

Irrigation Perturbed Optimal Linear-Control Problem

Ana P. Lemos-Paião^{1,*}, Sofia O. Lopes², M. d. R. de Pinho²

1. Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, Portugal

2. SYSTEC-ARISE Research Center for Systems and Technologies, Faculty of Engineering, University of Porto, Portugal

Abstract: We analyse the irrigation problem as an optimal control problem within the context of agricultural planning. By formulating irrigation as an optimal control system, we aim to optimize the water usage, while ensuring efficient agricultural productivity. A key parameter in our model is the percentage of water loss due to deep percolation, denoted by β , which is difficult to estimate and sensitive to environmental perturbations. As a result, the problem becomes a parametric optimal linear-control one with an inequality state constraint. The main goal of this work consists in investigating how perturbations of β affect the optimal solution and the optimal cost, with the help of parameter sensitivity analysis. To achieve this, we reformulate the parametric optimal linear-control problem as a finite-dimensional non-linear programming one for which we derive necessary and second-order sufficient optimality conditions. Additionally, we design a parameter sensitivity analysis. Numerical results are provided to illustrate the designed parameter sensitivity analysis which shows that as β increases, the irrigation period of time and the irrigation water flow rate also increase.

Keywords: Parametric optimal control, parameter sensitivity analysis, non-linear programming methods, agricultural irrigation problem.

2020 Mathematics Subject Classification: 49M05, 49N05, 49K40, 90C30, 90C31.

1. Introduction

According to the Food and Agriculture Organization, freshwater resources are rapidly declining and water scarcity is nowadays a critical challenge for sustainable development. Droughts exacerbate this crisis specially in vulnerable regions [11]. It is then of foremost importance to implement measures to improve water-use efficiency and agricultural productivity.

Over the past two decades, different optimization approaches for the efficient use of the irrigation water have been proposed. In this respect, see, e.g., [7, 12, 13, 14, 15, 16, 17, 18, 23] where soil water balance models are used, or [2, 19, 25]. These last three are based on Richards equation. Approaches based on the maximization of biomass yield at harvest, as in, e.g., [4], are also of high relevance in this respect.

In this paper, we focus on minimizing the irrigation water used to produce healthy crops within the framework of optimal control. In general and in such setting, the amount of water used is treated as a control variable and the aim consists in minimizing the amount of irrigation water used to produce a healthy crop.

Our starting point is an optimal control problem proposed in [17], where the main goal is to maintain adequate soil moisture for the development of a healthy plant from sowing to harvest. The problem is formulated in as a state-constrained optimal control problem, linear in both the cost function and the dynamics.

Although the Pontryagin Maximum Principle is a powerful tool to characterize the analytical solution of optimal control problems (see [10, 24]), the presence of state constraints may obstruct its identification. The situation may be more difficult when the dynamic model depends on unknown or hard-to-estimate parameters and the objective is to determine the solution as a function of these parameters.

Our goal is to determine solutions of the optimal control problem proposed in [17] as a function of

*Corresponding author.

E-mail addresses: A. P. Lemos-Paião*: anapaiao@ua.pt; S. O. Lopes: solpes@fe.up.pt; M. d. R. de Pinho: mrpinho@fe.up.pt
<https://doi.org/10.68381/jca34030>

Received date: October 30, 2025; Accepted date: April 23, 2026

© 2027 Published by Banach Press LLC. (Dedicated to Alexander Plakhov on the occasion of his 65th birthday.)

a parameter. One may hope the use of numerical methods by direct method to be of help. Numerical methods consist on discretization of the optimal control problem leading to a non-linear programming problem. However, the resulting non-linear programming problem is, in general, a large-scale problem (see, e.g., [3, 22]) with a large number of variables due to the discretization step needed to ensure that numerical solutions are good approximations of the analytical ones. When the structure of the optimal solution is known, as in our case of interest [17], the alternative is to construct a parametric problem with a reduced number of variables, and to solve it by the indirect method. The presence of hard-to-estimate parameters calls for a *parameter sensitivity analysis* to study the variations of optimal solution and optimal cost in terms of such parameters. To do that, and at first sight, we could take advantage of the linear-control framework of our problem. In principle, this would allow us to appeal to various results of convex analysis to study how changes of the parameter affect the stability of trajectories, controls and multipliers. However, the presence of the state constraint, although convex, prevents the direct application of known parametric sensitivity analysis for reasons we explain next. Indeed, there exists an impressive and interesting panoply of research works on sensitivity analysis for different optimization problems (see, e.g., [9]). Optimal linear-control problems with state constraints are not an exception. In general, sensitivity analysis is conducted transcribing parametric optimal linear-control problems into non-linear programming problems, whose solutions allow a prediction on the switching structure of the original problem (see, e.g., [6, 21]). In [6] optimal control problems with mixed state-control constraints and with the control appearing linearly in the objective function are considered. They propose two methods for transcribing parametric optimal linear-control problems into non-linear programming formulations, together with an approach for conducting sensitivity analysis. Such transcription methods are designed for mixed state-control constrained problems and can not be applied to our irrigation problem of interest. In [21], Maurer and Vossen study parametric optimal control problems with state constraints and with the control appearing linearly in the objective function, assuming that the optimal control has finitely many bang-bang and boundary arcs and, moreover, assuming that the control is discontinuous at junction times. However, this last discontinuity assumption of the boundary control at junction times in [21] is a feature we can not guarantee for our optimal control problem with a L^1 cost. On the contrary, the analysis in [17] and numerical findings indicate that the optimal control may be continuous at some junction times.

Thus, in the absence of any applicable scheme from the literature, we need a new approach to conduct sensitivity analysis of the irrigation problem considered in [17]. The main difficulty of the approaches to sensitivity analysis as those proposed in [6, 21] consists in transcribing the parametric optimal linear-control problem into a non-linear programming one. Here, and based on the transcription methods proposed in [6, 21], we reformulate our parametric optimal linear-control problem of interest as a finite dimensional non-linear programming problem. For this problem, and following [5], we derive necessary and sufficient optimality conditions, and we establish a parameter sensitivity analysis. We then illustrate our results numerically by considering the irrigation problem of interest.

This paper is organized as it follows. In Section 2, we present the optimal control problem and the structure of its optimal solution. In Section 3, we reformulate the optimal control problem into a low dimensional non-linear optimization problem. Furthermore, we discretize it and we derive necessary and second-order sufficient optimality conditions. Parameter sensitivity analysis associated with the optimal solution and the optimal value function is also carried out, in Section 4. Finally, in Section 5, we provide numerical simulations in order to illustrate our findings.

2. Parametric optimal linear-control problem

The non-parametric optimal linear-control problems applied to agricultural irrigation proposed in [12, 17] play a central role in this paper. In this section, we present the irrigation problem and we briefly summarize the characterization of its optimal solution profile as done in [12, 17].

2.1. Formulation

$$\mathbf{OC}(\beta) \quad \begin{cases} \min & \int_0^{t_f} u(t) dt \\ \text{s. t.} & \dot{x}(t) = u(t) + \text{rfall}(t) - \text{evtp}(t) - \beta x(t) \text{ a.e. } t \in [0, t_f], \\ & x(0) = x_0, \\ & u(t) \in [0, \gamma], \forall t \in [0, t_f], \\ & x_{\min} - x(t) \leq 0, \forall t \in [0, t_f], \end{cases}$$

where

- t represents time;
- t_f is the planning irrigation time;
- x is the state variable with $x(t)$ denoting the amount of water in the soil at time t ;
- u is the control with $u(t)$ expressing the flow rate of irrigation water at t ;
- γ is the maximum flow rate of irrigation water entering the system;
- $\text{rfall}(t)$ is the daily precipitation at t ;
- $\text{evtp}(t)$ is the estimated evapotranspiration for the type of crop at t ;
- $\beta \in [\beta_{\min}, \beta_{\max}]$ represents the percentage of water loss due to the run-off and deep infiltration;
- x_0 is the initial quantity of the water in the soil;
- $x_{\min}$ is the hydrological need of the crop.

We remark that $\text{evtp}(t)$ at t is calculated based on the model [1] and it is assumed that real data from a weather station is used. Here, we refrain from adding any discussion on the formulation of the irrigation problem, or alternative approaches. Instead, we refer the reader to [12, 17, 18] and the references therein. Observe that $\mathbf{OC}(\beta)$ is a parametric optimal control problem with the parameter β taking values within the interval $[\beta_{\min}, \beta_{\max}]$.

2.2. Characterization of the solution

Next, we recall the main results of [12, 17] characterizing the optimal solution.

For each $\beta \in [\beta_{\min}, \beta_{\max}]$, the existence of solution of $\mathbf{OC}(\beta)$ problem is guaranteed and any optimal control is globally optimal, due to the convexity of $\mathbf{OC}(\beta)$. However, because the cost function is not strictly convex, the uniqueness of the solution cannot be guaranteed.

Gathering the results in [17, Proposition 1], the normality is ensured. Moreover, from the results in [17, Section 3.4] and complemented by [12], we characterize the optimal solution of $\mathbf{OC}(\beta)$, for each $\beta \in [\beta_{\min}, \beta_{\max}]$, by means of the Pontryagin Maximum Principle (Theorem 9.3.1 of [26]). This requires the following assumptions (see [12, 17]):

- (A1) the upper bound of the control γ is large and never reached;
- (A2) function g , defined by $g(t) = \text{rfall}(t) - \text{evtp}(t)$, is Lipschitz continuous for all $t \in [0, t_f]$;
- (A3) the optimal trajectory $\bar{x}$ does not have *contact points* (means that the trajectory does not touch the boundary at isolated points) and it has a unique boundary interval $[t_{\text{in}}, t_{\text{out}}]$, with $t_{\text{in}} \geq 0$ and $t_{\text{out}} \leq t_f$ ¹, and $\beta x_{\min} - g(t) > 0$ for all $t \in]t_{\text{in}}, t_{\text{out}}[$;
- (A4) the multiplier μ associated with the state constraint is a Borel regular measure decomposed by the Lebesgue Decomposition Theorem as $\mu = \mu_{ab} + \mu_{\text{sing}} + \mu_d$, with $\mu_{\text{sing}} = 0$ (here μ_{ab} is an absolutely continuous measure with respect to the Lebesgue measure, μ_{sing} is a singular measure and μ_d is a discrete measure).

