

# On the Resolution of Ties in Fair Convex Allocation Problems

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**Abstract:** We study the emergence of indistinguishable, but structurally distinct, outcomes in convex resource allocation models with fairness-oriented objectives. Such outcomes arise when differently prioritized users receive identical coverage despite differences in initial conditions. We formalize this phenomenon as indeterminate priority (IP) outcomes and provide a structural taxonomy, based on Karush-Kuhn-Tucker conditions, that characterizes when they can occur and how they depend on resource scarcity and overlap. We are motivated by applications where visibly distinct outcomes are desired for prioritized users. To this end, we introduce a preference for distinctness such that users are discouraged from ties in final coverage whenever they compete for shared resources. We do so via a smooth pairwise penalty that perturbs the objective by penalizing ties in final coverages, and show that for sufficiently small coefficients this perturbation preserves the underlying fairness structure. As a case study, we apply our framework to a previously developed vaccine distribution model from Texas. IP outcomes arise in this real-world instance under both equal and unequal weights, and persist even under increased resource availability. By appropriately choosing the perturbation parameter's scale, our approach yields distinct interpretable solutions without distorting the fairness objective. We further illustrate how the choice of penalty shape and perturbation magnitude influences the extent of separation among tied users.

**Keywords:** Convex optimization, fairness, tie-breaking, distinct solutions, perturbation.

**2010 Mathematics Subject Classification:** 90C25, 91B32, 90C46.

## 1. Background

This work is inspired by the longstanding contributions of Prof. Alexander Plakhov in optimal control, convex geometry, and variational analysis. His seminal studies on Newton's problem of minimal resistance [15] and his results in optimal mass transport [16] demonstrate the power of convex-analytic techniques in resolving complex physical and mathematical trade-offs. Our analysis likewise relies on convex structure – specifically, to diagnose and rectify ambiguous allocation outcomes – echoing the spirit that permeates Prof. Plakhov's research. Just as his work has advanced the theory of mathematical optimization while retaining applied relevance (e.g., the dynamics of moving bodies [18]), we study tie-breaking in convex resource allocation with an eye toward policy applications.

Optimization models for resource allocation are pervasive in operations research and mathematical economics, see, e.g., [5, 10, 22]. Many such models employ convex objective functions combined with eligibility constraints [3, 8], and their practical motivation is often grounded in notions of fairness across differently weighted users [2, 9].

Singh [23] formalizes a fairness criterion for one such class of models. The framework considers multiple users  $i \in I$ , each with an initial coverage level  $f_i \in [0, 1]$  and a final coverage level  $y_i \in [0, 1]$  after allocation. Substitutable resources  $k \in K$  are available in limited quantities  $b_k$ , and users are eligible to receive only a subset of these. Specifically, the sets  $K_i \subseteq K, \forall i$  and  $I_k \subseteq I, \forall k$  denote the subsets of resources that user  $i$  is eligible for and the subset of users that resource  $k$  is suitable for, respectively. Each user has a weight  $w_i > 0$

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encoding their priority and represents a group of agents of population size  $n_i$  (i.e., a collection of  $n_i$  form user  $i$ ). Similar to a social welfare framework (see, e.g., [6, 12]), the goal is to minimize a weighted total loss function:

$$L(y) = \sum_{i \in I} w_i n_i F(y_i), \quad (1)$$

where  $F$  is a convex, decreasing function capturing the loss from a shortage (i.e., under-coverage). This yields the following convex optimization model:

$$z_F^* = \min_{x, y} \sum_{i \in I} w_i n_i F(y_i) \quad (2a)$$

$$\text{s.t.} \quad f_i + \frac{1}{n_i} \sum_{k \in K_i} x_{ik} = y_i, \quad \forall i \in I, \quad : \alpha_i \quad (2b)$$

$$y_i \in [0, 1], \quad \forall i \in I, \quad : \lambda_i \quad (2c)$$

$$\sum_{i \in I_k} x_{ik} \leq b_k, \quad \forall k \in K, \quad : \nu_k \quad (2d)$$

$$x_{ik} \geq 0, \quad \forall i \in I, \forall k \in K_i. \quad : \mu_{ik} \quad (2e)$$

The decision variables of model (2) are the number of resources of type  $k$  to provide to user  $i$ ,  $x_{ik}$ , and the corresponding final coverage of user  $i$ ,  $y_i$ . Further, the following definition of fairness is considered in [23].

