

The magic and mystery of Newton's least resistance problem

Alexander Plakhov^{1,2,*}

1. Center for R&D in Mathematics and Applications, University of Aveiro, Portugal

2. Institute for Information Transmission Problems, Russia

Abstract: The purpose of this article is to briefly describe the recent results (including those of the author and his collaborators) obtained on the problem of least aerodynamic resistance for convex and nonconvex bodies, for the case of translation and for motion with rotation, as well as on the free motion of bodies within the framework of Newtonian aerodynamics.

Keywords: Newton's aerodynamic problem, single impact condition, optimal mass transport, Kakeya needle problem, Magnus effect, spinning rough disc, free motion of bodies.

2020 Mathematics Subject Classification: 37N05, 49K30, 49Q10

1. Introduction

In the spring of 2002, I was teaching a course called Seminar to a group of students at the University of Aveiro. This course allowed for a free choice of topics to be studied and discussed, and I chose Calculus of Variations, based on the popular book by V. M. Tikhomirov, *Stories about maxima and minima* [31]. One of the topics was Newton's aerodynamic problem: two students were required to understand the topic and present it in class. My task was to ask a few questions about the topic to determine the students' understanding. I had no idea that the topic of this lesson would determine the main direction of my research for more than two decades.

Newton's problem is based on the following mechanical model. A solid body (by which we mean a domain with a piecewise smooth boundary in $\mathbb{R}^d$, where d is usually 2 or 3) moves through a medium of point particles. The particles in the medium are initially at rest, do not interact with each other, and upon collision with the surface of the body, are reflected back in the perfectly elastic manner. Although this model is very simple, it has proved to be a rich source of many interesting problems and an area of application for many modern mathematical methods.

When considering this model, one might imagine an artificial Earth satellite at an altitude between 100 and 1000 km, either moving forward or rotating, and pose problems about finding the most streamlined shape, or about the trajectory it would describe during free motion. One might also imagine a body with a mirrored surface onto which light rays fall, within the framework of geometric optics, and pose problems about finding invisible bodies, as well as bodies of minimal visibility in cases where full visibility is impossible. The problem of bodies with the greatest visibility – retroreflectors – is also relevant.

The range of mathematical methods used to solve these problems is impressive: it includes, of course, the calculus of variations, but also the theory of billiards, the optimal mass transfer problem, the Kakeya needle problem, convex geometry, and so on.

The problem, originally formulated by Newton in [14], in modern terms looks as follows. A convex body translates with fixed velocity v in a rarefied medium with constant density ρ . In a frame of reference associated with the body, each particle arrives, reflects off the surface of the body, and then flies away forever. The body is defined by the relations $x = (x_1, x_2) \in \Omega$, $0 \leq z \leq u(x)$ in an orthogonal coordinate system with the coordinates x_1, x_2, z , and the particles fall vertically downward onto the body with velocity

*Corresponding author.

E-mail addresses: A. Plakhov*: plakhov@ua.pt

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$(0, 0, -v)$. Here Ω is a compact convex set with nonempty interior in $\mathbb{R}^2$, which we will refer to as a convex domain, and $u : \Omega \rightarrow \mathbb{R}$ is a concave function of two variables describing the forward (i.e., subject to particle impacts) part of the body's surface. Under these conditions, the z -component of the braking force acting on the body from the medium (usually called *resistance*) can be described by the formula $F = 2\rho v^2 R(u)$, where $R(u)$ is given by

$$R(u) = \int_{\Omega} \frac{1}{1 + |\nabla u(x)|^2} dx. \quad (1)$$

Newton in [14] solved the problem of least resistance in the class of axially symmetric bodies with fixed length M along the direction of motion and fixed width $2L$. This problem amounts to minimizing the functional $R(u)$ in (1) in the class of radial functions $u(x) = \varphi(|x|)$, where $|x| = \sqrt{x_1^2 + x_2^2}$, and the function $\varphi : [0, L] \rightarrow \mathbb{R}$ is non-increasing, concave, and satisfies $0 \leq \varphi \leq M$. In this case $\Omega = \{x : |x| \leq L\}$ is a circle of radius L . The solution looks like a truncated cone with a slightly inflated lateral surface (see Fig. 1).

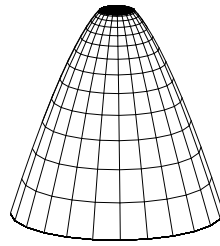


Figure 1. A solution to the axisymmetric Newton problem with $M = 2L$.

