

Rotating Rod and Ball Revisited: Comparison of Billiard and Sliding Motions

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Abstract: We continue the study of the mechanical system composed of a rod (straight line) and a ball (massless point) on the plane, where the rod rotates uniformly about a fixed point and the ball reflects elastically when colliding with the rod and moves freely between successive reflections. A sliding motion of the ball along the rod is also allowed. In a previous paper it was proved that the following regimes of motion are possible: a billiard motion, a sliding motion, and the transient regimes: a billiard motion followed by sliding and sliding followed by a billiard motion. Additionally, the asymptotic behavior of the system was studied. – In this paper we conduct a more detailed study of the dynamics. It is shown, in particular, that billiard motion is asymptotically indistinguishable from sliding one; when the number of iterations is large, the billiard particle and the corresponding sliding particle practically merge, but the billiard particle can be distinguished by vibration with very high frequency and very low amplitude.

Keywords: Billiard with moving boundary, rotating rod, elastic impact, sliding.

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1. Introduction

Here we study the billiard in a rotating half-plane. More precisely, we consider a mechanical system consisting of a straight line (the rod) uniformly rotating about one of its points, and a massless point (the ball) located outside the line. The point is elastically reflected upon collision with the line and moves freely between successive impacts. This paper continues the study begun in the papers [16] and [17].

Billiards with time-dependent boundaries have been a subject of intensive research in the last few decades. Several papers studied billiard in domains with oscillating boundaries [1, 8, 11, 12, 13, 14], while others [3, 6, 9, 10] concentrated on billiard in rotating domains. The surge of interest in this topic was initially motivated by developing models of the so-called Fermi acceleration. There has been much interest in the question of whether the energy of the system grows to infinity as a result of multiple reflection from the boundary of a bounded domain.

In our study, however, the domain is unbounded, and the energy always goes to infinity, so the answer to this question is trivial. Instead, we concentrate on a detailed study of the asymptotic behavior of the system's dynamics.

We believe that a comparison of our results with classical results on Fermi acceleration in time-dependent billiards is of interest. In particular, previous studies (see, e.g., [11, 13]) found that the velocity increases at most linearly as a function of the number of collisions and can increase exponentially or according to a power law as a function of physical time. In our model the velocity is asymptotically proportional to $n^{3/2}$ and to e^t , where n is the number of collisions and t is time.

Our interest in this topic arose from the model of Newtonian aerodynamics going back to Newton's Principia [19]. In this model, a body moves in a medium consisting of point particles. The medium is extremely rarefied, so mutual interaction of particles is neglected, and collisions of particles with the body are perfectly elastic. Interest in this topic has been revived since the early 1990s, and since then many papers have appeared, mostly dealing with the problem of minimal resistance in various classes of bodies

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[2, 4, 5, 18, 20, 21, 22, 24, 25].

The study of free motion of bodies and, more widely, of interaction of bodies with the medium in Newtonian aerodynamics is also of interest. This topic is poorly understood (see, however, the papers [7, 15, 23]). One of the simplest planar bodies is a line segment, and we consider here the extreme case when a segment of infinite length interacts with a particle of the medium.

In the previous paper the following important results have been obtained. Five regimes of motion are possible: a billiard motion, a sliding motion, the two transient regimes: a billiard motion followed by sliding and sliding followed by a billiard motion, and finally, a constant motion when the ball is at the center of rotation. Asymptotic formulas in the case of billiard motion for time intervals between consecutive hits and for positions of the points of hits have been obtained.

Numerical simulations show that, over time, a billiard motion becomes undistinguishable from a sliding one. This does not follow directly from the results of our previous paper, and this fact prompted us to conduct a more elaborated study.

In this paper we conduct a detailed study of asymptotical behavior of the key quantities. Our main result is Theorem 3.1 stating, in particular, that billiard motion is asymptotically indistinguishable from sliding one: the difference between these two types of motion vanishes exponentially.

2. Basic formulas

Consider a rod rotating uniformly counterclockwise with unit angular velocity about the origin in the complex plane $\mathbb{C}$ and a massless point (the ball) moving freely outside the rod and reflecting elastically when hitting the rod. Sliding motion of the ball along the rod is also allowed. In a rotating reference system the rod coincides with the real axis $\mathbb{R}$, and the ball is always contained in the closed upper half-plane $\mathbb{C}_0^+ := \{x + iy : y \geq 0\}$.

Let $z(t)$ be the position of the ball at the instant t in the rotating coordinate system, and correspondingly, $z(t)e^{it}$ be its position in an inertial coordinate system.

The dynamics of the rod is governed by two equations. The first one,

$$\ddot{z} + 2i\dot{z} - z \begin{cases} = 0, & \text{if } z \in \mathbb{C}^+ := \{x + iy : y > 0\} \\ \in i\mathbb{R}_0^+, & \text{if } z \in \mathbb{R} \end{cases}$$

should be understood in the sense of distributions. It means that the acceleration of the ball equals zero, when it is located outside the rod, and is orthogonal to the rod and points inward to the upper half-plane, when the ball is on the rod. The second equation is as follows:

$$\text{if } z(t) \in \mathbb{R} \text{ then } \dot{z}(t+0) = \dot{z}(t-0)^*;$$

it means that the ball reflects elastically when hitting the rod. Here and in what follows, $*$ means complex conjugation.

One easily sees that between consecutive reflections the ball moves in the upper half-plane, $z(t) \in \mathbb{C}^+$ according to

$$z(t) = (\zeta + vt)e^{-it}$$

with appropriate complex ζ and v .