¹ This means that $\bar{x}(t) = x_{\min}$ for all $t \in [t_{\text{in}}, t_{\text{out}}]$ and $\bar{x}(t) \neq x_{\min}$ for $t \notin [t_{\text{in}}, t_{\text{out}}]$.

Assumption (A3) is confirmed by the simulations reported in [12]. Observe also that (A4) is not restrictive, since this is the situation in most, if not all, practical problems (see, e.g., [10, 20]). Moreover, and as shown in [12], the bounded variation multiplier q has a single jump in $t = t_{\text{out}}$ and it is absolutely continuous in the remaining instants of time.

The optimal control is, then, given by:

$$\bar{u}(t) = \begin{cases} 0, & t \in [0, t_{\text{in}}[, \\ \beta x_{\min} - g(t), & t \in]t_{\text{in}}, t_{\text{out}}[, \\ 0, & t \in]t_{\text{out}}, t_f], \end{cases} \quad (1)$$

where, as stated before, $g(t) = \text{rfall}(t) - \text{evtp}(t)$ for all $t \in [0, t_f]$. In what follows, and to simplify the exposition, we write $g(t)$ instead of $\text{rfall}(t) - \text{evtp}(t)$. Given (1), the optimal state is, then, given by

$$\bar{x}(t) = \begin{cases} e^{-\beta t} \left(x_0 + \int_0^t e^{\beta s} g(s) ds \right), & t \in [0, t_{\text{in}}[, \\ x_{\min}, & t \in [t_{\text{in}}, t_{\text{out}}], \\ e^{-\beta(t-t_{\text{out}})} \left(\int_{t_{\text{out}}}^t e^{\beta(s-t_{\text{out}})} g(s) ds + x_{\min} \right), & t \in]t_{\text{out}}, t_f]. \end{cases} \quad (2)$$

Under (A3), we are assuming that there is an unique boundary interval $[t_{\text{in}}, t_{\text{out}}]$. Consequently, the optimal trajectory and the corresponding optimal control are unique. So, small variations of β lead to small variations in the optimal control, if the state trajectory is stable, under small variations of β . We emphasize that optimal control problem under study is convex with bounded control, as well as the dynamics is Lipschitz continuous and depends continuously on β .

If assumption (A3) were not imposed, the optimal trajectories could lie on the boundary on several disjoint intervals for some $\beta \in [\beta_{\min}, \beta_{\max}]$. As highlighted in [12], the analysis can nevertheless be carried out by decomposing the time interval and performing a piecewise analytical study.

3. Induced optimization problem

We first recall that $\text{OC}(\beta)$ problem has an unique control variable which appears linearly in the objective function and in the dynamical equation. Since the parameter β can be subject to perturbations, our aim is to carry out a parameter sensitivity analysis around β . As stated in the Introduction, we can not apply known results from sensitivity analysis, because the multiplier q is not discontinuous at t_{in} . So, a fresh analysis is called for. In this respect, the first step is to define a non-linear optimization problem, induced by $\text{OC}(\beta)$. Once this is done, we then proceed following the approach in [5].

3.1. Formulation

Here, we transcribe the $\text{OC}(\beta)$ problem into a finite-dimensional non-linear optimization problem, denoted by $\text{NLP}(\beta)$. Such transcription is possible, since we know the structure of the optimal solution of $\text{OC}(\beta)$ problem.

The variables of $\text{NLP}(\beta)$ problem are two instants of time: t_1 and t_2 that represent the moments in which the trajectory enters and leaves the boundary, respectively. This is of foremost importance, since we transcribe the optimal control problem into a non-linear programming one with only two variables.

Proposition 3.1. *Suppose that the optimal solution $(\bar{x}, \bar{u})$ of $\text{OC}(\beta)$ problem, defined as in (2) and (1), satisfies assumptions (A1)–(A4). Then $z^* := [t_{\text{in}} \ t_{\text{out}}]^T$ solves the following finite-dimensional non-linear programming problem, given by*

$$\text{NLP}(\beta) \quad \begin{cases} \max & \int_{t_1}^{t_2} (\beta x_{\min} - g(s)) ds \\ \text{s. t.} & e^{-\beta t_1} \left(x_0 + \int_0^{t_1} e^{\beta s} g(s) ds \right) = x_{\min}. \end{cases}$$

Remark 3.2. In the problem $\mathbf{NLP}(\beta)$ we seek values $t_1, t_2 \in [0, t_f]$ maximizing the functional subject to equality $e^{-\beta t_1}(x_0 + \int_0^{t_1} e^{\beta s} g(s) ds) = x_{\min}$. Thus, the goal of $\mathbf{NLP}(\beta)$ problem is to determine the maximum interval of time during which the soil needs to be irrigated. As we work under assumption (A3), there is only one such interval in $[0, t_f]$. $\square$

Proof. We now prove that the optimal solution of $\mathbf{NLP}(\beta)$, $z^* = [t_1^* \ t_2^*]^T$, is equal to $[t_{\text{in}} \ t_{\text{out}}]^T$, where t_{in} and t_{out} are the entry and exit times of the optimal trajectory of $\mathbf{OC}(\beta)$ problem.

First, observe that $[t_{\text{in}} \ t_{\text{out}}]^T$ is an admissible solution to $\mathbf{NLP}(\beta)$, since it verifies the equality constraint of $\mathbf{NLP}(\beta)$, due to the continuity of the optimal state given by (2). Moreover, it is an easy task to conclude that

$$t_1^* < t_2^*.$$

In what follows we prove that $t_1^* = t_{\text{in}}$ and $t_2^* = t_{\text{out}}$, by contradiction. So, let us assume that $z^* = [t_1^* \ t_2^*]^T$ solves $\mathbf{NLP}(\beta)$, with $t_1^* \neq t_{\text{in}}$ and/or $t_2^* \neq t_{\text{out}}$.

Case 1: To prove that $t_1^* = t_{\text{in}}$, let us exclude the cases $t_1^* < t_{\text{in}}$ and $t_1^* > t_{\text{in}}$.

Let us exclude the case $t_1^* < t_{\text{in}}$. In fact, if this would hold, by definition of t_{in} , we would have

$$\bar{x}(t_1^*) = e^{-\beta t_1^*} \left(x_0 + \int_0^{t_1^*} e^{\beta s} g(s) ds \right) > x_{\min},$$

violating the equality constraint of $\mathbf{NLP}(\beta)$ problem.

It remains to prove that $t_1^* > t_{\text{in}}$ is false. To do so, we start by assuming that $t_1^* > t_{\text{in}}$.

Since

$$\beta x_{\min} - g(t) > 0, \quad \forall t \in]t_{\text{in}}, t_{\text{out}}[,$$

we have

$$\int_{t_{\text{in}}}^{t_1^*} (\beta x_{\min} - g(s)) ds > \int_{t_1^*}^{t_2^*} (\beta x_{\min} - g(s)) ds$$

contradicting the optimality of z^* . As we prove that both $t_1^* < t_{\text{in}}$ and $t_1^* > t_{\text{in}}$ are false, then we prove that $t_1^* = t_{\text{in}}$.

Case 2: To prove that $t_2^* = t_{\text{out}}$, let us exclude the cases $t_2^* < t_{\text{out}}$ and $t_2^* > t_{\text{out}}$.

Let us prove, by contradiction, that the case $t_2^* < t_{\text{out}}$ is false. In fact, if $t_2^* < t_{\text{out}}$, then

$$\int_{t_{\text{in}}}^{t_2^*} (\beta x_{\min} - g(s)) ds < \int_{t_{\text{in}}}^{t_{\text{out}}} (\beta x_{\min} - g(s)) ds,$$

contradicting the optimality of z^* .

Now, it remains to exclude the case $t_2^* > t_{\text{out}}$. If we consider that $t_2^* > t_{\text{out}}$, then

$$\int_{t_{\text{in}}}^{t_2^*} (\beta x_{\min} - g(s)) ds = \int_{t_{\text{in}}}^{t_{\text{out}}} (\beta x_{\min} - g(s)) ds + \int_{t_{\text{out}}}^{t_2^*} (\beta x_{\min} - g(s)) ds.$$

From (A3), we get that $\int_{t_{\text{in}}}^{t_{\text{out}}} (\beta x_{\min} - g(s)) ds > 0$. Now, it remains to analyse the signal of $\int_{t_{\text{out}}}^{t_2^*} (\beta x_{\min} - g(s)) ds$.

As $\bar{x}$ is a continuous function, from Section 2.2, we know that $\bar{x}(t_{\text{out}}) = x_{\min}$ and $\bar{x}(t_2^*) > x_{\min}$, since we are assuming that $t_2^* \in]t_{\text{out}}, t_f]$.