**Definition 1.1.** [23] Consider two users  $i, j$  with  $w_i > w_j$ ,  $y_i < 1$  and  $y_i \leq y_j$ . Then, an optimal solution of model (2) is *fair* if  $x_{jk} = 0 \forall k \in K_i \cap K_j$ .  $\square$

Singh [23] proves the fairness guarantee of Definition 1.1 under mild assumptions. Intuitively, in an optimal solution, a higher-weighted user should not achieve a lower coverage than a lower-weighted one unless the latter already began with a substantially larger  $f_i$ . Definition 1.1 enforces a slightly stronger notion than this intuition. Examples of such functions include  $F(z) = (1-z)^m$  for  $m > 1$ , and  $F(z) = -\ln(z+\varepsilon)$  for small  $\varepsilon > 0$  [23].

However, even when prioritization is respected, the model may produce *indeterminate* outcomes: situations where two or more users receive identical final coverage. We formally define this notion in Section 2. These ties can arise despite differences in weights, initial coverages, or eligibility sets. Although mathematically optimal and consistent with the fairness framework, some of these solutions obscure the intended prioritization and may complicate public justification of the allocation; see, e.g., [1].

**Example 1.2.** Consider an instance of model (2) with  $F(z) = (1-z)^2$  with two users ( $|I| = 2$ ). Let  $w_1 = 2$ ,  $w_2 = 1$ ,  $n_1 = n_2 = 100$ , and  $f_1 = f_2 = 0.6$ . In both parts below, there is a shared resource  $k = 1$  and an additional resource that is available to only one of the users.

- (a) Consider two resources with capacities  $b_1 = 40, b_2 = 40$  with both users eligible for resource 1, but only user 2 can use resource 2; i.e.,  $K_1 = \{1\}, K_2 = \{1, 2\}$ . Then, an optimal solution assigns 40 units of resource 1 to user 1 and 40 units of resource 2 to user 2, giving  $y_1 = y_2 = 1$ .
- (b) Consider two resources with capacities  $b_1 = 40, b_2 = 40$  with both users eligible for resource 1, but only user 1 can use resource 2; i.e.,  $K_1 = \{1, 2\}, K_2 = \{1\}$ . Then, an optimal solution assigns 40 units of resource 2 to user 1 and 40 units of resource 1 to user 2, giving  $y_1 = y_2 = 1$ .

In both cases, different allocations of the shared resource lead to the same optimal coverage. The final coverage levels coincide because the total resource is abundant enough to cover both users. Such a tie is *benign* in the sense that all competing users achieve full coverage ( $y_i = 1$ ), hence breaking the tie requires withholding resources from at least one user, strictly worsening the primary loss  $L(y)$  without any compensating gain.  $\square$

Such ties can also occur in a simpler single-resource setting where resources are scarce as the following example shows.

**Example 1.3.** Consider an instance of model (2) with  $F(z) = (1-z)^2$  with two users ( $|I| = 2$ ) and one resource ( $|K| = 1$ ), with both users eligible. Let  $w_1 = 2, w_2 = 1, n_1 = n_2 = 100, f_1 = 0.1$ , and  $f_2 = 0.4$ .

- (a) If  $b_1 = 30$ , the optimal solution allocates all units to user 1, giving the result  $y_1 = y_2 = 0.4$ .

(b) If  $b_1 = 150$ , both users reach full coverage  $y_1 = y_2 = 1$ .