In [17] the existence and uniqueness of the motion $(z(t), \dot{z}(t))$, $t \in \mathbb{R}$ was proved under given initial conditions $(z(0), \dot{z}(0^+)) \in (\mathbb{C}^+ \times \mathbb{C}) \cup (\mathbb{R} \times \mathbb{C}_0^+)$. Further, it was proved that only five regimes of motion are possible: billiard, sliding, two transient regimes: from billiard to sliding and from sliding to billiard, and also the regime corresponding to the ball at rest at the origin.

Let us briefly recall the basic formulas. The sliding regimes and the sliding parts of transient regimes are governed by the following equations:

- $z(t) = \kappa \sinh(t - t_0)$, $t \in \mathbb{R}$, where $\kappa > 0$, $t_0 \in \mathbb{R}$, as well as two limiting regimes $z(t) = \kappa e^t$ and $z(t) = -\kappa e^{-t}$, $t \in \mathbb{R}$, where $\kappa > 0$ (sliding);
- $z(t) = \kappa \cosh(t - t_0)$, $t \geq t_0$, where $\kappa > 0$, $t_0 \in \mathbb{R}$ (sliding for $t \geq t_0$ and billiard for $t < t_0$, transition at $t = t_0$);
- $z(t) = -\kappa \cosh(t - t_0)$, $t \leq t_0$, where $\kappa > 0$, $t_0 \in \mathbb{R}$ (sliding for $t \leq t_0$ and billiard for $t > t_0$, transition at $t = t_0$).

The formulas for billiard regimes (and for the billiard part of transient regimes) are more complicated. Let $t_n < t_{n+1}$ be instants of consecutive reflections; then the points of hits $z(t_n)$ and $z(t_{n+1})$ belong to $\mathbb{R}$ and $z(t + t_n) \in \mathbb{C}^+$ when $0 < t < t_{n+1} - t_n$. In this interval we have $z(t + t_n) = (r_n + v_n t)e^{-it}$, where $r_n = z(t_n) \in \mathbb{R}$ and $v_n \in \mathbb{C}$.

We define x_n and y_n by $x_n + iy_n := \dot{z}(t_n + 0)$; in other words,

$$x_n = \operatorname{Re} \dot{z}(t_n) \quad \text{and} \quad y_n = \operatorname{Im} \dot{z}(t_n + 0).$$

One obviously has $y_n \geq 0$. Further, we define $r_n := z(t_n)$ and $\tau_n := t_{n+1} - t_n$. We get

$$\begin{aligned} z(t + t_n) &= [r_n + x_n t + i(r_n + y_n)t]e^{-it} \\ &= [(r_n + x_n t) \cos t + (r_n + y_n)t \sin t] + i[(r_n + y_n)t \cos t - (r_n + x_n t) \sin t]. \end{aligned} \quad (1)$$

Denote $g_n(t) := \operatorname{Im} z(t + t_n) = (r_n + y_n)t \cos t - (r_n + x_n t) \sin t$.

The function g_n is positive on $(0, \tau_n)$ and equals zero at $t = \tau_n$. Thus, τ_n satisfies

$$(r_n + y_n)\tau_n \cos \tau_n = (r_n + x_n \tau_n) \sin \tau_n. \quad (2)$$

From (1) and (2) one obtains several useful recursive formulas. Indeed,

$$\begin{aligned} r_{n+1} &= z(t_n + \tau_n) = \operatorname{Re} z(t_n + \tau_n) = (r_n + x_n \tau_n) \cos \tau_n + (r_n + y_n)\tau_n \sin \tau_n \\ &= (r_n + y_n)\tau_n \left(\frac{\cos^2 \tau_n}{\sin \tau_n} + \sin \tau_n \right) = \frac{(r_n + y_n)\tau_n}{\sin \tau_n}. \end{aligned}$$

Further, for $0 < t < \tau_n$ one has

$$\operatorname{Re} \dot{z}(t_n + t) = [x_n + (r_n + y_n)t] \cos t + (y_n - x_n t) \sin t, \quad (3)$$

$$\operatorname{Im} \dot{z}(t_n + t) = -[x_n + (r_n + y_n)t] \sin t + (y_n - x_n t) \cos t, \quad (4)$$

hence

$$\begin{aligned} x_{n+1} &= \operatorname{Re} \dot{z}(t_n + \tau_n - 0) = [x_n + (r_n + y_n)\tau_n] \cos \tau_n + (y_n - x_n \tau_n) \sin \tau_n \\ &= x_n \cos \tau_n + (r_n + x_n \tau_n) \sin \tau_n + (y_n - x_n \tau_n) \sin \tau_n = x_n \cos \tau_n + (r_n + y_n) \sin \tau_n \\ &= x_n \cos \tau_n + \frac{(r_n + x_n \tau_n) \sin^2 \tau_n}{\tau_n \cos \tau_n} = \frac{x_n \tau_n + r_n \sin^2 \tau_n}{\tau_n \cos \tau_n} = \frac{r_n + y_n}{\sin \tau_n} - \frac{r_n \cos \tau_n}{\tau_n}. \end{aligned}$$