More details on the solution of Newton's problem can be found in the already cited book [31] or in almost any textbook on the calculus of variations.

I came up with the following question for the student giving the presentation. The concavity of the function u guarantees that each particle reflects from the body only once, and therefore formula (1) is correct. Are there any non-concave functions for which uniqueness of reflections also holds (and therefore the formula is valid)? [Answer: yes, there are.]

Reflecting further on this problem, I wondered what would happen if we allowed multiple reflections of particles from the body, and unexpectedly came up with a design – a system of channels that allows one to construct a nonconvex body with arbitrarily low resistance. The corresponding result on the least resistance problem for nonconvex bodies was published in [15].

Turning then to a search for literature on the subject, I was astonished to discover that this topic, which had seemed definitively closed for about 300 years (from 1687 to 1993), suddenly came to life ten years ago, starting with the seminal paper by Buttazzo and Kawohl [4], and that in [2] it was shown that the solution to the problem of minimizing (1) in the class of convex (not necessarily radial) functions of given amplitude and width does not coincide with Newton's solution! Thus, although axial symmetry is present in the formulation of the problem, and the class of admissible functions has it, the solution does not. (These papers were preceded by the work of the first author's student Guasoni, which was then published as Tese Laurea and in which, in particular, for certain parameters of the problem, a form more streamlined than Newton's was found for the first time [9].)

Below we will consider a number of problems arising within the framework of this model, the results obtained, and open questions.

Here, for the sake of keeping the article to a reasonable length, we will not touch upon the extensive topics related to zero resistance and invisibility, camouflage (i.e., bodies of least visibility), as well as retroreflectors (i.e., bodies of greatest visibility).

2. Problem of minimal resistance for convex bodies

Consider the problem of minimizing (1) in the class of convex (generally speaking, not radial) functions $u : \Omega \rightarrow \mathbb{R}$ satisfying $0 \leq u \leq M$. Here $\Omega \subset \mathbb{R}^2$ is a convex domain. We can also consider the three-dimensional body bounded above by the graph of u , that is, defined by the equations $x \in \Omega$, $0 \leq z \leq u(x)$.

Not much is actually known about the solution to this problem. We know that a solution exists, but it does not coincide with Newton's solution, and thus has less resistance. The upper level set $L_M = \{x : u(x) = M\} \subset \Omega$ is not a point, that is, it either has a nonempty interior or is a segment [9]. The lateral

surface of the graph of the function can be foliated by segments [2, 24, 13]. The optimal function u is equal to zero on the boundary [25]. The slope at regular points of the lateral surface is at least 45 degrees [3]. If the upper level set is not a segment, then the slope of the lateral surface at almost all points near the upper level set is 45 degrees [23, 26].

One can also consider the more general minimization problem:

$$R(u) = \int_{\Omega} f(\nabla u(x)) dx, \quad (2)$$

where f is some smooth function and Ω is a convex domain. This problem arises naturally if the particles are reflected inelastically, obeying some other law (as is usually the case in reality). In this case, the function f is determined by the law of reflection. Most of the results listed above (except for the angle of inclination of the lateral surface) are also valid for this problem under some rather mild conditions on Ω and f .

In the articles [12, 32], optimal shapes for the minimization problem (1) are numerically found in the case where Ω is the unit circle. Fig. 2 shows a reconstruction of the solution for the case where the top level set is a square.

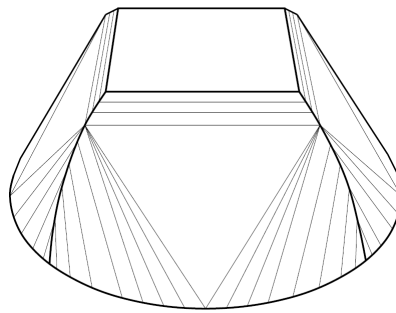


Figure 2. An optimal body where the upper level set is a square. It can be seen that the surface of the body is the union of a family of line segments and several planar parts — triangles.

There are open questions concerning the optimal function u and its graph $\text{graph}(u)$, as well as the upper level set L_M , which we will formulate as problems or conjectures. Generally speaking, they concern the general problem (2) with an arbitrary convex domain Ω . We will indicate cases with narrower settings separately.