In both cases the solution respects the priority ordering (i.e., the higher-weight user is not neglected and, when resources are scarce, receives all units of the shared resource); however, the final coverages are non-distinct. Part (b) is benign for the same reason as Example 1.2: the resource is sufficiently abundant for both users to reach full coverage  $y_1 = y_2 = 1$ . By contrast, in part (a) the resource is scarce and both users remain under-covered with  $y_1 = y_2 = 0.4 < 1$ ; the higher-priority user receives all units of the shared resource and the final coverage levels coincide. In this instance, the tie is structurally unavoidable as feasibility bounds together with the fairness criterion jointly force equality of final coverages.  $\square$

Examples 1.2 and 1.3 show that ties in final coverages can arise both in abundant and scarce-resource regimes. When resources are abundant enough that all competing users reach  $y_i = 1$ , ties are benign in the sense that breaking them would require deliberately making some user worse off. In contrast, when resources are scarce and some users remain under-covered, ties may obscure how the priority structure is reflected in the solution – even though the fairness criterion of Singh [23] is satisfied. However, even in this situation, certain scarce ties may be structurally unavoidable when feasibility bounds and fairness jointly force equality of final coverages (as in Example 1.3(a)). As such, Examples 1.2 and 1.3 illustrate situations where ties are either benign or structurally unavoidable. Our work considers both such forms as not needing reconciliation. The next example illustrates what is potentially a provocative tie.

**Example 1.4.** Consider an instance of model (2) with  $F(z) = (1 - z)^2$  with three users ( $|I| = 3$ ) and two resources ( $|K| = 2$ ). Let  $w_1 = 2, w_2 = 1, w_3 = 0.5, n_1 = n_2 = n_3 = 100$ , and initial coverages  $f_1 = 0.1, f_2 = 0.2, f_3 = 0.0$ . Consider the first user is only eligible for the first resource, the third user is only eligible for the second resource, while the second user is eligible for both.

Consider resource capacities  $b_1 = 30$  and  $b_2 = 20$ . Then, an optimal solution assigns all units of resource 1 to user 1 and all units of resource 2 to user 2, giving  $y_1 = 0.4, y_2 = 0.4, y_3 = 0$ . The tie between users 1 and 2 is both non-benign (since, both users remain under-covered) and structurally avoidable (since, user 2 has access to a resource not shared with user 1, distinct feasible allocations exist that separate the final coverages without violating the fairness criterion). This is precisely the type of ties that matter for our analysis.  $\square$

Provocative ties of the form in Example 1.4 arise in realistic allocation settings where fairness (in the sense of [23]) is preserved, however optimal solutions obscure the intended prioritization among users. Whether in public health, education, or disaster relief, decision-makers are often under public scrutiny and must justify allocation outcomes to stakeholders. The question is then how can resources be re-assigned, if possible, in a nearly-optimal fashion that is still fair? Consider the following illustrative scenarios:

- (a) **Public health:** During vaccine allocation, groups such as frontline workers, the elderly, and high-risk individuals may be assigned different weights. If two groups with clearly different risk profiles receive identical coverage, the allocation may appear opaque or unjustified – even if mathematically fair; see, e.g., Yi and Marathe [26].
- (b) **Educational grants:** A funding agency may prioritize applicants from underrepresented backgrounds by assigning higher weights. If the resulting grant amounts are equal across groups, the process may be perceived as arbitrary or inconsistent with stated goals; see also Corbett-Davies et al. [4].
- (c) **Disaster relief:** Ties in aid distribution between regions with differing levels of damage and vulnerability may undermine the perceived responsiveness or credibility of relief efforts; see also Mazepus and van Leeuwen [11].

In each of these examples, decision-makers may be challenged to explain why, under resource scarcity, distinct priorities did not lead to visibly distinct outcomes. This motivates our focus on *distinctness*: promoting visibly distinct outcomes for users with different weights whenever such refinements are compatible with feasibility and fairness. In discrete optimization, such distinctness may be imposed explicitly, e.g., via so-called all-different constraints [24], or via big- $M$  formulations that force  $y_i \neq y_j$ . Symmetry in feasible or optimal solutions of integer programs is another closely related notion in the discrete optimization community, see, e.g., [13]. In our setting, however, the feasible region already encodes highly basic resource

constraints while the objective function seeks to ensure fairness in the allocation. Our focus is on not altering the feasible set and ensuring distinctness, where possible, via the objective function alone.