Further,

$$\begin{aligned} y_{n+1} &= -\operatorname{Im} \dot{z}(t_n + \tau_n - 0) = [x_n + (r_n + y_n)\tau_n] \sin \tau_n - (y_n - x_n \tau_n) \cos \tau_n \\ &= \frac{(r_n + y_n)\tau_n \cos \tau_n - r_n \sin \tau_n}{\tau_n} + (r_n + y_n)\tau_n \sin \tau_n - (y_n - x_n \tau_n) \cos \tau_n \\ &= (r_n + x_n \tau_n) \cos \tau_n - \frac{r_n \sin \tau_n}{\tau_n} + (r_n + y_n)\tau_n \sin \tau_n \\ &= \frac{(r_n + y_n)\tau_n \cos^2 \tau_n}{\sin \tau_n} + (r_n + y_n)\tau_n \sin \tau_n - r_n \frac{\sin \tau_n}{\tau_n} \\ &= (r_n + y_n) \frac{\tau_n}{\sin \tau_n} - r_n \frac{\sin \tau_n}{\tau_n}. \end{aligned}$$

Introducing the normalized variables

$$a_n = \frac{x_n}{r_n}, \quad \beta_n = \frac{y_n}{r_n}, \quad b_n = 1 + \beta_n = 1 + \frac{y_n}{r_n},$$

one comes to the formulas

$$r_{n+1} = \frac{b_n \tau_n}{\sin \tau_n} r_n = r_n (\beta_n + 1) \frac{\tau_n}{\sin \tau_n}; \quad (5)$$

$$a_{n+1} = \frac{1}{\tau_n} - \frac{\cos \tau_n \sin \tau_n}{b_n \tau_n^2} = \frac{1}{\tau_n} - \frac{\cos \tau_n \sin \tau_n}{(\beta_n + 1) \tau_n^2}; \quad (6)$$

$$b_{n+1} = 2 - \frac{1}{b_n} \left(\frac{\sin \tau_n}{\tau_n} \right)^2 \Rightarrow \beta_{n+1} = 1 - \frac{1}{\beta_n + 1} \left(\frac{\sin \tau_n}{\tau_n} \right)^2. \quad (7)$$

Besides, from (2) one easily derives the useful formula

$$b_n \tau_n = (1 + a_n \tau_n) \tan \tau_n \Rightarrow (\beta_n + 1) \tau_n = (1 + a_n \tau_n) \tan \tau_n. \quad (8)$$

Substituting (8) in (6), after some algebra one obtains

$$a_{n+1} = a_n \frac{\cos^2 \tau_n}{1 + a_n \tau_n} + \tau_n \left(\frac{\sin \tau_n}{\tau_n} \right)^2. \quad (9)$$

The following asymptotic formulas concerning the billiard motion were obtained in Theorem 1 of the previous paper [17].

- The coordinates of impacts r_n satisfy

$$\frac{r_{n+1}}{r_n} = 1 + \frac{3}{2n}(1 + o(1)), \quad \text{and so,} \quad r_n = \pm |n|^{3/2+o(1)} \quad \text{as } n \rightarrow \pm\infty. \quad (10)$$

- The instants of impacts t_n and the time intervals between them $\tau_n = t_{n+1} - t_n$ satisfy

$$\tau_n = \frac{3}{2|n|}(1 + o(1)), \quad \text{and so,} \quad t_n = \pm \frac{3}{2} \ln |n|(1 + o(1)) \quad \text{as } n \rightarrow \infty. \quad (11)$$

- The maximal distance between the ball and the rod between consecutive impacts satisfies

$$\max\{\operatorname{Im} z(t), t_n \leq t \leq t_{n+1}\} = \frac{9}{16n^2} |r_n|(1 + o(1)) \quad \text{as } n \rightarrow \pm\infty. \quad (12)$$

In this paper the information about the system dynamics will be significantly refined. In particular, we will examine the question of whether it is possible to distinguish a billiard motion from sliding simply by observing their asymptotic behavior.

3. Asymptotic behavior of the system

The main result of this paper is the following theorem.

Theorem 3.1. *In the case of billiard motion we have*

$$t_n = c_0 \pm \frac{3}{2} \ln |n| + O(|n|^{-1}) \quad \text{as } n \rightarrow \pm\infty, \quad (13)$$

$$r_n = \pm c_1 |n|^{3/2} + c_2 |n|^{1/2} + O(|n|^{-1/2+\varepsilon}) \quad \text{as } n \rightarrow \pm\infty, \quad (14)$$

$$z(t) = \pm c e^{|t|} + O(e^{-|t|/3}) \quad \text{as } t \rightarrow \pm\infty, \quad (15)$$

$$\operatorname{Re} \dot{z}(t) = c e^{|t|} + O(e^{-|t|/3}) \quad \text{as } t \rightarrow \pm\infty, \quad (16)$$

$$\limsup_{t \rightarrow \pm\infty} (|\operatorname{Im} \dot{z}(t)| e^{-|t|/3}) = k, \quad (17)$$

where $\varepsilon > 0$ is arbitrary, the constants c, c_0, c_1, c_2, k ($c > 0, c_1 > 0, k = \frac{3\varepsilon}{2} e^{2c_0/3}$) depend on the choice of the sign (+ or -), and the two signs $\pm$ in each formula agree.

In the case of transitional motion, corresponding formulas are valid: for transition from sliding to billiard (from billiard to sliding) one should take $t \rightarrow +\infty$ and $n \rightarrow +\infty$ ($t \rightarrow -\infty$ and $n \rightarrow -\infty$).