Problem 1. Describe the set of singular points of the optimal function. Is it true that it is nowhere dense in Ω ? Based on numerical experiments, it appears that this set is the union of a finite number of simple-looking curves in Ω .

Problem 2. Describe the flat regions of the lateral surface of the graph, that is, the (convex) sets with nonempty interior on which u is affine. Numerical simulations for the special problem (1), when Ω is a circle, seem to indicate that (at least for some values of the parameter M) these regions are symmetrically located triangles, and the function u is regular on the boundary of these triangles.

Problem 3. Describe the set of outward unit normals at regular points of the graph of u . Is it true that it is the union of a finite number of curves on S^2 ?

Problem 4. Describe the top level set L_M . Numerical simulations for problem (1) and for the unit circle Ω show that this set is a symmetrically located regular polygon (possibly a 2-gon, i.e., a line segment), and the number of sides is a monotonically decreasing function of the height M , tending to infinity as M tends to zero, and equal to 2 when M exceeds some threshold value $M^* \approx 1.5$.

Problem 5. For problem (1) and for the case where Ω is a circle, prove that if the upper level set is a segment, then the slope angles at regular points of the lateral surface are separated from 45 degrees by a positive constant. This conjecture is based on the results of numerical experiments.

Discussion of open issues can also be found in [10].

3. Problem of minimal resistance under single impact condition

As noted above, formula (1) is valid not only for concave u , but also for a broader class of piecewise differentiable functions. Specifically, the function must satisfy the condition that each particle falling in

the vertical direction is reflected from the graph of the function only once. In the aforementioned article [4], an important observation was made: there exist nonconvex (and radial) functions u whose resistance is less than Newtonian (see Fig. 3). Thus, we arrive at the following problem: minimize the functional (1) in the class of functions $u : \Omega \rightarrow \mathbb{R}$, $0 \leq u \leq M$, satisfying the single impact condition (SIC).

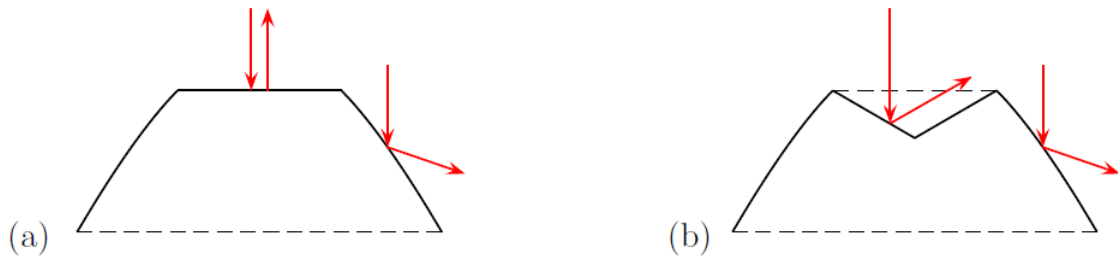


Figure 3. Figure (a) shows the central vertical section of the optimal convex radial function, and figure (b) shows the section of the radial function satisfying the SIC: the horizontal segment is replaced by two segments forming angles of 30 degrees to the horizontal.

This problem has been studied in many papers; see, for example, [5, 6, 7, 10, 1]. For radial functions with the SIC, the solution was found in [5]. In [6], it was proved that in the class of (generally non-radial) functions satisfying the SIC, the minimum is not attained, and a lower bound for the resistance was established, $R(u) \geq \phi(\Omega, M)$, where

$$\phi(\Omega, M) = \int_{\Omega} \frac{1}{2} \left(1 - \frac{M}{\sqrt{M^2 + d^2(x)}} \right) dx, \quad (3)$$

with $d(x) = \text{dist}(x, \partial\Omega)$. This estimate is derived based on the consideration that each incident particle is reflected at a height no greater than M , and the horizontal distance from the point of reflection to the point where the reflected particle intersects the horizontal plane $z = 0$ is no less than $d(x)$.

It turned out that this lower bound is exact, and a minimizing sequence of functions was constructed in [21, 22] using a construction similar to that devised by Besicovitch for solving the Kakeya problem. Namely, for any positive ε we construct a function whose graph can be described as a set of mirrors and corridors. The mirrors are located at height $M - \varepsilon \leq u \leq M$, and the area of the projection of the union of all the corridors onto the horizontal plane does not exceed ε . A particle with initial coordinate x , reflected from a mirror, then moves above one of the corridors and intersects the horizontal plane $z = 0$ at a point x_1 outside Ω , with the distance between x and x_1 not exceeding $d(x) + \varepsilon$.