Thus,

$$\bar{x}(t_2^*) - \bar{x}(t_{\text{out}}) > 0 \quad \text{for} \quad t_2^* \in]t_{\text{out}}, t_f]. \quad (3)$$

On the other hand, we also know that

$$\bar{x}(t_2^*) - \bar{x}(t_{\text{out}}) = \int_{t_{\text{out}}}^{t_2^*} \dot{\bar{x}}(s) ds = \int_{t_{\text{out}}}^{t_2^*} (g(s) - \beta \bar{x}(s)) ds,$$

since $\bar{u}(t) = 0$ for all $t \in]t_{\text{out}}, t_f]$. As we have (3) and $\bar{x}(t) \geq x_{\min}$ for all $t \in [0, t_f]$, then we get that

$$0 < \int_{t_{\text{out}}}^{t_2^*} (g(s) - \beta \bar{x}(s)) ds \leq \int_{t_{\text{out}}}^{t_2^*} (g(s) - \beta x_{\min}) ds.$$

Consequently, we obtain

$$\int_{t_{\text{out}}}^{t_2^*} (\beta x_{\min} - g(s)) ds < 0$$

which contradicts the intention of maximization of the cost function of $\mathbf{NLP}(\beta)$. As we prove that both $t_2^* < t_{\text{out}}$ and $t_2^* > t_{\text{out}}$ are false, then we prove that $t_2^* = t_{\text{out}}$. This concludes the proof. $\square$

The formulation of our $\mathbf{OC}(\beta)$ problem has a big disadvantage. This is the nature of the function g , which is unknown. However, values of g at some set of discrete points may be estimated. Assuming this holds, we consider an approximation of the cost function of $\mathbf{NLP}(\beta)$ in terms of the new variables $\xi_1 = t_1$ and $\xi_2 = t_2 - t_1$ as it follows:

$$\int_{t_1}^{t_2} (\beta x_{\min} - g(s)) ds \simeq \xi_2 \left[\beta x_{\min} - \frac{1}{N-1} \sum_{k=1}^{N-1} g \left(\xi_1 + \frac{\xi_2(k-1)}{N-1} \right) \right].$$

Additionally, we approximate the function $\int_0^{t_1} e^{\beta s} g(s) ds$ as follows

$$\int_0^{t_1} e^{\beta s} g(s) ds \simeq \frac{\xi_1}{N-1} \sum_{k=1}^{N-1} e^{\beta \frac{\xi_1(k-1)}{N-1}} g \left(\frac{\xi_1(k-1)}{N-1} \right).$$

Here, and throughout, we consider $N \in \mathbb{N}$ sufficiently big.

Lemma 3.3. *For each $\beta \in [\beta_{\min}, \beta_{\max}]$ and for N sufficiently big, the $\mathbf{NLP}(\beta)$ problem can be approximated by the discretized problem defined as:*

$$\widetilde{\mathbf{NLP}}(\beta) \quad \begin{cases} \max & \xi_2 \left[\beta x_{\min} - \frac{1}{N-1} \sum_{k=1}^{N-1} g \left(\xi_1 + \frac{\xi_2(k-1)}{N-1} \right) \right] \\ \text{s. t.} & e^{-\beta \xi_1} \left[x_0 + \frac{\xi_1}{N-1} \sum_{k=1}^{N-1} e^{\beta \frac{\xi_1(k-1)}{N-1}} g \left(\frac{\xi_1(k-1)}{N-1} \right) \right] = x_{\min}, \end{cases}$$

where $\tilde{z} := [\xi_1 \ \xi_2]^T$ is the optimization variable of $\widetilde{\mathbf{NLP}}(\beta)$.

Remark 3.4. It is worth noting that the convex optimal control problem $\mathbf{OC}(\beta)$ is transcribed into a non-linear programming problem $\widetilde{\mathbf{NLP}}(\beta)$ involving only two optimization variables, which represents a significant advantage. However, the resulting $\widetilde{\mathbf{NLP}}(\beta)$ formulation is not convex, despite the convexity of the original $\mathbf{OC}(\beta)$ problem. Recall that the unique equality constraint of $\widetilde{\mathbf{NLP}}(\beta)$ is not affine. $\square$

3.2. Necessary optimality conditions

Here, we now establish the necessary optimality conditions for $\widetilde{\mathbf{NLP}}(\beta)$. We do so, applying known necessary conditions for non-linear programming problems to $\widetilde{\mathbf{NLP}}(\beta)$.

Let us start by defining the Lagrangian function $L : \mathbb{R}^3 \times [\beta_{\min}, \beta_{\max}] \rightarrow \mathbb{R}$ associated to the problem $\widetilde{\mathbf{NLP}}(\beta)$ problem as

$$L(\xi_1, \xi_2, \mu, \beta) = L_A(\xi_1, \xi_2, \beta) + \mu L_B(\xi_1, \xi_2, \beta),$$

where

$$L_A(\xi_1, \xi_2, \beta) := -\xi_2 \left[\beta x_{\min} - \frac{1}{N-1} \sum_{k=1}^{N-1} g \left(\xi_1 + \frac{\xi_2(k-1)}{N-1} \right) \right]$$

and

$$L_B(\xi_1, \xi_2, \beta) := e^{-\beta \xi_1} \left[x_0 + \frac{\xi_1}{N-1} \sum_{k=1}^{N-1} e^{\beta \frac{\xi_1(k-1)}{N-1}} g \left(\frac{\xi_1(k-1)}{N-1} \right) \right] - x_{\min}.$$

Note that $\widetilde{\text{NLP}}(\beta)$ has only one constraint and this constraint is an equality one defined by:

$$C_a(\xi_1, \xi_2, \beta) = L_B(\xi_1, \xi_2, \beta). \quad (4)$$

In the following proposition, we obtain the necessary optimality conditions for $\widetilde{\text{NLP}}(\beta)$ for a fixed parameter $\beta \in [\beta_{\min}, \beta_{\max}]$.

Proposition 3.5. (Necessary Optimality Conditions) *Let $\hat{\xi} = [\hat{\xi}_1 \quad \hat{\xi}_2]^T$ be an optimal solution of $\widetilde{\text{NLP}}(\beta)$ for which $\text{rank}(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)) = 1$, for a given parameter $\beta \in [\beta_{\min}, \beta_{\max}]$. Then, there is a uniquely determined multiplier $\hat{\mu} \in \mathbb{R}$ given by:*

$$\hat{\mu} = \frac{-\hat{h}_2 \sum_{k=1}^{N-1} g'(\hat{\xi}_1 + \hat{h}_2 k^-)}{\sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \{g(\hat{h}_1 k^-) + \hat{h}_1 [k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-)]\} - x_0 \beta e^{-\beta \hat{\xi}_1}}, \quad (5)$$

and, additionally, the following equation is verified:

$$\frac{1}{N-1} \sum_{k=1}^{N-1} \{g(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-)\} = \beta x_{\min}, \quad (6)$$

where $\hat{h}_1 = \frac{\hat{\xi}_1}{N-1}$, $\beta_k^N = \beta(N-k)$, $k^- = k-1$, $\hat{h}_2 = \frac{\hat{\xi}_2}{N-1}$ and g' is the first derivative of function g .

Proof. For a given parameter $\beta \in [\beta_{\min}, \beta_{\max}]$, let us assume that $\hat{\xi} = [\hat{\xi}_1 \quad \hat{\xi}_2]^T$ is an optimal solution of $\widetilde{\text{NLP}}(\beta)$ for which $\text{rank}(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)) = 1$. Thus, from the Jacobian

$$\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta) = \begin{bmatrix} \frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) & \frac{\partial L_B}{\partial \xi_2}(\hat{\xi}, \beta) \end{bmatrix},$$

where

$$\begin{cases} \frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) = \sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \{g(\hat{h}_1 k^-) + \hat{h}_1 [k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-)]\} \\ \quad - x_0 \beta e^{-\beta \hat{\xi}_1}, \\ \frac{\partial L_B}{\partial \xi_2}(\hat{\xi}, \beta) = 0, \end{cases} \quad (7)$$

with $\hat{h}_1 = \frac{\hat{\xi}_1}{N-1}$, $\beta_k^N = \beta(N-k)$ and $k^- = k-1$, we get that

$$\sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \{g(\hat{h}_1 k^-) + \hat{h}_1 [k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-)]\} \neq x_0 \beta e^{-\beta \hat{\xi}_1}.$$

Under such assumptions, there is a uniquely determined multiplier $\hat{\mu} \in \mathbb{R}$ verifying the following Karush-Kuhn-Tucker (KKT) conditions (see [5, Theorem 1]):

$$\frac{\partial L}{\partial \xi}(\hat{\xi}, \hat{\mu}, \beta) = 0. \quad (8)$$

As we have (7) and

$$\begin{cases} \frac{\partial L_A}{\partial \xi_1}(\hat{\xi}, \beta) = \hat{h}_2 \sum_{k=1}^{N-1} g'(\hat{\xi}_1 + \hat{h}_2 k^-), \\ \frac{\partial L_A}{\partial \xi_2}(\hat{\xi}, \beta) = \frac{1}{N-1} \sum_{k=1}^{N-1} \{g(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-)\} - \beta x_{\min}, \end{cases} \quad (9)$$

where $\hat{h}_2 = \frac{\hat{\xi}_2}{N-1}$, then

$$(8) \Leftrightarrow \begin{cases} \frac{\partial L_A}{\partial \xi_1}(\hat{\xi}, \beta) + \hat{\mu} \frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) = 0, \\ \frac{\partial L_A}{\partial \xi_2}(\hat{\xi}, \beta) + \hat{\mu} \frac{\partial L_B}{\partial \xi_2}(\hat{\xi}, \beta) = 0, \end{cases} \Leftrightarrow \begin{cases} \hat{\mu} = -\frac{\frac{\partial L_A}{\partial \xi_1}(\hat{\xi}, \beta)}{\frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta)}, \\ \frac{\partial L_A}{\partial \xi_2}(\hat{\xi}, \beta) = 0, \end{cases}$$

that is equivalent to

$$\left\{ \begin{array}{l} \hat{\mu} = \frac{-\hat{h}_2 \sum_{k=1}^{N-1} g'(\hat{\xi}_1 + \hat{h}_2 k^-)}{\sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \left\{ g(\hat{h}_1 k^-) + \hat{h}_1 \left[k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-) \right] \right\} - x_0 \beta e^{-\beta \hat{\xi}_1}}, \\ \frac{1}{N-1} \sum_{k=1}^{N-1} \left\{ g(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-) \right\} = \beta x_{\min}, \end{array} \right.$$

which implies (5) and (6). This concludes the proof. $\square$

Remark 3.6. We remark that since we do not know a general analytical expression for function g , then it is not possible to determine an analytical expression for $g(\hat{h}_1 k^-)$, $g'(\hat{h}_1 k^-)$, $g(\hat{\xi}_1 + \hat{h}_2 k^-)$ and $g'(\hat{\xi}_1 + \hat{h}_2 k^-)$ which appear in Proposition 3.5 and in its proof. $\square$

3.3. Second-order sufficient optimality conditions

For the parameter sensitivity analysis, the verification of second-order sufficient optimality conditions is crucial. Besides identifying the solution of $\widetilde{\text{NLP}}(\beta)$ from the set of admissible solutions satisfying (5) and (6), second-order sufficient optimality conditions also allow us to prove that the optimal state and control of $\widetilde{\text{NLP}}(\beta)$ are differentiable with respect to β : an essential property for the study of parameter sensitivity analysis associated with $\widetilde{\text{NLP}}(\beta)$ (see [5]). Those conditions for $\widetilde{\text{NLP}}(\beta)$ are stated in the following proposition.