Remark 3.2. The theorem implies that for any billiard motion $z(t)$, $\dot{z}(t)$ there exists a sliding motion of the form $z_s(t) = c e^t$, or $\kappa \sinh(t - t_0)$, or $\kappa \cosh(t - t_0)$ with $t \geq t_0$, such that z_s is asymptotically exponentially close to z in the future, that is, $|z(t) - z_s(t)| = O(e^{-t/3})$ as $t \rightarrow +\infty$. However, their derivatives are not asymptotically close! Indeed, $\dot{z}_s(t)$ is real, while the imaginary part of $\dot{z}(t)$, by (17), is unbounded in any neighborhood of $+\infty$. These facts are illustrated in Figure 2. To put it philosophically, when the number of iterations is large, the billiard and sliding particles practically merge, but the billiard particle can be distinguished by vibration with very high frequency and intensity and very low amplitude in the direction orthogonal to the rod. $\square$

The illustrative Figure 1 shows, in an inertial frame of reference, the trajectories of a billiard motion $z(t)e^{it}$ and the sliding motion κe^{t+it} approximating it. The billiard trajectory is chosen so as the particle is at rest, $z(t)e^{it} = 1$, in the time interval $t \in [-\pi/2, \pi/2]$.

It suffices to consider the limit $t \rightarrow +\infty$; the case $t \rightarrow -\infty$ follows using the fact that if $(z(t), \dot{z}(t))$ is a motion, then $(-z(-t)^*, \dot{z}(-t)^*)$ is also a motion.

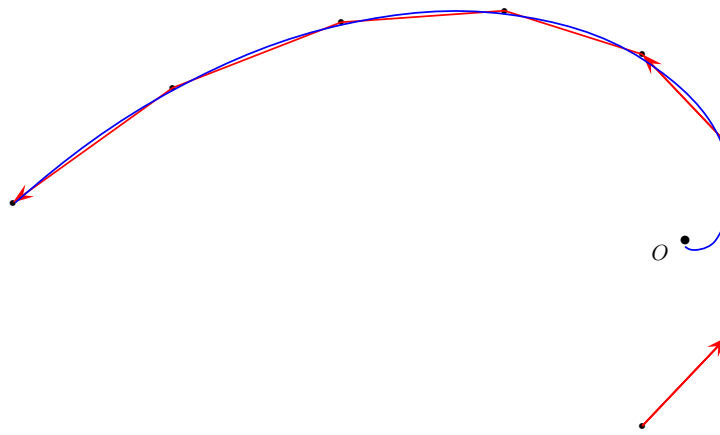


Figure 1. The trajectories of a billiard motion and of the approximating sliding motion in an inertial reference system. The origin is at the point O .

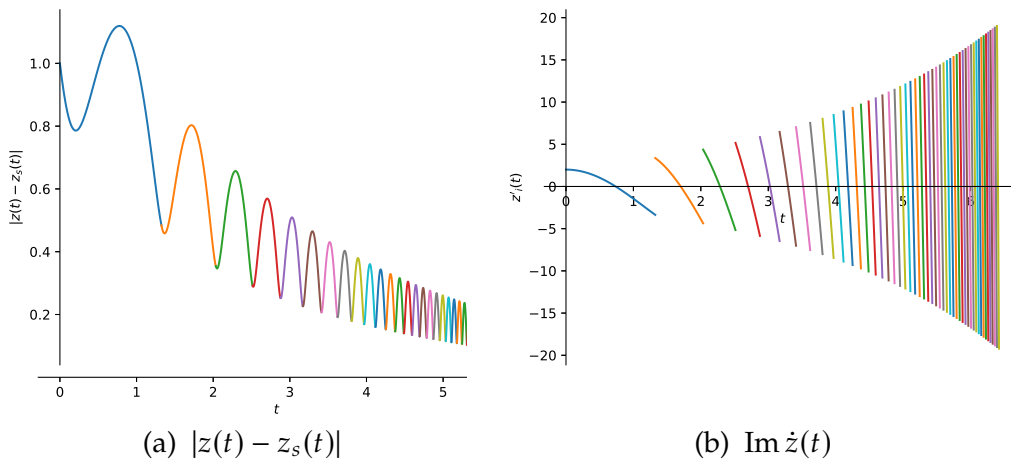


Figure 2. There are shown (a) the difference $|z(t) - z_s(t)|$ between the billiard motion of the previous figure and the sliding motion $z_s(t) = \kappa e^t$ approximating it, and (b) $\text{Im } \dot{z}(t)$. The breaks of the graphs in (b) correspond to the impacts of the ball on the rod.

According to Lemma 9 in [17], there is n_0 such that $r_{n_0} > 0$. Since we consider billiard motions and transitions from sliding to billiard, there is an infinite sequence of hits $r_{n_0}, r_{n_0+1}, \dots$. Without loss of generality assume that $n_0 = 1$, and so, $r_1 > 0$.

Define the auxiliary quantities λ_n , ψ_n , and θ_n by

$$\tau_n = \frac{3}{2n} (1 + \lambda_n), \quad a_n = 1 - \psi_n, \quad b_n = 1 + \frac{3}{2n} (1 + \theta_n), \quad (18)$$

Lemmas 6 and 7 from [17] give the following estimates for them:

$$\lambda_n \rightarrow 0, \quad \psi_n \rightarrow 0, \quad \text{and} \quad \theta_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

These estimates will be further specified in the following Lemmas 3.3–3.7.

In what follows, when writing $o(\cdot)$ or $O(\cdot)$, we will always mean in the limit $n \rightarrow \infty$.