The minimum resistance for functions satisfying the SIC is many times smaller than the minimum in the class of convex functions. However, this enormous advantage in streamlining cannot be achieved in real life: the surfaces of functions with the SIC that are close to minimizing ones, known today, have such a complex structure that they cannot be constructed in practice, since the significant structural details are much smaller than the dimensions of atoms.

4. Problem of minimal resistance with multiple collisions

Let us now consider (generally speaking, nonconvex) bodies with arbitrary geometry, such that incident particles can be reflected from them several times. Here, too, the results are quite unexpected. It turns out that the infimum of resistance in many reasonable classes of bodies is zero. Following the article [21], let us consider several classes of bodies.

Let $U_{\Omega, M}$ be the class of piecewise differentiable functions $u : \Omega \rightarrow \mathbb{R}$ satisfying the inequalities $0 \leq u \leq M$ and possessing regular scattering, understood in the following sense: for almost all $x \in \Omega$, a particle with the initial coordinate x is reflected a finite number of times from the graph of u at regular points. Thus, for almost all $x \in \Omega$, the final velocity of the particle $v(x; u) = (v_1(x; u), v_2(x; u), v_3(x; u))$ is defined. In this case, we define the resistance by the following formula,

$$R(u) = \int_{\Omega} \frac{1 + v_3(x; u)}{2} dx,$$

which generalizes the original definition (1). As above, $R(u)$ is the z -component of the braking force acting on the body from the medium.

Recall that with the function u we associate a three-dimensional body defined by the equations $x \in \Omega$, $0 \leq z \leq u(x)$. Each such body is contained in the cylinder $\Omega \times [0, M]$ and contains its lower end $\Omega \times \{0\}$.

Let $B_{\Omega, M}$ denote the class of bodies B with piecewise C^1 boundary, satisfying the inclusions $\Omega \times \{0\} \subset B \subset \Omega \times [0, M]$ and possessing regular scattering in the sense of the previous definition. Its resistance is defined by a similar formula

$$\mathcal{R}(B) = \int_{\Omega} \frac{1 + v_3(x; B)}{2} dx,$$

where $v(x; B) = (v_1(x; B), v_2(x; B), v_3(x; B))$ denotes the final velocity of the particle with the initial coordinate x .

It turns out that the minimum resistance in the class $U_{\Omega, M}$ is the same as in the class of functions satisfying the SIC, that is, $\phi(\Omega, M)$; allowing multiple collisions does not improve it. The situation is completely different for the class of bodies $B_{\Omega, M}$, that is, for bodies that are not necessarily subgraphs of functions. In this class, the infimum resistance is zero, and to achieve it, it is sufficient to allow only two collisions. Namely, the following theorem is true.

Theorem 4.1.

- (a) $\inf_{u \in U_{\Omega, M}} \mathcal{R}(u) = \phi(\Omega, M).$
- (b) $\inf_{B \in B_{\Omega, M}} \mathcal{R}(B) = 0;$

moreover, this infimum is attained on the subclass of bodies B , from which each particle is reflected no more than twice.

In point (b), the construction of the minimizing sequence of bodies is quite complicated and is also based on Besicovitch's ideas (and so, cannot be implemented in practice).

Note that in the case where Ω is a unit circle, there is a fairly simple construction using pairs of arcs of confocal parabolas that allows one to design a minimizing sequence of bodies with two reflections if $M > 1/2$, and a minimizing sequence of bodies with three reflections if $M \leq 1/2$ (for a more detailed discussion of this idea, see Figs. 5 and 6 of [19] and the explanations therein). The next problem, which seems quite easy, concerns a possible generalization of this construction to arbitrary domains Ω .

Problem 6. For arbitrary convex domains Ω , find the value $M^*(\Omega) > 0$ and construct a minimizing sequence of (simple-looking) bodies such that for $M > M^*(\Omega)$ the number of reflections from them is no more than double, and otherwise no more than triple. It would also be interesting to generalize this result to arbitrary (not necessarily convex) compact domains with nonempty interior and piecewise smooth boundary.

5. Motion with slow rotation

We have seen that, in the case of translational motion, the force of resistance of the medium can be reduced to zero by choosing appropriate shapes of the bodies. However, in reality, a body in motion usually performs at least slight rotational movements. What can be said about the shapes of least resistance in this case?