Proposition 3.7. (Sufficient Optimality Conditions) *Let $\hat{\xi}$ and $\hat{\mu}$ be, respectively, an admissible point to $\widetilde{\text{NLP}}(\beta)$ and the corresponding multiplier which satisfy (5) and (6) for a given parameter $\beta \in [\beta_{\min}, \beta_{\max}]$. If*

$$\left\{ \begin{array}{l} \frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) \neq 0, \\ \frac{1}{(N-1)^2} \sum_{k=1}^{N-1} \left\{ 2k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-) + (k^-)^2 \hat{h}_2 g''(\hat{\xi}_1 + \hat{h}_2 k^-) \right\} > 0, \end{array} \right. \quad (10)$$

then $\hat{\xi}$ is a strong local minimum of $\widetilde{\text{NLP}}(\beta)$.

Proof. Let us start by considering that $\hat{\xi}$ and $\hat{\mu}$ are, respectively, an admissible point to $\widetilde{\text{NLP}}(\beta)$ and the corresponding multiplier satisfying (5) and (6), for a given parameter $\beta \in [\beta_{\min}, \beta_{\max}]$. From [5, Theorem 2], if $\text{rank}(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)) = 1$ and $H^T \frac{\partial^2 L}{\partial \xi^2}(\hat{\xi}, \hat{\mu}, \beta) H$ has a positive eigenvalue, where H is a 2×1 matrix with full column rank whose column spans the kernel of $\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)$, then $\hat{\xi}$ is a strong local minimum of $\widetilde{\text{NLP}}(\beta)$. As we are assuming that $\frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) \neq 0$, then we are ensuring that $\text{rank}(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)) = 1$ due to (4) and (7). So, it remains to prove that

$$\begin{aligned} & H^T \frac{\partial^2 L}{\partial \xi^2}(\hat{\xi}, \hat{\mu}, \beta) H \text{ has a positive eigenvalue} \\ \Leftrightarrow & \frac{1}{(N-1)^2} \sum_{k=1}^{N-1} \left\{ 2k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-) + (k^-)^2 \hat{h}_2 g''(\hat{\xi}_1 + \hat{h}_2 k^-) \right\} > 0. \end{aligned} \quad (11)$$

To do that, let us first obtain matrix H . The kernel of $\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)$ is given by:

$$\ker \left(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta) \right) = \left\{ v \in \mathbb{R}^2 : \frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)v = \begin{bmatrix} 0 & 0 \end{bmatrix} \right\}.$$

Analysing the condition of the kernel, we have

$$\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)v = \begin{bmatrix} 0 & 0 \end{bmatrix} \Leftrightarrow \frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta)v_1 = 0 \Leftrightarrow v_1 = 0 \wedge v_2 \in \mathbb{R},$$

as we are considering that $\text{rank}\left(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)\right) = 1$ which is equivalent to $\frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) \neq 0$, since $\frac{\partial L_B}{\partial \xi_2}(\hat{\xi}, \beta) = 0$. Thus, we have that $\ker\left(\frac{\partial C_a}{\partial \xi}(\hat{\xi}, \beta)\right) = \langle (0, 1) \rangle$ which implies that $H = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Consequently, we obtain

$$H^T \frac{\partial^2 L}{\partial \xi^2}(\hat{\xi}, \hat{\mu}, \beta)H = \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta)$$

and

$$\begin{aligned} \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta) &= \frac{\partial}{\partial \xi_2} \left(\frac{1}{N-1} \sum_{k=1}^{N-1} [g(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-)] - \beta x_{\min} \right) \\ &= \frac{1}{(N-1)^2} \sum_{k=1}^{N-1} \left\{ 2k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-) + (k^-)^2 \hat{h}_2 g''(\hat{\xi}_1 + \hat{h}_2 k^-) \right\} \end{aligned}$$

which proves (11). This concludes the proof. $\square$

4. Parameter sensitivity analysis

From now on, β_0 is a fixed nominal value of parameter β and B_0 is a neighbourhood of β_0 . In this section, we provide a parameter sensitivity analysis with respect to the optimal solution and the optimal cost of $\widetilde{\text{NLP}}(\beta)$. We do that by obtaining approximate solutions and costs to $\widetilde{\text{NLP}}(\beta)$ for $\beta \in B_0$, determined by first-order Taylor expansion around the nominal solution and nominal cost of $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$ for a fixed β_0 .

Proposition 4.1. *For a given nominal value β_0 , let $\hat{\xi} = [\hat{\xi}_1 \quad \hat{\xi}_2]^T$ and $\hat{\mu}$ be, respectively, an admissible point and a multiplier for the unperturbed problem $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$, satisfying necessary and sufficient optimality conditions of Propositions 3.5 and 3.7. Then, there is a neighbourhood $B_0 \subseteq [\beta_{\min}, \beta_{\max}]$ of β_0 and there are unique continuously differentiable functions $\xi : B_0 \rightarrow \mathbb{R}^2$ and $\mu : B_0 \rightarrow \mathbb{R}$ such that:*

- (A) $\xi(\beta_0) = \hat{\xi}$ and $\mu(\beta_0) = \hat{\mu}$;
- (B) the equality constraint of $\widetilde{\text{NLP}}(\beta)$ is satisfied for all $\beta \in B_0$;
- (C) $\text{rank}\left(\frac{\partial C_a}{\partial \xi}(\xi(\beta), \beta)\right) = 1$ for all $\beta \in B_0$;
- (D) for all $\beta \in B_0$, the pair $(\xi(\beta), \mu(\beta))$ satisfies second-order sufficient optimality conditions of Proposition 3.7 for the perturbed problem $\widetilde{\text{NLP}}(\beta)$. In particular, $(\xi(\beta), \mu(\beta))$ is a strong local minimum of $\widetilde{\text{NLP}}(\beta)$.

Moreover, we get the following approximations for functions $\xi = [\xi_1 \quad \xi_2]^T$ and μ , for all $\beta \in B_0$:

$$\begin{cases} \begin{bmatrix} \xi_1(\beta) \\ \xi_2(\beta) \end{bmatrix} \approx \begin{bmatrix} \hat{\xi}_1 + \frac{d\xi_1}{d\beta}(\beta_0)(\beta - \beta_0) \\ \hat{\xi}_2 + \frac{d\xi_2}{d\beta}(\beta_0)(\beta - \beta_0) \end{bmatrix}, \\ \mu(\beta) \approx \hat{\mu} + \frac{d\mu}{d\beta}(\beta_0)(\beta - \beta_0), \end{cases} \quad (12)$$

where the sensitivity differentials $\frac{d\xi_1}{d\beta}(\beta_0)$, $\frac{d\xi_2}{d\beta}(\beta_0)$ and $\frac{d\mu}{d\beta}(\beta_0)$ are given by:

$$\left\{ \begin{array}{l} \frac{d\xi_1}{d\beta}(\beta_0) = \frac{1}{D} \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0); \\ \frac{d\xi_2}{d\beta}(\beta_0) = \frac{1}{D} \left(\frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \right)^2 \frac{\partial^2 L}{\partial \xi_2 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ - \frac{1}{D} \frac{\partial^2 L}{\partial \xi_2 \partial \xi_1}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0); \\ \frac{d\mu}{d\beta}(\beta_0) = \frac{1}{D} \left\{ \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \frac{\partial^2 L}{\partial \xi_1 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \right. \\ - \frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \frac{\partial^2 L}{\partial \xi_2 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ + \left(\frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) \right)^2 \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0) \\ \left. - \frac{\partial^2 L}{\partial \xi_1^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0) \right\}; \end{array} \right. \quad (13)$$