To this end, we pursue a different route and revisit a classical idea in optimization: adding a small, separable penalty term to the objective to discourage equal coverage levels. This perturbation leaves the feasible region unchanged and selects distinct refinements of the primary-optimal set whenever such refinements exist. Our approach is related to a lexicographic ordering – where one first minimizes the primary loss  $L(y)$  and then refines within its optimal sets – however, we seek a single penalty-based formulation. As we show later, for appropriately small penalization the priority structure remains respected.

The following are the main contributions of this work:

- (i) We define and classify indeterminate priority outcomes in fairness seeking convex resource-allocation models with attention to how eligibility structures interact.
- (ii) We derive conditions under which a perturbation breaks non-distinct outcomes while preserving the original fairness guarantees of Definition 1.1.
- (iii) We validate the approach using a case study derived from a real-world vaccine allocation problem during the 2009 H1N1 pandemic, showing that indeterminate outcomes arise both under equal and unequal weights, and across a range of resource regimes.

The structure of the rest of the work is as follows. Section 2 formalizes the problem we study and presents a taxonomy of priorities. Section 3 introduces the perturbed model and studies its key properties. Section 4 presents our numerical study, while Section 5 concludes with a summary and implications for both theory and practice.

## 2. Problem setup

We study the distinctness of optimal allocations under the following assumptions:

- Assumption 2.1.** (a) Distinct weights: Users are relabeled such that we have  $w_1 > w_2 > \dots > w_{|I|}$ ; i.e.,  $w_j > w_i$  for all pairs  $j < i$ .
- (b) Unmet initial demand: Every user has  $f_i < 1$ .
- (c) Scarce resources: For each  $k \in K$ ,  $\sum_{i \in I_k} x_{ik} = b_k < \sum_{i \in I_k} n_i(1 - f_i)$ , implying every resource is fully utilized [23, Theorem 3].
- (d) Loss function: The function  $F: [0, 1] \rightarrow \mathbb{R}$  is continuously differentiable, strictly convex, and strictly decreasing; i.e.,  $F'(z)$  exists, is strictly increasing, and satisfies  $F'(z) < 0$  for all  $z \in [0, 1]$ .  $\square$

Assumption 2.1 enforces a strict priority ordering and rules out trivial saturation of users (e.g., via resource abundance or via completely met initial demand). Further, we restrict attention to user pairs that actually compete for at least one common resource. Suppose a pair  $(i, j)$  ties at optimality; i.e.,  $y_i = y_j$ , and  $K_i \cap K_j \neq \emptyset$ .

If  $x_{ik} = x_{jk} = 0$  for all shared  $k \in K_i \cap K_j$ , two cases arise:

- (i) neither user receives any allocation (i.e., their coverage is entirely due to  $f_i, f_j$ ), or
- (ii) each user receives allocations only from non-shared resources.

Case (ii) entails no direct competition while case (i) contains no meaningful priority comparison; thus, we exclude such pairs from our analysis and focus on pairs that receive some shared resource.

**Definition 2.2.** A pair  $(i, j)$  with  $i > j$  is a *relevant pair* if, for some  $k \in K_i \cap K_j$ , at least one of them receives a positive allocation of the shared resource; i.e.,

$$\max\{x_{ik}, x_{jk}\} > 0.$$

$\square$

For relevant pairs, we distinguish three types of priority relations.

**Definition 2.3.** For a relevant pair  $(i, j)$ , exactly one of the following holds at optimality:

$$\begin{aligned} \text{Consistent Priority (CP): } & y_j > y_i, \\ \text{Violated Priority (VP): } & y_j < y_i, \\ \text{Indeterminate Priority (IP): } & y_j = y_i. \end{aligned} \quad \square$$

**Definition 2.4.** An optimal allocation  $(x, y)$  of model (2) is *distinct* if  $y_j \neq y_i$  for every relevant pair  $i > j$ . Equivalently, no IP cases occur.  $\square$

The following simple observation, used later in our perturbation analysis, characterizes VP pairs.