This problem is of a completely different nature. It is typically quite difficult, and optimal mass transfer methods are often used to solve it. We will highlight several cases for study. Below, it is assumed that the rotational movements are very slow, so that the effect of rotation can be ignored when calculating the resistance at any given moment. We are interested in minimizing the time-averaged value of the resistance.

5.1. Motion with uniform oscillations

A two-dimensional body moves through a medium on the plane and performs uniform oscillations of fixed amplitude A . The body has a flat nose in the form of a segment perpendicular to the direction of motion. We pose the problem of roughening the nose so that its average resistance is minimal. This problem is solved in [29] using the methods of optimal mass transport.

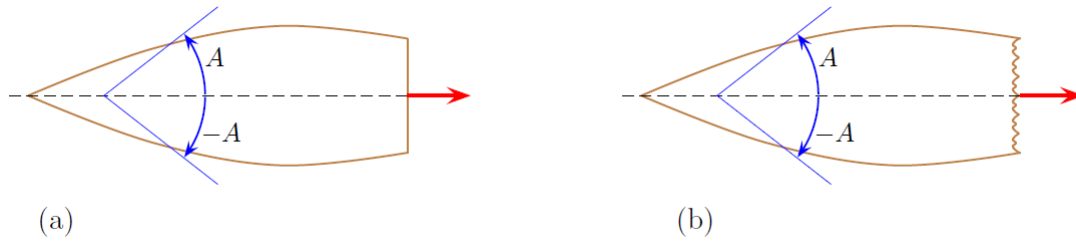


Figure 4. (a) A body with a flat blunt nose makes rotations of amplitude A in the course of motion. (b) The same body with its nose roughened.

Specifically, we define the average resistance as $\bar{R}_A = \int_{-A}^A R(\varphi) d\varphi$, where $R(\varphi)$ is the resistance of the nose at a rotation angle φ . For each value of the rotation amplitude $A \in [0, \pi]$, we compare the resistance of the optimally roughened nose with its original resistance; see Fig. 4. Thus, we are interested in the infimum of the ratio $\bar{R}_A / \bar{R}_A^0$, where $\bar{R}_A^0$ is the resistance of the original nose – segment. This infimum is denoted by $\kappa(A)$, and it is a monotonically increasing function of A . It has been found that in the absence of rotation, the resistance of the nose can be reduced by at most half, that is, $\kappa(0) = 1/2$, and for small values of the rotation amplitude A , $\kappa(A) - 1/2$ is proportional to $\sqrt{A}$. The maximum amplitude $A = \pi$ is equivalent to uniform rotation, and the value $\kappa(\pi) = 0.987820\dots$ is found numerically based on an analytical formula. Thus, the resistance of a uniformly rotating nose can be reduced by at most approximately 1.22%.

5.2. The least resistance of a slowly rotating body of fixed area in $\mathbb{R}^2$

Consider the problem of the smallest average resistance for a slowly uniformly rotating body of fixed area on the plane, see [16, 17, 18]. We define the average resistance of body B as the quantity $\bar{R}(B) = \int_{S^1} R(B, v) dv$, where $R(B, v)$ denotes the resistance of B when moving in the direction v . It is easy to see that in the class of convex bodies, the solution is a disc, and this result can be generalized to higher dimensions.

In the class of connected (generally speaking, nonconvex) bodies, the answer is unexpected: the solution is a rough disc with a specific shape of roughness. The ratio of the resistance of a rough disc to the resistance of a smooth disc of the same area is $m_2 = 0.987820\dots$. Not surprisingly, the two quantities coincide, $m_2 = \kappa(\pi)$.

A rough disc is defined as follows. A sequence of connected two-dimensional sets of equal area is constructed, converging to a disc in the Hausdorff sense. The boundary of the n th set is piecewise smooth and is invariant under rotation around the origin by π/n . That is, the boundary of each set is formed by periodic repetition of a dimple – the part of the boundary enclosed within an angle of π/n with its vertex at the origin.

Essentially, the problem amounts to studying the billiard scattering (that is, the measure that describes the joint distribution of the angle of entry and the angle of exit) on a dimple, and then to solving a particular problem of optimal mass transfer, which consists in minimizing the resistance depending on this measure.