with

$$\left\{ \begin{array}{l} \frac{\partial^2 L}{\partial \xi_1^2}(\hat{\xi}, \hat{\mu}, \beta_0) = \hat{h}_2 \sum_{k=1}^{N-1} g''(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{\mu} x_0 \beta^2 e^{-\beta \hat{\xi}_1} \\ + \hat{\mu} \sum_{k=1}^{N-1} \left\{ \frac{e^{-\hat{h}_1 \beta_k^N}}{(N-1)^2} [2k^- g'(\hat{h}_1 k^-) - 2\beta_k^N g(\hat{h}_1 k^-) \right. \\ + \hat{h}_1 (k^-)^2 g''(\hat{h}_1 k^-) \\ \left. - \hat{h}_1 \left\{ \beta_k^N (k^- g'(\hat{h}_1 k^-) + \beta_k^N g(\hat{h}_1 k^-)) \right\} \right\}; \\ \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) = \frac{\sum_{k=1}^{N-1} k^- \{2g'(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g''(\hat{\xi}_1 + \hat{h}_2 k^-)\}}{(N-1)^2}; \\ \frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) = \frac{\partial^2 L}{\partial \xi_2 \partial \xi_1}(\hat{\xi}, \hat{\mu}, \beta_0) \\ = \frac{\sum_{k=1}^{N-1} \{g'(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g''(\hat{\xi}_1 + \hat{h}_2 k^-)\}}{N-1}; \\ \frac{\partial C_a}{\partial \xi_j}(\hat{\xi}, \beta_0) = \frac{\partial L_B}{\partial \xi_j}(\hat{\xi}, \beta_0), \quad j = 1, 2; \\ \frac{\partial^2 L}{\partial \xi_1 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) = \hat{\mu} \sum_{k=1}^{N-1} \frac{k-N}{N-1} e^{-\beta_k^N \hat{h}_1} \left\{ [2\hat{h}_1 - (\hat{h}_1)^2 \beta_k^N] g(\hat{h}_1 k^-) \right. \\ \left. + k^- g'(\hat{h}_1 k^-) \right\} + \hat{\mu} x_0 e^{-\beta \hat{\xi}_1} (\beta \hat{\xi}_1 - 1); \\ \frac{\partial^2 L}{\partial \xi_2 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) = -x_{\min}; \\ \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0) = (\hat{h}_1)^2 \sum_{k=1}^{N-1} (k-N) e^{-\beta_k^N \hat{h}_1} g(\hat{h}_1 k^-) - x_0 \hat{\xi}_1 e^{-\beta \hat{\xi}_1}; \end{array} \right. \quad (15)$$

and

$$D = -\frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \left(\frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \right)^2.$$

Remark 4.2. The Proposition 4.1 ensures the stability of the minimizer $\xi(\cdot)$ and the corresponding multiplier $\mu(\cdot)$, since ξ and μ are unique and continuously differentiable functions of β . Furthermore, System (12) characterizes the parametric dependence of both the solution and the multiplier. $\square$

Proof. Let us suppose that $\hat{\xi} = [\hat{\xi}_1 \quad \hat{\xi}_2]^T$ and $\hat{\mu}$ are, respectively, an admissible point and a multiplier for the unperturbed problem $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$, satisfying necessary and sufficient optimality conditions of Propositions 3.5 and 3.7. Then, [5, Theorem 3] and [8, Theorem 5.2] asserts that there is a neighbourhood $B_0 \subseteq [\beta_{\min}, \beta_{\max}]$ of β_0 and that there are unique continuously differentiable functions $\xi : B_0 \rightarrow \mathbb{R}^2$ and $\mu : B_0 \rightarrow \mathbb{R}$ with the following properties:

- (A) $\xi(\beta_0) = \hat{\xi}$ and $\mu(\beta_0) = \hat{\mu}$;
- (B) the active sets are constant in B_0 , that is, the equality constraint of $\widetilde{\text{NLP}}(\beta)$ is satisfied for all $\beta \in B_0$;
- (C) $\text{rank} \left(\frac{\partial C_a}{\partial \xi} \right) (\xi(\beta), \beta) = 1$ for all $\beta \in B_0$;
- (D) for all $\beta \in B_0$, the pair $(\xi(\beta), \mu(\beta))$ satisfies second-order sufficient optimality conditions of Proposition 3.7 for the perturbed problem $\widetilde{\text{NLP}}(\beta)$. In particular, $(\xi(\beta), \mu(\beta))$ is a strong local minimum of $\widetilde{\text{NLP}}(\beta)$.

In what follows, we provide approximate solutions to $\widetilde{\text{NLP}}(\beta)$, determined by the first-order Taylor expansion around the nominal solution to the unperturbed problem $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$ for a fixed β_0 . The differentiability properties of optimal solutions of $\widetilde{\text{NLP}}(\beta)$ for $\beta \in B_0$, provided from [5, Theorem 3], are of fundamental importance for real-time approximation of perturbed solution $\xi(\beta) = [\xi_1(\beta) \quad \xi_2(\beta)]^T$ and perturbed multiplier $\mu(\beta)$, by their first-order Taylor expansions given by:

$$\begin{cases} \begin{bmatrix} \xi_1(\beta) \\ \xi_2(\beta) \end{bmatrix} \approx \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} + \begin{bmatrix} \frac{d\xi_1}{d\beta}(\beta_0) \\ \frac{d\xi_2}{d\beta}(\beta_0) \end{bmatrix} (\beta - \beta_0), \\ \mu(\beta) \approx \hat{\mu} + \frac{d\mu}{d\beta}(\beta_0)(\beta - \beta_0), \end{cases}$$

where $\frac{d\xi}{d\beta}(\beta_0) = \begin{bmatrix} \frac{d\xi_1}{d\beta}(\beta_0) & \frac{d\xi_2}{d\beta}(\beta_0) \end{bmatrix}^T$ and $\frac{d\mu}{d\beta}(\beta_0)$ are sensitivity differentials obtained from

$$\begin{aligned} \begin{bmatrix} \frac{d\xi}{d\beta}(\beta_0) \\ \frac{d\mu}{d\beta}(\beta_0) \end{bmatrix} &= -M \begin{bmatrix} \frac{\partial^2 L}{\partial \xi \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0) \end{bmatrix} \\ &= - \begin{bmatrix} \frac{\partial^2 L}{\partial \xi_1^2}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \\ \frac{\partial^2 L}{\partial \xi_2 \partial \xi_1}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial C_a}{\partial \xi_2}(\hat{\xi}, \beta_0) \\ \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) & \frac{\partial C_a}{\partial \xi_2}(\hat{\xi}, \beta_0) & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 L}{\partial \xi_1 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ \frac{\partial^2 L}{\partial \xi_2 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0) \end{bmatrix} \\ &= - \begin{bmatrix} \frac{\partial^2 L}{\partial \xi_1^2}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \\ \frac{\partial^2 L}{\partial \xi_2 \partial \xi_1}(\hat{\xi}, \hat{\mu}, \beta_0) & \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) & 0 \\ \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 L}{\partial \xi_1 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ \frac{\partial^2 L}{\partial \xi_2 \partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ \frac{\partial C_a}{\partial \beta}(\hat{\xi}, \beta_0) \end{bmatrix}, \end{aligned}$$

since

$$\frac{\partial C_a}{\partial \xi_2}(\hat{\xi}, \beta_0) = \frac{\partial L_B}{\partial \xi_2}(\hat{\xi}, \beta_0) = 0,$$

with $M = [m_{ij}]_{3 \times 3}$ such that:

- $m_{11} = 0;$
- $m_{12} = 0;$
- $m_{13} = -\frac{1}{D} \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0);$
- $m_{21} = 0;$
- $m_{22} = -\frac{1}{D} \left(\frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \right)^2;$
- $m_{23} = \frac{1}{D} \frac{\partial^2 L}{\partial \xi_2 \partial \xi_1}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0);$
- $m_{31} = -\frac{1}{D} \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0);$
- $m_{32} = \frac{1}{D} \frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0);$
- $m_{33} = \frac{1}{D} \left\{ \frac{\partial^2 L}{\partial \xi_1^2}(\hat{\xi}, \hat{\mu}, \beta_0) \frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) - \left(\frac{\partial^2 L}{\partial \xi_1 \partial \xi_2}(\hat{\xi}, \hat{\mu}, \beta_0) \right)^2 \right\};$

and

$$D = \det(M) = -\frac{\partial^2 L}{\partial \xi_2^2}(\hat{\xi}, \hat{\mu}, \beta_0) \left(\frac{\partial C_a}{\partial \xi_1}(\hat{\xi}, \beta_0) \right)^2.$$

So, one gets the sensitivity differentials of (13). Furthermore, from (4), (7) and (9), we obtain (14)–(15). This concludes the proof. $\square$

Proposition 4.1 allows us to know approximate solutions to $\widetilde{\text{NLP}}(\beta)$ for all $\beta \in B_0$. In the next Proposition we deduce an approximation of the perturbed optimal cost of $\widetilde{\text{NLP}}(\beta)$, for all $\beta \in B_0$.

Proposition 4.3. Assume that $\tilde{G}(\hat{\xi}, \beta_0)$ is the optimal cost of the unperturbed problem $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$. Then, an approximation of the perturbed optimal cost $\tilde{G}(\xi(\beta), \beta)$ is given by:

$$\tilde{G}(\hat{\xi}, \beta_0) + \left[\hat{\mu} e^{-\beta \hat{\xi}_1} \left\{ \hat{h}_1 \sum_{k=1}^{N-1} e^{\beta \hat{h}_1 k^-} (\hat{h}_1 k^- - \hat{\xi}_1) g(\hat{h}_1 k^-) - x_0 \hat{\xi}_1 \right\} - \hat{\xi}_2 x_{\min} \right] (\beta - \beta_0),$$

for all $\beta \in B_0$.

Proof. Let $\tilde{G}(\hat{\xi}, \beta_0) = L_A(\hat{\xi}, \beta_0)$ be the optimal cost of the unperturbed problem $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$. Taking into account reference [5], the first-order Taylor expansion around optimal cost of $\widetilde{\text{NLP}}(\beta) |_{\beta=\beta_0}$, given by

$$\tilde{G}(\xi(\beta), \beta) \approx \tilde{G}(\hat{\xi}, \beta_0) + \frac{d\tilde{G}}{d\beta}(\hat{\xi}, \beta_0)(\beta - \beta_0),$$

provides an approximation of the perturbed optimal cost $\tilde{G}(\xi(\beta), \beta)$ for all $\beta \in B_0$, where

$$\begin{aligned} \frac{d\tilde{G}}{d\beta}(\hat{\xi}, \beta_0) &= \frac{\partial L}{\partial \beta}(\hat{\xi}, \hat{\mu}, \beta_0) \\ &= \hat{\mu} e^{-\beta \hat{\xi}_1} \left\{ \hat{h}_1 \sum_{k=1}^{N-1} e^{\beta \hat{h}_1 k^-} (\hat{h}_1 k^- - \hat{\xi}_1) g(\hat{h}_1 k^-) - x_0 \hat{\xi}_1 \right\} - \hat{\xi}_2 x_{\min}. \end{aligned}$$

This concludes the proof. $\square$

5. Numerical example

In this section, under assumptions (A1)–(A4) and values of Table 1, we provide numerical simulations associated with the $\text{OC}(\beta)$ and $\widetilde{\text{NLP}}(\beta)$ problems: see Subsection 5.1. With respect to $\widetilde{\text{NLP}}(\beta)$, we verify, numerically, the necessary and sufficient optimality conditions derived in Propositions 3.5 and 3.7: see Subsection 5.2.

