi_i - D w_{xx}^n(\phi_i)_{xx}) \, dx \, dt \\ &\quad - \int_0^L \mu(w_t^n(t) - w^1) \phi_i \, dx \quad \text{for } t \in (0, T), \end{aligned}$$

and (19) leads to boundedness of p :

$$\|p(w^n)\|_{L^2(Q)} \leq c_2 \quad (c_2 > 0). \quad (20)$$

Rewriting (15) integrated over time in the other way:

$$\int_Q \mu w_{tt}^n \phi_i \, dx \, dt = \int_Q (q \phi_i - D w_{xx}^n(\phi_i)_{xx} - p(w^n) \phi_i) \, dx \, dt,$$

from (19) and (20) we estimate the second order time derivative :

$$\|w_{tt}^n\|_{L^2(Q)} \leq c_3 \quad (c_3 > 0). \quad (21)$$

The uniform estimates (19)–(21) follow from solution to the Galerkin system (15) and the existence of an accumulation point $w \in V$ such that

$$\begin{aligned} w_t^n &\rightharpoonup w_t \star\text{-weakly in } L^\infty(0, T; L^2(0, L)), \\ w_{xx}^n &\rightharpoonup w_{xx} \star\text{-weakly in } L^\infty(0, T; L^2(0, L)), \\ w_{tt}^n &\rightharpoonup w_{tt} \text{ weakly in } L^2(Q) \text{ as } n \rightarrow \infty. \end{aligned} \quad (22)$$

In (22) Sobolev embedding provides $w^n \rightarrow w$ strongly in $L^\infty(Q)$. Then on taking the limit in (15) we get the variational equation (12). $\square$

To validate Theorem 3.1, in the reference example (8)–(10) $S > 0$ and

$$S(w)_t = p(w)w_t - k((1 + \kappa)|w_t| + w_t)S(w) \leq p(w)w_t,$$

thus the assumption (14) holds true.

4. Granular foundation

For given $w(t)$, assume that $(0, T)$ is split by points $0 = t_0 < t_1 < \dots$ into monotony intervals $(t_{i-1}, t_i) \subset (0, T)$ accounting for alternative cases:

$$\begin{cases} \{t_{i-1} < t < t_i : w_t(t) < 0\} =: \mathcal{I}_i^{\{-1\}}, \\ \{t_{i-1} < t < t_i : w_t(t) = 0\} =: \mathcal{I}_i^{\{0\}}, \\ \{t_{i-1} < t < t_i : w_t(t) > 0\} =: \mathcal{I}_i^{\{+1\}}. \end{cases} \quad (23)$$

Then the plastic part ξ in (10) contributes piecewise linearly:

$$\xi(w) = \begin{cases} -(2 + \kappa)w_t & \text{in } \mathcal{I}_i^{\{-1\}}, \\ 0 & \text{in } \mathcal{I}_i^{\{0\}}, \\ \kappa w_t & \text{in } \mathcal{I}_i^{\{+1\}}. \end{cases} \quad (24)$$

Respectively, we calculate the integrals:

$$-\int_{t_{i-1}}^t \xi(w)(s) \, ds = \begin{cases} (2 + \kappa)(w(t) - w(t_{i-1})), & t \in \mathcal{I}_i^{\{-1\}}, \\ 0, & t \in \mathcal{I}_i^{\{0\}}, \\ -\kappa(w(t) - w(t_{i-1})), & t \in \mathcal{I}_i^{\{+1\}}, \end{cases} \quad (25)$$

whose summation over

$$(0, t) = \left(\bigcup_{t_i < t} \mathcal{I}_i^{\{-1\}}\right) \cup \left(\bigcup_{t_i < t} \mathcal{I}_i^{\{0\}}\right) \cup \left(\bigcup_{t_i < t} \mathcal{I}_i^{\{+1\}}\right) \cup (t_i, t) \quad (26)$$

determines the functions $S(w)$ in (9) and $p(w)$ in (8).

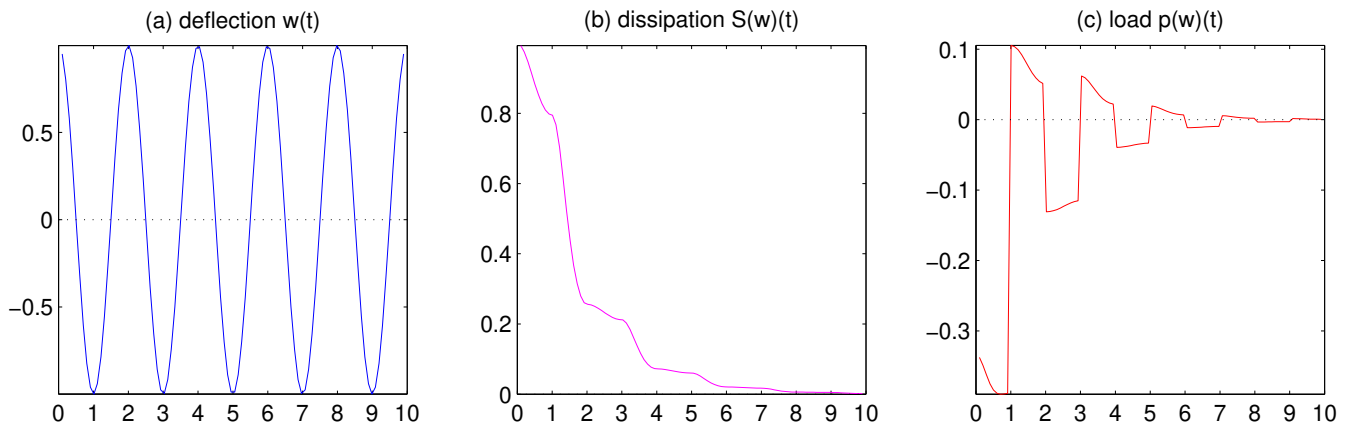


Figure 2. Periodic vibration in time t : deflection w (a); dissipation $S(w)$ (b); load $p(w)$ (c).

For example we take in (23)–(25) the periodic vibration $w(t) = \cos(\pi t)$, illustrated versus time $t \in (0, 10)$ in Figure 2(a).

The parameters are taken as $\kappa = 1$ and $k = 0.2$. The corresponding dissipation $S(w)(t)$ and load $p(w)(t)$ computed in Matlab are depicted in Figure 2(b) and Figure 2(c). From these figures we conclude that the example reaction (8)–(10) catches plastic and ageing phenomena. It possesses hardening behavior towards solidification of the granular foundation in the limit of periodic beam vibrations applied.

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