5.3. The problem of optimal roughening in $\mathbb{R}^d$, $d \geq 2$

The body moves forward and at the same time slowly uniformly rotates (somersaults). We are interested in the average resistance $\bar{R}$ given by the same formula as in the previous section, with the only difference that the integration is performed over S^{d-1} . By a rough body obtained by roughening a convex body $C \in \mathbb{R}^d$ we mean a sequence of sets C_n with piecewise smooth boundary, contained in C (and not necessarily connected), where the volume of $C \setminus C_n$ tends to zero, and such that the sequence $\bar{R}(C_n)$ converges to some limit $\bar{R}(\mathfrak{C})$. This limit will be called the resistance of the rough body $\mathfrak{C}$.

We are interested in $\inf_{\mathfrak{C}} \frac{\bar{R}(\mathfrak{C})}{\bar{R}(C)} = m_d(C)$, where the infimum is taken over all possible roughenings of the body C . It turns out that the answer does not depend on C , but only on the dimension d , that is, $m_d(C) = m_d$. The sequence m_d is monotonically decreasing. The value of m_2 has already been indicated above; further, we have $m_3 = 0.969445\dots$ and $m_d \rightarrow 0.791\dots$ as $d \rightarrow \infty$. These results are presented in [19, 20].

Note, however, that the dimples known to us of that produce near-optimal resistance have extremely complex shapes that cannot be implemented in practice. Thus, we come to the following problems.

Problem 7. (a) Find a two-dimensional dimple with a simple shape and resistance smaller than 1.

(b) The same problem for a three-dimensional dimple.

The following problems also remain open.

Problem 8. Solve the problem of the least average resistance of a slowly rotating body

(a) in the class of (generally speaking, disconnected) bodies of fixed area in $\mathbb{R}^2$;

(b) in the class of connected bodies of fixed volume in $\mathbb{R}^d$, $d \geq 3$.

Extending the class of bodies to disconnected ones in point (a) makes the problem much more difficult. On the contrary, narrowing the class of bodies to connected ones does not help at all in point (b). In both points, it is not even known whether the infimum is positive or equal to zero. In the paper [27], using optimal mass transfer methods, a lower bound of the resistance was found in the class of bodies of fixed volume in $\mathbb{R}^d$, $d \geq 2$, enclosed in a fixed sphere. It is unknown, however, whether this lower bound is exact.

6. The problem of free motion of bodies

It seems strange, but apparently no one has previously studied the free motion of bodies within the framework of Newtonian aerodynamics. Below, we present the results we obtained for free motion in a medium on the plane. Of course, here we assume that the body not only translates but also rotates, and the rotation cannot be considered slow.

6.1. Motion of a spinning rough disc

If there is no roughness, the disc always moves in a straight line. In the presence of roughness, a deflecting transverse component of the drag force usually occurs. This phenomenon is called the Magnus effect. If the deflection is in the direction of rotation, the effect is called direct; otherwise, it is called inverse; see Fig. 5. Of course, the magnitude (and even direction) of the deflecting force is determined by the shape of the roughness.



Figure 5. The force acting on a spinning disc translating with the velocity $\vec{v}$. (a) Magnus effect. (b) Reverse Magnus effect.

The article [11] considers the question of possible trajectories of the disc. Under certain heuristic assumptions, the following result is obtained. Any rectifiable curve of finite length on the plane, with an appropriate choice of roughness, can be approximated arbitrarily closely in the Hausdorff sense by the trajectory of the center of the disc.

The following question is considered in the papers [28, 30]. Let a rough disc of radius 1 move with a constant (say, unit) linear velocity and a constant (counterclockwise) angular velocity ω . Determine the possible forces acting on the disc.

Since the answer depends on the shape of the roughness, the answer is not a single vector, but a set of vectors $\vec{R}_\omega(\nu)$. Here ν denotes the scattering law – the measure generated by the scattering of particles on the disc boundary.

We assume that the boundary of the disc is formed by periodic repetition of one part of the boundary region, a dimple. The probability measure ν specifies the joint distribution of the pair (angle of incidence, angle of exit) for billiard scattering on the dimple. Let $\mathcal{N}$ denote the set of all such measures. For a fixed value of ω , the closure of the set of all possible forces $\mathcal{R}_\omega = \{\vec{R}_\omega(\nu) : \nu \in \mathcal{N}\}$ is convex. The sets $\mathcal{R}_\omega$ are found numerically based on mathematical formulas for several values of ω ; see Fig. 6.1.