Table 1. Parameter values

Parameter	Value	SI Unity
N	2001	–
β_0	0.15	–
$\beta_{\min}$	0.13	–
$\beta_{\max}$	0.1625	–
t_f	9	day
x_0	44.2362	mm
$x_{\min}$	18.7200	mm
γ	1000	mm/day (per m ²)
g_1	-4.3415	mm/day (per m ²)
g_2	-3.3820	mm/day (per m ²)
g_3	-3.9805	mm/day (per m ²)
g_4	-4.3890	mm/day (per m ²)
g_5	-4.1230	mm/day (per m ²)
g_6	-4.0565	mm/day (per m ²)
g_7	-4.0090	mm/day (per m ²)
g_8	11.0315	mm/day (per m ²)
g_9	10.1385	mm/day (per m ²)
g_{10}	11.5635	mm/day (per m ²)

Finally, in Subsection 5.3, we analyse the sensitivity of the optimal solution and of the optimal cost with respect to parameter β , according to the sensitivity analysis developed in Section 4.

We recall that $g(t) = \text{rfall}(t) - \text{evtp}(t)$, where $\text{rfall}(t)$ is the daily precipitation and $\text{evtp}(t)$ is the estimated evapotranspiration for the type of crop, at t . The precipitation data is collected from a weather station that takes place in *Carrazeda de Ansiães* – Portugal. On the other hand, the evapotranspiration is obtained from the product between the crop's coefficient and the referenced evapotranspiration which is calculated according to [1], under the data of the weather station. Note that the jump in the values of the vector $[g_1 \ \cdots \ g_{10}]$, corresponding to the passage from the seventh to the eighth day, is due to sudden rainfall. We remark that we only have daily information with respect to g . However, as we intend to work numerically with a mesh-grid with more than 10 points, we approximate these values through a continuous logistic function with the help of a TI-*n*spire CX II-T CAS graphic calculator (see [12]):

$$g(t) \approx \frac{14.9566}{1 + 8.41547 \times 10^{97} \times e^{-36.5776t}} - 4.04542. \quad (16)$$

5.1. Numerical results related to the problems $\text{OC}(\beta)$ and $\widetilde{\text{NLP}}(\beta)$

In this section, we present the numerical solutions to the discretized optimal control problem, obtained using both the direct method and the new reformulation based on the indirect method, as analysed in Proposition 3.1 and Lemma 3.3.

The numerical results are shown in Figure 1 to the trajectory and in Figure 2 to the control. The dark green dashed curve of the optimal trajectory and optimal control are obtained via the direct method, where the time at which the trajectory enters the boundary (t_{in}) is, approximately, 2.96 and the time at which it exits the boundary (t_{out}) is, approximately, 6.16: see Figures 1 and 2. The light green solid curves, of Figures 1 and 2, represent the optimal solutions obtained by using the indirect method. They yield the values

$$(\xi_1, \xi_2, \mu) \approx (2.95819, 3.20249, 1.00037). \quad (17)$$

These values are observed to be close to t_{in} and $t_{\text{out}} - t_{\text{in}}$, respectively. Considering Equations (1) and (2), as well as (17), the corresponding optimal trajectory and optimal control can be determined. Consequently, the light green solid curves are drawn in Figures 1 and 2.

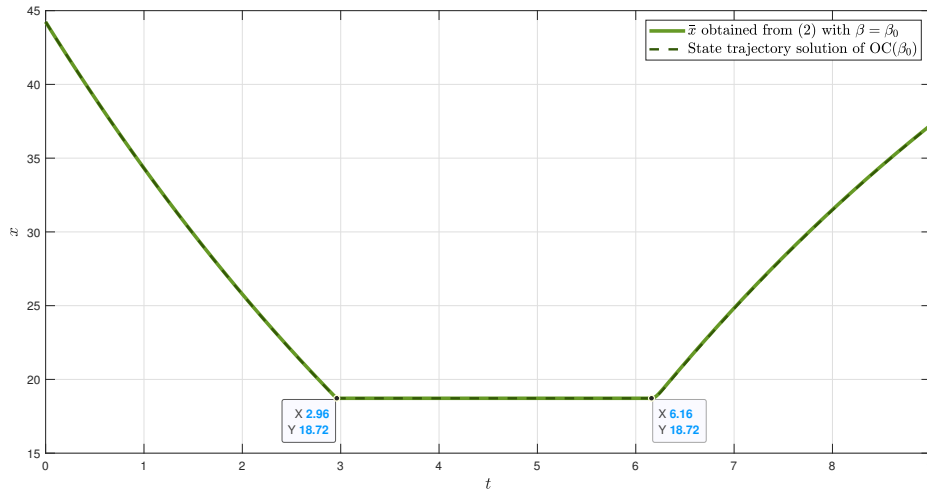


Figure 1. (Colour only in online version) Optimal trajectory obtained from (2) and from $OC(\beta)$ with $\beta = \beta_0$.

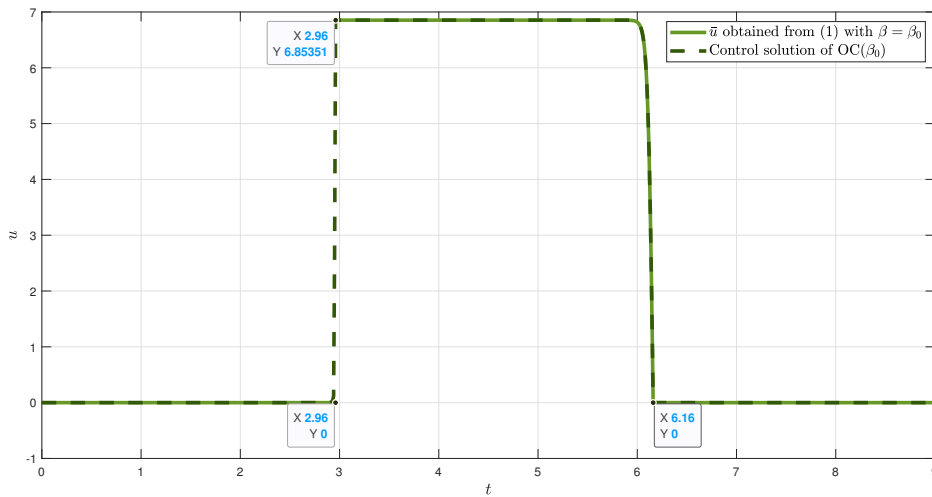


Figure 2. (Colour only in online version) Optimal control obtained from (1) and from $OC(\beta)$ with $\beta = \beta_0$.

It can be observed from Figures 1 and 2 that the solutions obtained through both the direct and indirect methods are in agreement, confirming the theoretical results established in Proposition 3.1. We emphasize that in the direct form, the problem is large and depends on several variables and constraints. This contrasts with the indirect form that consider the $NLP(\beta)$ problem of Proposition 3.1, where there are only two optimization variables, ξ_1 and ξ_2 , and only one equality constraint.

Finally, from the analysis of Figures 1 and 2, as is expected and according to the results of the research works carried out in [12, 17], irrigation is required when the humidity of the soil is equal to the minimum water requirements of the plant, $x_{\min}$.

5.2. Verification of the optimality conditions

Beyond the necessary and sufficient optimality conditions, derived in Propositions 3.5 and 3.7, are fundamental for the study of sensitivity analysis, they also allow us to ensure that the numerical solution found is, in fact, a minimizer. As stated above, in the indirect method, the optimal solution for the nominal value $\beta = \beta_0$ is given by (17).

Now, let us verify if the necessary optimality conditions (5) and (6), derived in Proposition 3.5, are satisfied for (17), by considering (16), $\beta = \beta_0$, $\hat{\xi}_1 = 2.95819$ and $\hat{\xi}_2 = 3.20249$. For the example introduced in the current section, with optimal solution (17), we have:

$$\hat{h}_2 \sum_{k=1}^{N-1} g'(\hat{\xi}_1 + \hat{h}_2 k^-) \approx 6.8544$$

and

$$\sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \{g(\hat{h}_1 k^-) + \hat{h}_1 [k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-)]\} - x_0 \beta e^{-\beta \hat{\xi}_1} \approx -6.8526.$$

This means that:

$$\frac{-\hat{h}_2 \sum_{k=1}^{N-1} g'(\hat{\xi}_1 + \hat{h}_2 k^-)}{\sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \{g(\hat{h}_1 k^-) + \hat{h}_1 [k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-)]\} - x_0 \beta e^{-\beta \hat{\xi}_1}} \approx 1.00026,$$

which is an approximate value to $\mu \approx 1.00037$ (see (17)). So, condition (5) is numerically checked. The remaining necessary optimality condition (6), derived in Proposition 3.5, is also numerically verified, since we get:

$$\frac{1}{N-1} \sum_{k=1}^{N-1} \{g(\hat{\xi}_1 + \hat{h}_2 k^-) + \hat{h}_2 k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-)\} \approx 2.8073$$

which is an approximate value of $\beta x_{\min}$ that is equal to $0.15 \times 18.7200 = 2.8080$. Thus, we conclude that for the example introduced in the current section, with optimal solution (17), the necessary optimality conditions (5) and (6) are satisfied.