Lemma 3.3. *We have*

$$\psi_n = \frac{9}{8} n^{-2} + o(n^{-2}), \quad \text{hence} \quad a_n = 1 - \frac{9}{8} n^{-2} + o(n^{-2}), \quad (19)$$

and

$$b_n = 1 + \tau_n + \frac{1}{3} \tau_n^2 - \frac{9}{16} n^{-3} + o(n^{-3}). \quad (20)$$

Proof. Substituting the decompositions

$$(1 + a_n \tau_n)^{-1} = 1 - a_n \tau_n + a_n^2 \tau_n^2 - a_n^3 \tau_n^3 + O(n^{-4}), \quad \cos^2 \tau_n = 1 - \tau_n^2 + O(n^{-4}),$$

and

$$\tau_n \left(\frac{\sin \tau_n}{\tau_n} \right)^2 = \tau_n - \frac{\tau_n^3}{3} + O(n^{-4})$$

in (9), after some algebra one obtains

$$1 - a_{n+1} = (1 - a_n) \left[1 - \tau_n (a_n + 1) (1 - a_n \tau_n + a_n^2 \tau_n^2) \right] + \frac{\tau_n^3}{3} + O(n^{-4}),$$

hence

$$\psi_{n+1} = \psi_n \left[1 - \frac{3}{n} + o(n^{-1}) \right] + \frac{9}{8n^3} + o(n^{-3}). \quad (21)$$

Define the values ω_n by $\psi_n = \frac{9}{8} n^{-2} (1 + \omega_n)$. It suffices to show that $\omega_n = o(1)$.

From equation (21) one obtains

$$1 + \omega_{n+1} = (1 + \omega_n) \left[1 - \frac{3}{n} + o(n^{-1}) \right] \left(1 + \frac{2}{n} + o(n^{-1}) \right) + n^{-1} + o(n^{-1}),$$

hence

$$\omega_{n+1} = \left(1 - \frac{1}{n} + o(n^{-1}) \right) \omega_n + o(n^{-1}).$$

Fix $0 < \alpha < 1/2$ and take an n_0 such that for $n \geq n_0$, both $o(n^{-1})$ in the above formula satisfy $|o(n^{-1})| \leq \alpha/n$. For $n \geq n_0$ one has

$$|\omega_{n+1}| \leq \left(1 - \frac{1}{2n} \right) |\omega_n| + \frac{\alpha}{n}, \quad \text{whence} \quad |\omega_{n+1}| - 2\alpha \leq \left(1 - \frac{1}{2n} \right) (|\omega_n| - 2\alpha).$$

It follows that either the values $|\omega_n| - 2\alpha$ always remain positive, and therefore, tend to zero, or one value $|\omega_n| - 2\alpha$ and all the following ones are negative. Thus, the limit superior of $|\omega_n|$ does not exceed 2α . Since α can be made arbitrarily small, one concludes that $\omega_n \rightarrow 0$. Equation (19) is proved.

Now using (19) and (8) and taking into account that $\frac{\tan \tau_n}{\tau_n} = 1 + \frac{1}{3} \tau_n^2 + O(n^{-4})$, one obtains

$$b_n = \left[1 + \left(1 - \frac{9}{8} n^{-2} + o(n^{-2}) \right) \tau_n \right] \left(1 + \frac{1}{3} \tau_n^2 + O(n^{-4}) \right),$$

whence follows formula (20). Lemma 3.3 is proved. $\square$

Lemma 3.4. For any $\varepsilon > 0$ we have

$$\theta_n = O(n^{-1+\varepsilon}), \quad \lambda_n = O(n^{-1+\varepsilon}), \quad \text{and} \quad \theta_n - \lambda_n = \frac{1}{2n} + O(n^{-2+\varepsilon}), \quad (22)$$

and hence,

$$\tau_n = \frac{3}{2n} \left(1 + \theta_n - \frac{1}{2n} + O(n^{-2+\varepsilon}) \right). \quad (23)$$

Additionally we have

$$b_n = 1 + \frac{3}{2n} \left(1 + \lambda_n + \frac{1}{2n} + \frac{\lambda_n}{n} - \frac{3}{8n^2} \right) + o(n^{-3}). \quad (24)$$

Proof. It follows from (18) and (20) that

$$\lambda_n = \theta_n + O(n^{-1}), \quad (25)$$

and so,

$$\tau_n = \frac{3}{2n} (1 + \theta_n) + O(n^{-2}). \quad (26)$$

From (7), using (26) and the latter relation in (18), one derives

$$\frac{3}{2(n+1)} (1 + \theta_{n+1}) = 1 - \frac{1}{1 + \frac{3}{2n} (1 + \theta_n)} \left[1 - \frac{1}{3} \left(\frac{3}{2n} (1 + \theta_n) + O(n^{-2}) \right)^2 + O(n^{-4}) \right],$$

whence

$$\begin{aligned} \frac{3}{2n} \left(1 + \frac{1}{n} \right)^{-1} (1 + \theta_{n+1}) &= 1 - \left[1 - \frac{3}{2n} (1 + \theta_n) + \frac{9}{4n^2} (1 + \theta_n)^2 + O(n^{-3}) \right] \times \\ &\quad \times \left[1 - \frac{3}{4n^2} (1 + \theta_n)^2 + O(n^{-3}) \right] \\ &= \frac{3}{2n} (1 + \theta_n) - \frac{9}{4n^2} (1 + \theta_n)^2 + \frac{3}{4n^2} (1 + \theta_n)^2 + O(n^{-3}). \end{aligned}$$

It follows that

$$\begin{aligned} 1 + \theta_{n+1} &= \left[1 + \theta_n - \frac{1}{n} (1 + \theta_n)^2 + O(n^{-2}) \right] \left(1 + \frac{1}{n} \right) \\ &= 1 + \theta_n - \frac{1}{n} (\theta_n + \theta_n^2) + O(n^{-2}), \end{aligned}$$

and so,

$$\theta_{n+1} = \left(1 - \frac{1+\theta_n}{n} \right) \theta_n + O(n^{-2}). \quad (27)$$