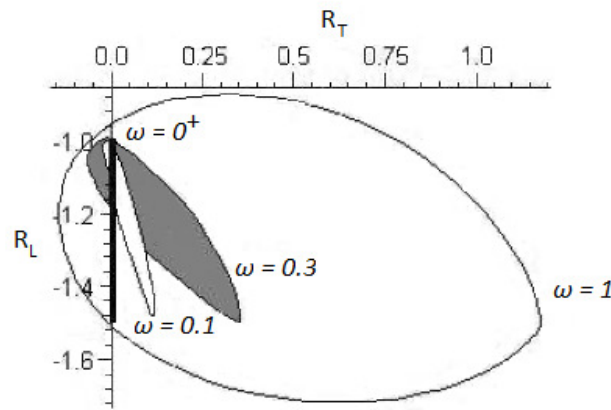


Figure 6. The convex sets $\mathcal{R}_\omega$ with $\omega = 0^+, 0.1, 0.3$ and 1 are shown. The set $\mathcal{R}_{0^+}$ is the vertical segment with the endpoints $(0, -0.9878\dots)$ and $(0, -1.5)$. The coordinate system is chosen so that the vertical axis is aligned with the direction of the disc's translational motion. The disc rotates counterclockwise.

The expression for $R_\omega(v)$ is relatively simple when $\omega = 1$ and looks as follows (formula (2.21) in [30]):

$$\vec{R}_1(v) = 3 \iint_{x \geq 0} \sin^2 x \begin{bmatrix} \cos(2x - y) + \cos x \\ -\sin(2x - y) - \sin x \end{bmatrix} dv(x, y)$$

in the coordinate system chosen so that the vertical axis is aligned with the direction of the disc's translational motion. Here $x, y \in [-\pi/2, \pi/2]$ mean the angle of incidence and the angle of exit for a single particle; the measure ν is invariant with respect to exchange of the variables x and y , and its projections on the x - and y -axes are $\frac{1}{2} \cos x \, dx$ and $\frac{1}{2} \cos y \, dy$, respectively.

Interestingly, depending on the roughness, both the direct and inverse Magnus effects can occur. Apparently, the inverse effect is observed more often; at least, it occurs for all specific simple shapes of roughness that we have considered. This brings us to the following problems.

Problem 9. (a) Find a hollow with a simple shape and a value ω for which the proper Magnus effect takes place.

(b) Do there exist a single hollow and two different values ω_1 and ω_2 corresponding to the inverse and proper Magnus effect, respectively?

6.2. Motion of a rod under a simplifying assumption

Another simple shape is a rod, by which we mean a line segment. The mass of the rod is distributed symmetrically about its center. We assume that each particle on the plane reflects off the rod at most once and then leaves the plane and flies away forever. This assumption greatly simplifies the dynamics. This study is conducted in [8].

For almost all initial data, the motion of the rod stabilizes to a translation with the velocity directed along the rod itself. Interestingly, the asymptotic behavior of the angular velocity of the rod depends on the mass distribution. Namely, consider the quantity $B = \frac{3I}{ma^2}$, where m is the mass of the rod, $2a$ is its length, and I is its moment of inertia. The equality $B = 1$ is realized when the mass is uniformly distributed along the rod. If the mass is mainly concentrated near the ends (center) of the rod, we have $B > 1$ ($B < 1$). We found that ω decreases asymptotically as $1/t^2$ if $B > 1$, as $1/(t^2 \ln t)$ if $B = 1$, or as $1/t^{2/B}$ if $B < 1$.

Note that the study of the dynamics in $\mathbb{R}^2$ without the simplifying assumption that particles leave the plane after the first collision is much more difficult, since multiple reflections must be taken into account.

Problem 10. Describe the dynamics of rotating bodies on the plane, for example, of a rod or a square with uniform distribution of mass; see Fig. 7.

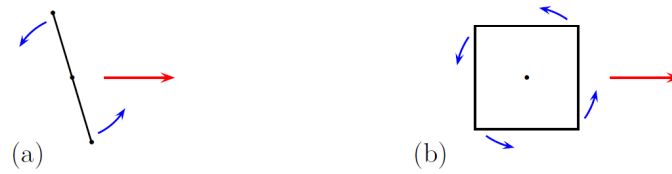


Figure 7. Rotating figures: (a) a rod; (b) a square.

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