**Proposition 2.5.** Under Assumption 2.1, for every relevant pair  $(i, j)$  with a VP, we have

$$w_i F'(y_i) - w_j F'(y_j) > 0.$$

**Proof.** From Assumption 2.1(d),  $F$  is strictly convex and strictly decreasing,  $F'$  is strictly increasing and negative on  $[0, 1]$ . For a VP pair we have  $w_j > w_i$  and  $y_j < y_i$ . Then

$$w_i F'(y_i) - w_j F'(y_j) = (w_i - w_j) F'(y_i) + w_j (F'(y_i) - F'(y_j)).$$

The first term is positive since  $w_i - w_j < 0$  and  $F'(y_i) < 0$ ; the second term is positive since  $w_j > 0$  and  $F'(y)$  is strictly increasing. Hence, the sum is positive.  $\square$

Under Assumption 2.1, model (2) is a strictly convex optimization problem with linear constraints and a non-empty interior; thus, the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality. As labeled in model (2), let  $\alpha_i \in \mathbb{R}$ ,  $\nu_k \geq 0$  and  $\mu_{ik} \geq 0$  be the dual multipliers for constraints (2b), (2d) and (2e), respectively, and  $\lambda_i \geq 0$  be that for  $y_i \leq 1$ . Then, the KKT system is:

$$\text{stationarity: } \mu_{ik} = \nu_k + w_i F'(y_i) + \frac{\lambda_i}{n_i}, \quad \forall i \in I, k \in K_i \quad (3a)$$

$$\text{complementary slackness: } \mu_{ik} \cdot x_{ik} = 0, \quad \forall i \in I, k \in K_i \quad (3b)$$

$$\nu_k \cdot \left( b_k - \sum_{i \in I_k} x_{ik} \right) = 0, \quad \forall k \in K \quad (3c)$$

$$\lambda_i \cdot (1 - y_i) = 0, \quad \forall i \in I \quad (3d)$$

$$\text{primal feasibility: } x_{ik} \geq 0, \quad \sum_{i \in I_k} x_{ik} \leq b_k, \quad \forall i \in I, k \in K \quad (3e)$$

$$f_i + \frac{1}{n_i} \sum_{k \in K_i} x_{ik} = y_i \leq 1, \quad \forall i \in I \quad (3f)$$

For a relevant pair  $(i, j)$  sharing a resource  $k \in K_j \cap K_i$ , exactly three allocation patterns are possible:

$$(P1) \quad x_{ik} > 0, x_{jk} > 0,$$

$$(P2) \quad x_{ik} > 0, x_{jk} = 0,$$

$$(P3) \quad x_{ik} = 0, x_{jk} > 0,$$

The next result classifies which of these patterns can occur if an IP outcome arises.

**Proposition 2.6.** Under Assumption 2.1, let  $(i, j)$  be a relevant pair which satisfies  $y_i = y_j = \bar{y}$  in an optimal solution. Then, exactly one of the following holds:

(A)  $\bar{y} = 1$ : any pattern (P1)–(P3) can occur.

(B)  $\bar{y} < 1$ : only pattern (P3) can occur.

**Proof.** Choose  $k \in K_i \cap K_j$  with  $\max\{x_{ik}, x_{jk}\} > 0$ . Subtracting equation (3a) for users  $j$  and  $i$  and using  $y_j = y_i = \bar{y}$  gives

$$(w_i - w_j) F'(\bar{y}) = \mu_{ik} - \mu_{jk} + \frac{\lambda_j}{n_j} - \frac{\lambda_i}{n_i}.$$

From Assumption 2.1, the left-hand side is strictly positive.