Fix $0 < \varepsilon < 1$ and take a positive c such that $O(n^{-2})$ in the above equation satisfies $|O(n^{-2})| \leq cn^{-2}$, and consider the auxiliary sequence

$$z_n = \frac{\varepsilon}{2} \left(\frac{n}{n_0} \right)^{-1+\varepsilon}, \quad n \geq n_0,$$

where $n_0 = n_0(\varepsilon)$ is taken so as (i) for $n \geq n_0$, $|\theta_n| \leq \varepsilon/2$ and (ii) $n_0 \geq 4c/\varepsilon^2$. Let us first check that

$$\text{for } n \geq n_0, \quad z_{n+1} \geq \left(1 - \frac{1-z_n}{n} \right) z_n + cn^{-2}. \quad (28)$$

Indeed, one has (in the following formula, condition (ii) is used in the middle inequality)

$$\begin{aligned} z_{n+1} &= \frac{\varepsilon}{2} \left(\frac{n}{n_0} \right)^{-1+\varepsilon} \left(1 + \frac{1}{n} \right)^{-1+\varepsilon} \geq \frac{\varepsilon}{2} \left(\frac{n}{n_0} \right)^{-1+\varepsilon} \left(1 + \frac{-1+\varepsilon}{n} \right) \\ &\geq \frac{\varepsilon}{2} \left(\frac{n}{n_0} \right)^{-1+\varepsilon} \left(1 - \frac{1}{n} + \frac{\varepsilon}{2n} \right) + \frac{c}{n^2} \geq z_n \left(1 - \frac{1}{n} + \frac{z_n}{n} \right) + \frac{c}{n^2}. \end{aligned}$$

It suffices to prove by induction that for $n \geq n_0$, $|\theta_n| \leq z_n$. Indeed, the inequality for $n = n_0$ follows from condition (i) and the definition of z_n . Now, if the inequality is true for some $n \geq n_0$ then, using (27) and (28), one obtains

$$|\theta_{n+1}| \leq \left(1 - \frac{1}{n} + \frac{|\theta_n|}{n} \right) |\theta_n| + \frac{c}{n^2} \leq \left(1 - \frac{1}{n} + \frac{z_n}{n} \right) z_n + \frac{c}{n^2} \leq z_{n+1}.$$

Thus, we have $\theta_n = O(n^{-1+\varepsilon})$, and by (25), $\lambda_n = O(n^{-1+\varepsilon})$. The first two relations in (22) are proved.

Further, by (18) and (20) one has

$$1 + \frac{3}{2n} (1 + \theta_n) = 1 + \frac{3}{2n} (1 + \lambda_n) + \frac{1}{3} \left(\frac{3}{2n} \right)^2 + O(n^{-3+\varepsilon}),$$

whence we obtain the last equation in (22) and (23).

From (20) one obtains

$$b_n = 1 + \frac{3}{2n} \left(1 + \lambda_n + \frac{1+2\lambda_n+\lambda_n^2}{2n} - \frac{3}{8n^2} \right) + o(n^{-3}),$$

and using that $\lambda_n = o(n^{-1/2})$, one comes to (24). Lemma 3.4 is proved. $\square$

The following lemma refines Lemma 3.4.

Lemma 3.5. *For any $\varepsilon > 0$ holds true*

$$\theta_n = \frac{c}{n} + O(n^{-2+\varepsilon}), \quad (29)$$

and therefore,

$$\lambda_n = \frac{c-1/2}{n} + O(n^{-2+\varepsilon}), \quad (30)$$

where c is a constant.

Proof. The calculations performed here are similar to the calculations on asymptotic expansion at the beginning of Lemma 3.4. From (7), using (23) and the latter relation in (18), one obtains

$$\frac{3}{2(n+1)} (1 + \theta_{n+1}) = 1 - \frac{1}{1 + \frac{3}{2n} (1 + \theta_n)} \left[1 - \frac{1}{3} \left(\frac{3}{2n} \right)^2 (1 + \theta_n - \frac{1}{2n} + O(n^{-2+\varepsilon}))^2 + O(n^{-4}) \right],$$

whence

$$\begin{aligned} \frac{3}{2n} \left(1 + \frac{1}{n} \right)^{-1} (1 + \theta_{n+1}) &= 1 - \left[1 - \frac{3}{2n} (1 + \theta_n) + \frac{9}{4n^2} (1 + 2\theta_n) - \frac{27}{8n^3} + O(n^{-4+2\varepsilon}) \right] \times \\ &\quad \times \left[1 - \frac{3}{4n^2} \left(1 + 2\theta_n - \frac{1}{n} \right) + O(n^{-4+2\varepsilon}) \right] \\ &= \frac{3}{2n} (1 + \theta_n) - \frac{9}{4n^2} (1 + 2\theta_n) + \frac{27}{8n^3} + \frac{3}{4n^2} \left(1 - \frac{3}{2n} \right) + \frac{3\theta_n}{2n^2} - \frac{3}{4n^3} + O(n^{-4+2\varepsilon}) \\ &= \frac{3}{2n} + \frac{3\theta_n}{2n} - \frac{3}{2n^2} - \frac{3\theta_n}{n^2} + \frac{3}{2n^3} + O(n^{-4+2\varepsilon}); \end{aligned}$$

thus,

$$1 + \theta_{n+1} = \left(1 + \theta_n - \frac{1}{n} - \frac{2\theta_n}{n} + \frac{1}{n^2} + O(n^{-3+2\varepsilon}) \right) \left(1 + \frac{1}{n} \right),$$

hence

$$\theta_{n+1} = \theta_n \left(1 - \frac{1}{n} \right) + O(n^{-3+2\varepsilon}).$$