It remains to check if the second-order sufficient optimality conditions (10), derived in Proposition 3.7, are satisfied for (17), by considering (16), $\beta = \beta_0$, $\hat{\xi}_1 = 2.95819$ and $\hat{\xi}_2 = 3.20249$. Note that for the optimal solution (17), we obtain:

$$\begin{aligned} \frac{\partial L_B}{\partial \xi_1}(\hat{\xi}, \beta) &= \sum_{k=1}^{N-1} \frac{e^{-\hat{h}_1 \beta_k^N}}{N-1} \{g(\hat{h}_1 k^-) + \hat{h}_1 [k^- g'(\hat{h}_1 k^-) - \beta_k^N g(\hat{h}_1 k^-)]\} - x_0 \beta e^{-\beta \hat{\xi}_1} \\ &\approx -6.8526 \neq 0 \end{aligned}$$

and

$$\frac{1}{(N-1)^2} \sum_{k=1}^{N-1} \{2k^- g'(\hat{\xi}_1 + \hat{h}_2 k^-) + (k^-)^2 \hat{h}_2 g''(\hat{\xi}_1 + \hat{h}_2 k^-)\} \approx 135.7590 > 0.$$

Concluding, the second-order sufficient optimality conditions (10) are satisfied for the example introduced in the current section with optimal solution (17).

5.3. Sensitivity results

Here, we present the numerical results associated with the parameter sensitivity analysis. Starting from the nominal solution (17) obtained in Subsection 5.1, we apply the methodology developed in Section 4 to compute numerical solutions for different values of the parameter β within a neighbourhood, B_0 , of the nominal value β_0 . We highlight that by applying parameter sensitivity analysis techniques, we are able to design around 67 trajectories and controls for different β values in an acceptable computational time: see the corresponding numerical results in Figure 3. It is possible to observe that the trajectory and the control results for different values of $\beta \in B_0 = [\beta_{\min}, \beta_{\max}]$ exhibit a similar behaviour to the nominal solutions. However, the times t_{in} and t_{out} are different for each $\beta \in B_0$.

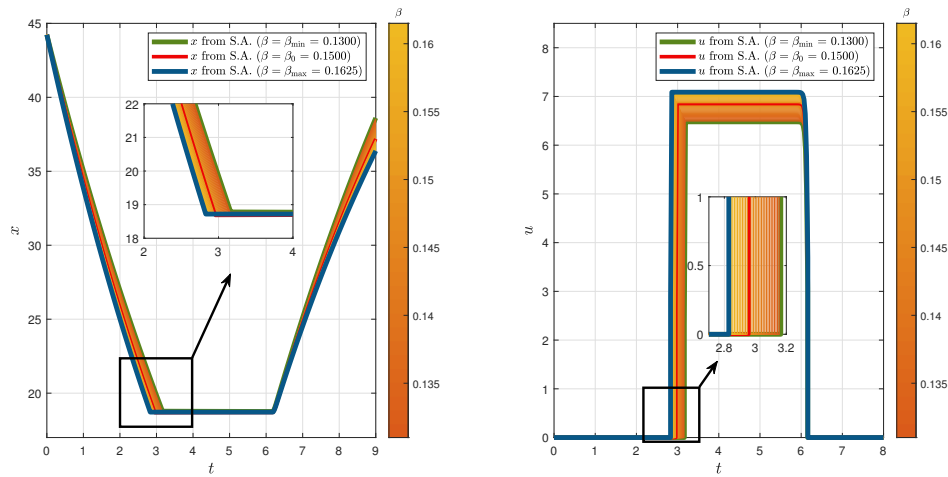


Figure 3. (Colour only in online version) Trajectory and control provided from Parameter Sensitivity Analysis.

From Figure 4, we can observe that as β increases, the cost function also increases. This is consistent with expectations, since a higher value of β implies greater water loss by percolation. Consequently, more water is required to irrigate the soil in order to meet the crop's water demand.

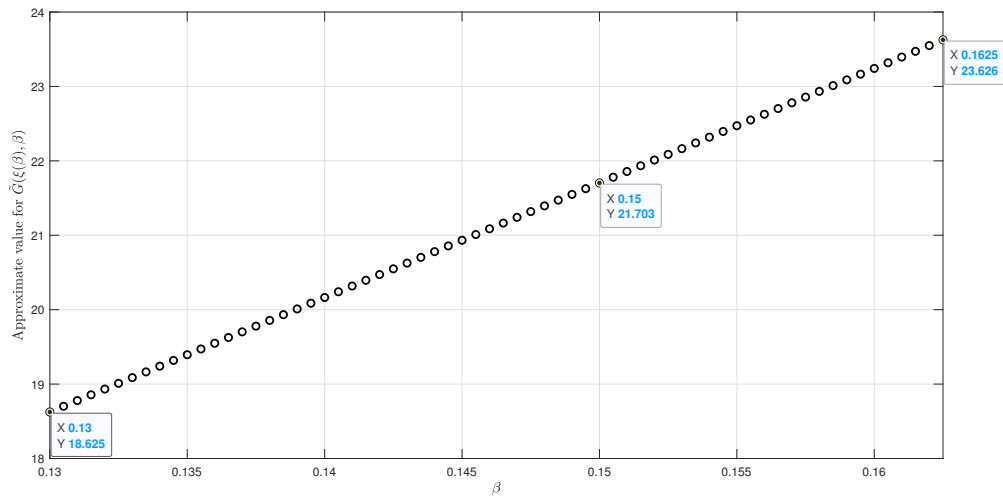


Figure 4. (Colour only in online version) The influence of β in the cost.

Figure 5 also provides results that are in agreement with what is expected for the entry and exit times, t_{in} and t_{out} , respectively, when the value of β increases. Observe that t_{in} decreases and t_{out} increases, as β increases, since an increase of the water loss by percolation implies starting the irrigation earlier and finishing it later. Moreover, from Figure 5, we remark that the amplitude of the variations of t_{in} and t_{out} are, respectively, given by:

$$|2.82919 - 3.16508| = 0.33589 \quad \text{and} \quad |6.1628 - 6.15989| = 0.00291.$$

The amplitude of the variations in t_{out} is smaller, because it rained on the eighth day. Therefore, the increase of β values causes less disturbance in t_{out} .

6. Conclusions

In this study, we analysed the irrigation problem as an optimal control one within the context of planning. By formulating irrigation as an optimal control system, we aim to optimize the water usage, while ensuring efficient agricultural productivity. The results emphasize the importance of strategic decision-making in water resource management, demonstrating that an optimal control approach can significantly enhance irrigation efficiency.

In this paper, we revisited the non-parametric optimal linear-control problem studied in [12]. It considers an inequality state constraint which brings great challenges in its resolution, since it is not possible

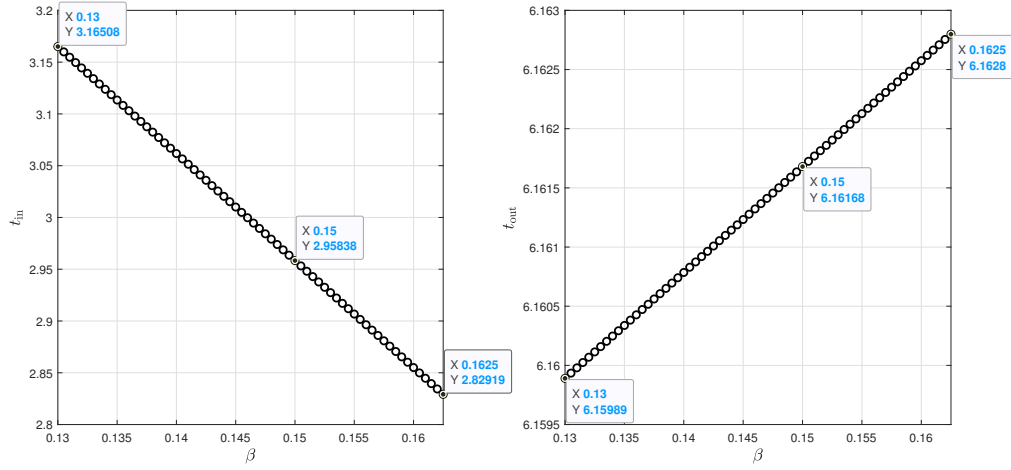


Figure 5. (Colour only in online version) The influence of β in t_{in} and t_{out} .

to obtain its analytical solution. Solving the problem of [12] by direct method leads to a large-scale problem with considerable computational effort. However, in this paper, we showed that the irrigation planning problem can be reformulated as a low-dimensional non-linear programming one, by characterizing the solution through the Pontryagin Maximum Principle.

As in any real-world problem modelling, there are parameters that are difficult to estimate, as well as sensitive to perturbations, and the irrigation problem of [12] is no exception. Its parameter β , which represents the percentage of water loss due to deep percolation, is particularly challenging to estimate, as it depends on soil characteristics that vary with rainfall, fertilizer use and crop types. Consequently, β is sensitive to environmental changes. To address this challenge, we studied a parametric optimal linear-control problem, denoted by $\mathbf{OC}(\beta)$, for all $\beta \in [\beta_{\min}, \beta_{\max}]$. This approach allows us to analyse the underlying mathematical application in a more robust and efficient manner.