- (A) If  $\bar{y} = 1$ ,  $\lambda_i$  and  $\lambda_j$  may be non-zero, and any sign on the right-hand side is possible; thus, all patterns (P1)–(P3) are compatible with optimality.
- (B) If  $\bar{y} < 1$ , complementary slackness implies  $\lambda_i = \lambda_j = 0$ . Then,  $\mu_{ik} > \mu_{jk}$ . Hence  $\mu_{ik} > 0$  and, by equation (3b),  $x_{ik} = 0$ . Thus, only pattern (P3) remains consistent.  $\square$

Examples 1.2–1.4 illustrate the three allocation patterns of Proposition 2.6. In Example 1.3(a), the shared resource is allocated solely to the higher-priority user, yielding pattern (P3); however, the resulting tie is structurally unavoidable because feasibility bounds and fairness jointly force equality. In Example 1.2(a), the higher-priority user again receives the shared resource giving pattern (P3); however, the tie is benign since both users reach full coverage. In Example 1.2(b), the lower-priority user receives the shared resource, yielding pattern (P2) which is again a benign tie. In Example 1.3(b), an optimal solution may allocate positive amounts of the shared resource to both users fully covering them yielding pattern (P1). Example 1.4 displays pattern (P3) in a non-benign setting where equality is not forced by feasibility; in this case, the tie is structurally avoidable because a priority-consistent separating direction exists within the feasible region (we analyze this in detail in Section 4). Proposition 2.6 shows that meaningful non-saturated ties ( $\bar{y} < 1$ ) have a very specific structure that is tied to fairness: any shared resource must be allocated exclusively to the higher-priority user, and IP outcomes arise only through interactions with non-shared resources and initial coverage. In the next section, we exploit this structure and introduce a perturbation of the objective that converts (where possible) any optimal solution with IP outcomes into a distinct one while preserving the fairness property of Definition 1.1.

### 3. Perturbations of the objective function

The analysis so far classifies when and how ties (i.e., IP outcomes) may arise in optimal solutions of model (2) using the concept of relevant priority pairs and the taxonomy of priorities stated in Definition 2.3. We now propose a refinement that preserves the fairness notion of Definition 1.1, yet seeks to eliminate ties. We do so via a perturbation that penalizes equal coverage levels without altering the feasible region. Such regularization has precedent not only in continuous optimization (e.g., via proximal terms to ensure strict convexity or to achieve faster convergence [21, 14]) but also in combinatorial optimization (e.g., to break symmetry in mixed-integer programs [19, 25]).

For a small  $\varepsilon > 0$ , define a pairwise penalty term

$$\Psi_\varepsilon(y) = \varepsilon \sum_{i,j \in I; j < i} h(y_i - y_j),$$

where the function  $h : [-1, 1] \rightarrow \mathbb{R}$  satisfies the following assumptions.

**Assumption 3.1.** The function  $h : [-1, 1] \rightarrow \mathbb{R}$  satisfies:

- (i)  $h$  attains a (strict) global maximum at 0; i.e.,  $h(0) > h(d)$  for all  $d \neq 0$ ;
- (ii)  $h(d) > 0$  for all  $d \in [-1, 1]$ .
- (iii)  $h$  is even and continuously differentiable on  $[-1, 1]$ ;
- (iv) there exists  $L > 0$  such that  $|h'(d)| \leq L$  for all  $d \in [-1, 1]$ .  $\square$

Representative examples of Assumption 3.1 include  $h(d) = 1/(\eta + d^{2p})$  for  $p > 0$  and  $h(d) = -\log(d^2 + \eta)$  for a suitable constant  $\eta > 0$ . Then, the tie-breaking problem is as follows.