Denote $z_n = (n-1)\theta_n$; then one has $z_{n+1} = z_n + O(n^{-2+2\varepsilon})$. It follows that the sequence z_n converges to a certain value c and, moreover, $z_n = c + O(n^{-1+2\varepsilon})$. Hence

$$\theta_n = \frac{z_n}{n-1} = \frac{c}{n-1} + O(n^{-2+2\varepsilon}) = \frac{c}{n} + O(n^{-2+2\varepsilon}).$$

Using the latter relation in (22), one comes to

$$\lambda_n = \frac{c}{n} - \frac{1}{2n} + O(n^{-2+2\varepsilon}).$$

Due to the arbitrariness of $\varepsilon > 0$, Lemma 3.5 is proved. □

Lemma 3.6. *We have*

$$r_n = c_0 n^{3/2} + c_1 n^{1/2} + O(n^{-1/2+\varepsilon}), \quad (31)$$

$$r_n = c_2 e^{t_n} \left(1 + \frac{9}{32} n^{-2} + o(n^{-2}) \right), \quad (32)$$

where $c_0 > 0$, c_1 , and $c_2 > 0$ are constants.

Proof. By formulas (5), (24), and (22) one has

$$\begin{aligned} \frac{r_{n+1}}{r_n} &= \frac{b_n \tau_n}{\sin \tau_n} = \frac{1 + \frac{3}{2n} \left(1 + \lambda_n + \frac{1}{2n} + \frac{\lambda_n}{n} - \frac{3}{8} n^{-2} \right) + o(n^{-3})}{1 - \frac{1}{6} \frac{9}{4n^2} (1 + 2\lambda_n + \lambda_n^2) + o(n^{-3})} \\ &= 1 + \frac{3}{2n} \left(1 + \lambda_n + \frac{3}{4n} + \frac{3\lambda_n}{2n} \right) + o(n^{-3}). \end{aligned}$$

It follows that

$$\begin{aligned} \ln r_{n+1} - \ln r_n &= \frac{3}{2n} \left(1 + \lambda_n + \frac{3}{4n} + \frac{3\lambda_n}{2n} \right) - \frac{1}{2} \frac{9}{4n^2} \left(1 + 2\lambda_n + \frac{3}{2n} \right) + \frac{1}{3} \frac{27}{8n^3} + o(n^{-3}) \\ &= \frac{3}{2n} \left(1 + \lambda_n - \frac{3}{8n^2} \right) + o(n^{-3}) = \tau_n - \frac{9}{16n^3} + o(n^{-3}). \end{aligned} \quad (33)$$

Hence, recalling that $\tau_n = t_{n+1} - t_n$, we get

$$\ln r_n = \ln r_1 + t_n - t_1 - \sum_{k=1}^{n-1} \left(\frac{9}{16k^3} + o(k^{-3}) \right) = \text{const} + t_n + \frac{9}{32} n^{-2} + o(n^{-2}). \quad (34)$$

Thus, we come to formula (32).

On the other hand, using (30), from (33) one derives

$$\ln r_n = \ln r_1 + \frac{3}{2} \sum_{k=1}^{n-1} \frac{1}{k} + \frac{3(c-1/2)}{2} \sum_{k=1}^{n-1} \left(\frac{1}{k^2} + O(k^{-3+\varepsilon}) \right).$$

By the Euler-Maclaurin formula one has

$$\sum_{k=1}^{n-1} \frac{1}{k} = \ln n + \gamma - \frac{1}{2n} + O(n^{-2}) \quad \text{and} \quad \sum_{k=1}^{n-1} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + O(n^{-2}),$$

where $\gamma \approx 0.577$ is Euler's constant, and so,

$$\ln r_n = \text{const}_1 + \frac{3}{2} \ln n + \frac{\text{const}_2}{n} + O(n^{-2+\varepsilon}). \quad (35)$$

From this formula it follows that

$$r_n = \hat{c} n^{3/2} e^{c_1/n + O(n^{-2+\varepsilon})} = \hat{c} n^{3/2} (1 + c_1/n + O(n^{-2+\varepsilon})).$$

Lemma 3.6 is proved. □

Lemma 3.7. *We have*

$$r_n = c_0 e^{t_n} \left(1 + O(e^{-\frac{4t_n}{3}}) \right), \quad (36)$$

and

$$t_n = c_1 + \frac{3}{2} \ln n + O(n^{-1}), \quad (37)$$

where $c_0 > 0$ and c_1 are constants.

Proof. One has

$$t_n = t_1 + \sum_{k=1}^{n-1} \tau_k \quad (38)$$

$$= t_1 + \sum_{k=1}^{n-1} \frac{3}{2k} (1 + \lambda_k) = \text{const} + \frac{3}{2} \ln n + O(n^{-1}). \quad (39)$$

It follows that

$$n = c e^{2t_n/3} (1 + O(n^{-1})), \quad (40)$$

where $c > 0$ is a constant, and by (32),

$$r_n = c_2 e^{t_n} \left(1 + \frac{9}{32c^2} e^{-4t_n/3} (1 + O(e^{-2t_n/3})) \right).$$

Lemma 3.7 is proved. $\square$

Now let us proceed to the proof of Theorem 3.1 in the limits $t \rightarrow +\infty$ and $n \rightarrow +\infty$. Formula (13) of the theorem follows from (37), and formula (14) follows from (31).