Our formulation of $\mathbf{OC}(\beta)$ distinguishes itself from other agricultural irrigation optimal linear-control problems, by explicitly accounting for variations in β . Then, we transcribed $\mathbf{OC}(\beta)$ into a finite-dimensional non-linear optimization problem, denoted by $\mathbf{NLP}(\beta)$. Such transcription was possible, since we knew the structure of the optimal solution of $\mathbf{OC}(\beta)$ problem. Additionally, we proved that problem $\mathbf{NLP}(\beta)$ can be approximated by a discretized one, denoted by $\widetilde{\mathbf{NLP}}(\beta)$, due to the difficulty of finding an analytical expression for function g , as well as of solving the equality constraint of $\mathbf{NLP}(\beta)$. Once this is done, we then proceeded the parameter sensitivity analysis with respect to $\widetilde{\mathbf{NLP}}(\beta)$, by following the approach in [5]. To study the sensitivity of β , we considered, as the reference solution, the optimal solution of $\widetilde{\mathbf{NLP}}(\beta)$ for the nominal value $\beta = \beta_0$. To do so, necessary and second-order sufficient optimality conditions were derived for $\widetilde{\mathbf{NLP}}(\beta)$ problem. These conditions are crucial for identifying the minimizer and performing sensitivity analysis with respect to a parameter. Then, we applied a first-order Taylor expansion around the nominal solution and nominal cost of $\widetilde{\mathbf{NLP}}(\beta)$ at $\beta = \beta_0$ to obtain approximate solutions.

To illustrate the theoretical results summarized above, we determined the numerical nominal solution of $\mathbf{OC}(\beta)$ in two different ways, using both direct and indirect methods, based on the parameter values from Table 1. As expected, the solutions coincided, validating the adopted approach. Finally, we provided a parameter sensitivity analysis with respect to $\widetilde{\mathbf{NLP}}(\beta)$, by verifying, numerically, necessary and second-order sufficient optimality conditions. Since computing the optimal solutions of $\mathbf{OC}(\beta)$ for different values of β is computationally expensive and time-consuming, the parameter sensitivity analysis with respect to $\widetilde{\mathbf{NLP}}(\beta)$ enables us to evaluate the variations in the optimal solution and the optimal cost without the need to repeatedly solve the problem for each perturbation of β .

From the results, we concluded that the computed optimal solutions exhibit similar behaviours for different values of β within a neighbourhood B_0 of the nominal value $\beta_0 = 0.15$, although the times t_{in} and t_{out} vary with β . As shown in Figures 4 and 5, we observe that as β increases, the entry time t_{in} decreases, as well as the exit time t_{out} and the cost function increase, indicating a longer irrigation period of time and a higher irrigation water flow rate.

Although the optimal solution and cost are sensitive to variations in β , small perturbations of β in B_0 did not affect the crop preservation state, since the minimum level $x_{\min}$ is always guaranteed for $\beta \in B_0$.

(see Figure 3). This ensures that, despite the variations in β , the irrigation solution remains effective in preserving the crops. Finally, we emphasize that different nominal values for β can be identified from an admissible set such that the corresponding neighbourhoods do not overlap and cover the entire set.

Acknowledgments: A. P. Lemos-Paião is supported by CIDMA (<https://ror.org/05pm2mw36>) under the Portuguese Foundation for Science and Technology (FCT, <https://ror.org/00snfq58>), Grants UID/04106/2025 (<https://doi.org/10.54499/UID/04106/2025>) and UID/PRR/04106/2025 (<https://doi.org/10.54499/UID/PRR/04106/2025>).

S. O. Lopes and M. d. R. de Pinho have been financially supported by the R&D Unit SYSTEC – Systems and Technologies Center (UID/00147) and the Associated Laboratory ARISE – Advanced Production and Intelligent Systems (ref. LA/P/0112/2020), both funded by FCT/MECI through national funds.

The authors are also grateful to the anonymous Referee for valuable comments and suggestions.

References

- [1] R. G. Allen, L. S. Pereira, D. Raes, M. Smith et al.: *Crop evapotranspiration. Guidelines for computing crop water requirements*, FAO irrigation and drainage paper 56 (1998), ISBN 92-5-104219-5.
- [2] M. Berardi, F. V. Difonzo, R. Guglielmi: *A preliminary model for optimal control of moisture content in unsaturated soils*, *Comp. Geosciences* 27/6 (2023) 1133–1144.
- [3] J. T. Betts, W. P. Huffman: *Exploiting sparsity in the direct transcription method for optimal control*, *Comp. Optim. Appl.* 14/2 (1999) 179–201.
- [4] K. Boumazza, N. Kalboussi, A. Rapaport, S. Roux, C. Sinfort: *Optimal control of a crop irrigation model under water scarcity*, *Optimal Control Appl. Methods* 42/6 (2021) 1612–1631.
- [5] C. Büskens, H. Maurer: *Sensitivity analysis and real-time optimization of parametric nonlinear programming problems*, in: *Online Optimization of Large Scale Systems*, M. Grötschel et al. (eds.), Springer, Berlin (2001) 3–16.
- [6] C. Büskens, H. J. Pesch, S. Winderl: *Real-time solutions of bang-bang and singular optimal control problems*, in: *Online Optimization of Large Scale Systems*, M. Grötschel et al. (eds.), Springer, Berlin (2001) 129–142.
- [7] H. Darouich, T. B. Ramos, L. S. Pereira, D. Rabino, G. Bagagiolo, G. Capello, L. Simionesei, E. Cavallo, M. Biddoccu: *Water use and soil water balance of mediterranean vineyards under rainfed and drip irrigation management: evapotranspiration partition and soil management modelling for resource conservation*, *Water* 14/4 (2022) 554.
- [8] A. V. Fiacco, Y. Ishizuka: *Sensitivity and stability analysis for nonlinear programming*, *Ann. Oper. Res.* 27/1 (1990) 215–235.
- [9] M. Grötschel, S. O. Krumke, J. Rambau (eds.): *Online Optimization of Large Scale Systems*, Springer, Berlin (2001).
- [10] R. F. Hartl, S. P. Sethi, R. G. Vickson: *A survey of the maximum principles for optimal control problems with state constraints*, *SIAM Review* 37/2 (1995) 181–218.
- [11] J. S. Kikstra et al.: *The IPCC Sixth Assessment Report WGIII climate assessment of mitigation pathways: from emissions to global temperatures*, *Geoscientific Model Development* 15/24 (2022) 9075–9109.
- [12] A. P. Lemos-Paião, S. O. Lopes, M. d. R. de Pinho: *Analytical study for different extremal state solutions of an irrigation optimal control problem with field capacity modes*, *Int. J. Appl. Comp. Math.* 8/2 (2022), art.no. 67, 27p.
- [13] A. P. Lemos-Paião, S. O. Lopes, M. d. R. de Pinho: *Note on optimal control problem applied to irrigation with sectioned soil*, in: *Dynamic Control and Optimization*, T. V. Tchemisova et al. (eds.), Aveiro 2021, Springer Proceedings in Mathematics and Statistics 407, Springer, Cham (2022) 19–40.
- [14] A. P. Lemos-Paião, S. O. Lopes, M. d. R. de Pinho: *A parametric optimal control problem applied to daily irrigation*, *Math. Modelling Nat. Phenomena* 20 (2025), art.no. 2, 33p.
- [15] A. P. Lemos-Paião, S. O. Lopes, M. d. R. de Pinho: *Sensitivity analysis for optimal control: application to an irrigation problem*, in: *CONTROLO 2024*, A. P. Aguiar et al. (eds.), Springer, Cham (2025) 186–197.

- [16] S. O. Lopes, F. A. C. C. Fontes, A. C. D. Caldeira, R. M. S. Pereira: *Optimal control of irrigation with field capacity modes: characterizing the minimal water consumption solution*, in: *13th APCA International Conference on Automatic Control and Soft Computing, CONTROLO 2018*, A. Cardoso et al. (eds.), IEEE (2018) 300–305.
- [17] S. O. Lopes, F. A. C. C. Fontes, R. M. S. Pereira, M. d. R. de Pinho, A. M. Gonçalves: *Optimal control applied to an irrigation planning problem*, *Math. Problems Eng.* 2016 (2016), art.no. 5076879, 10p.
- [18] S. O. Lopes, R. M. S. Pereira, P. A. Pereira, A. C. Caldeira, V. F. Fonte: *Optimal control applied to an irrigation planning problem: a real case study in Portugal*, *Int. J. Hydrology Sci. Technology* 9/2 (2019) 173–188.
- [19] G. J. Machado, R. M. S. Pereira, S. Clain, N. Araújo, S. O. Lopes: *A new stabilised scheme for the Richards' equation with evapotranspiration*, *Groundwater for Sustainable Development* 17 (2022), art.no. 100736.
- [20] H. Maurer: *On optimal control problems with bounded state variables and control appearing linearly*, *SIAM J. Control Optim.* 15/3 (1977) 345–362.
- [21] H. Maurer, G. Vossen: *Sufficient conditions and sensitivity analysis for optimal bang-bang control problems with state constraints*, in: *System Modeling and Optimization, Proc. 23rd IFIP Conf., Cracow 2007*, A. Korytowski et al. (eds.), IFIP Advances in Information and Communication Technology 312, Springer, Berlin (2009) 82–99.
- [22] L. T. Paiva, F. A. C. C. Fontes: *Adaptive time-mesh refinement in optimal control problems with state constraints*, *Discrete Cont. Dyn. Systems* 35/9 (2015) 4553–4572.
- [23] L. S. Pereira, P. Paredes, N. Jovanovic: *Soil water balance models for determining crop water and irrigation requirements and irrigation scheduling focusing on the FAO56 method and the dual K_c approach*, *Agricultural Water Manage.* 241 (2020), art.no. 106357.
- [24] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko: *The Mathematical Theory of Optimal Processes*, transl. from the Russian by K. N. Trirogoff, L. W. Neustadt (ed.), Wiley, New York (1962).
- [25] M. I. Romashchenko, V. O. Bohaienko, T. V. Matiash, V. P. Kovalchuk, A. V. Krucheniuk: *Numerical simulation of irrigation scheduling using fractional Richards equation*, *Irrigation Sci.* 39/3 (2021) 385–396.
- [26] R. B. Vinter: *Optimal Control*, Springer, Berlin (2010).