For $t = t_n$ by (36) we have $z(t_n) = r_n = c_0 e^{t_n} + O(e^{-t_n/3})$, and so, formula (15) is valid. It remains to consider the intermediate values $t \in (t_n, t_{n+1})$. It suffices to show that $\text{Re } z(t) = c e^t + O(e^{-t/3})$ and $\text{Im } z(t) = O(e^{-t/3})$. By (37), $n = c_1 e^{2t_n/3} (1 + o(1))$ and using (36) one finds that $r_n/n^2 = O(e^{-t_n/3})$. Using (12), one concludes that $\max\{\text{Im } z(t), t_n \leq t \leq t_{n+1}\} = O(e^{-t_n/3})$, and therefore, $\text{Im } z(t) = O(e^{-t/3})$.

Taking the derivative of both sides of (3) and using that $a_n = 1 + o(1)$, $b_n = 1 + o(1)$, $\tau_n = o(1)$, for $s = t - t_n \in (0, \tau_n)$ we get

$$\begin{aligned} \text{Re } \dot{z}(t_n + s) &= (r_n + 2y_n - x_n s) \cos s - [2x_n + (r_n + y_n)s] \sin s \\ &= r_n \left\{ (2b_n - 1 - a_n s) \cos s - (2a_n + b_n s) \sin s \right\} = r_n (1 + o(1)) > 0 \end{aligned} \quad (41)$$

for n sufficiently large. This implies that $\text{Re } \dot{z}(t)$ is contained between $\text{Re } \dot{z}(t_n) = x_n$ and $\text{Re } \dot{z}(t_{n+1}) = x_{n+1}$. Taking into account that $x_n = r_n a_n = r_n (1 + O(n^{-2}))$ by (19) and using that $n^{-2} = O(e^{-4t_n/3})$ and $0 < t - t_n < \tau_n = O(n^{-1})$, for an $s \in (0, t - t_n)$ one obtains

$$\begin{aligned} \text{Re } z(t) &= \text{Re } z(t_n) + \text{Re } \dot{z}(t_n + s)(t - t_n) = r_n + r_n (1 + O(n^{-2}))(t - t_n) \\ &= [c_0 e^{t_n} + O(e^{-t_n/3})] [1 + (t - t_n)(1 + O(n^{-2}))] = c_0 e^t + O(e^{-t/3}). \end{aligned}$$

Formula (15) is proved.

We have $\text{Re } \dot{z}(t_n) = x_n = a_n r_n$. Since by (15), $r_n = c e^{t_n} + O(e^{-t_n/3})$, by Lemma 3.3, $a_n = 1 + O(n^{-2})$, and by (40), $n = O(e^{2t_n/3})$, formula (16) is proved for $t = t_n$: $\text{Re } \dot{z}(t_n) = c e^{t_n} + O(e^{-t_n/3})$.

It remains to prove it for the intermediate values $t_n + t$, $t \in (0, \tau_n)$.

Indeed, using (3) and taking into account that we have $\tau_n = O(n^{-1}) = O(e^{-2t_n/3})$, $b_n - 1 = O(n^{-1}) = O(e^{-2t_n/3})$, $a_n = 1 + O(n^{-2}) = 1 + O(e^{-4t_n/3})$, and finally $\cos t = 1 + O(n^{-2}) = 1 + O(e^{-4t_n/3})$, one finds that for $0 < t < \tau_n$,

$$\begin{aligned} \text{Re } \dot{z}(t_n + t) &= r_n [(a_n + b_n t) \cos t + (b_n - 1 - a_n t) \sin t] \\ &= (c e^{t_n} + O(e^{-t_n/3})) [1 + t + O(e^{-4t_n/3})] = c e^{t_n + \ln(1+t)} + O(e^{-t_n/3}) \\ &= c e^{t_n + t} (1 + O(t^2)) + O(e^{-t_n/3}) = c e^{t_n + t} + O(e^{-(t_n + t)/3}). \end{aligned}$$

Formula (16) is proved.

By the latter relation in (18) and by (40) we have

$$b_n - 1 = \frac{3}{2n} (1 + o(1)), \quad n = c e^{2t_n/3} (1 + o(1)),$$

and therefore,

$$\begin{aligned}\operatorname{Im} \dot{z}(t_n + 0) &= y_n = r_n(b_n - 1) \\ &= \frac{3r_n}{2n} (1 + o(1)) = c e^{t_n} \frac{3}{2} e^{-2t_n/3} e^{2c_0/3} (1 + o(1)) = k e^{t_n/3} (1 + o(1)),\end{aligned}$$

where $k = \frac{3c}{2} e^{2c_0/3}$. Now using (4), we have

$$\operatorname{Im} \dot{z}(t_n + t) = r_n \left[(b_n - 1 - a_n t) \cos t - (a_n + b_n t) \sin t \right] = r_n \left[\frac{3}{2n} - 2t + o(n^{-1}) \right]$$

for $0 < t < \tau_n = \frac{3}{2n} + o(n^{-1})$, and therefore, $\operatorname{Im} \dot{z}$ varies from

$$\frac{3r_n}{2n} (1 + o(1)) = k e^{t_n/3} (1 + o(1)) \quad \text{to} \quad -\frac{3r_n}{2n} (1 + o(1)) = -k e^{t_n/3} (1 + o(1))$$

on the interval (t_n, t_{n+1}) . Formula (17) is proved, and so, Theorem 3.1 is completely proved.

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