$$\min_{x,y} \sum_{i \in I} w_i n_i F(y_i) + \Psi_\varepsilon(y) \tag{4a}$$

$$\text{s.t. } (2b) - (2e). \tag{4b}$$

The augmented objective in (4) discourages equal coverages: if a pair  $(i, j)$  has  $y_i = y_j$  and there exist feasible perturbations that separate  $y_i$  and  $y_j$  without increasing the primary loss  $\sum_i w_i n_i F(y_i)$  (or, only within a small enough tolerance), then the penalty term  $\Psi_\varepsilon(y)$  decreases because  $h$  attains a strict maximum at 0. Thus, whenever a distinct allocation is available as a refinement of the (near-)optimal solutions of the

primary objective (and, as we show later, is also compatible with the fairness structure), the perturbation tends to select such a distinct solution. In contrast, when equality is structurally forced by feasibility and fairness (as in Example 1.3), the perturbation cannot produce a distinct solution without increasing the primary loss function; hence, for  $\varepsilon$  sufficiently small (as given in Theorem 3.2), a tie remains optimal. Next, we present a set of additional restrictions on the penalty term that ensure optimal solutions remain fair.

**Theorem 3.2.** *Let Assumption 2.1 and Assumption 3.1 hold. For sufficiently small  $\varepsilon$ , an optimal solution of model (4) is fair in the sense of Definition 1.1; i.e., no Violated Priority (VP) outcome can occur.*

**Proof.** Assume, for contradiction, that in an optimal solution of model (4) a VP outcome occurs. Then there exists a relevant pair  $(i, j)$  with  $w_i < w_j$ ,  $y_i > y_j$ ,  $y_j < 1$  where  $x_{ik} > 0$  for some  $k \in K_i \cap K_j$ ; the strict inequality  $y_i > y_j$  follows from the definition of a VP outcome.

For the penalty term,  $\Psi_\varepsilon(y)$ , we have

$$\frac{\partial \Psi_\varepsilon(y)}{\partial y_i} = \varepsilon H_i(y), \quad \text{where} \quad H_i(y) = \sum_{r < i} h'(y_i - y_r) - \sum_{r > i} h'(y_r - y_i), \quad \forall i \in I.$$

For any feasible solution,  $y_r \in [0, 1]$ ,  $\forall r \in I$ , we have  $y_i - y_r \in [-1, 1]$ . Further, at an optimal solution with distinct coverages, we have  $y_i - y_r \in [-1, 1] \setminus \{0\}$  for all  $r \neq i$ . Each of the two sums defining  $H_i(y)$  contain at most  $|I| - 1$  terms. By Assumption 3.1, there exists  $L > 0$  such that  $|h'(d)| \leq L$  for all  $d \in [-1, 1]$ , hence

$$|H_i(y)| \leq 2L(|I| - 1), \quad \forall i \in I. \quad (5)$$

Let  $\underline{n} = \min_{i \in I} n_i > 0$ . Then, we have

$$\left| \varepsilon \left( \frac{H_i(y)}{n_i} - \frac{H_j(y)}{n_j} \right) \right| \leq \varepsilon \left( \frac{|H_i(y)|}{n_i} + \frac{|H_j(y)|}{n_j} \right), \quad (6a)$$

$$\leq \varepsilon \left( \frac{2L(|I| - 1)}{n_i} + \frac{2L(|I| - 1)}{n_j} \right), \quad (6b)$$

$$\leq \varepsilon \frac{4L(|I| - 1)}{\underline{n}}, \quad (6c)$$

where inequality (6a) holds by the triangle inequality, inequality (6b) holds from equation (5), and inequality (6c) holds directly by the definition of  $\underline{n}$ . Since the feasible region of models (2) and (4) is identical, the KKT conditions coincide with equation (3) except for the stationarity conditions, which become

$$\mu_{ik} = v_k + w_i F'(y_i) + \frac{\varepsilon}{n_i} H_i(y) + \frac{\lambda_i}{n_i}, \quad \forall i \in I, \quad k \in K_i.$$

For the relevant pair,  $(j, i)$ , under consideration, complementary slackness leads to  $\lambda_j = 0, \mu_{ik} = 0$ ; thus, the stationarity conditions imply

$$w_i F'(y_i) - w_j F'(y_j) = \mu_{ik} - \mu_{jk} - \varepsilon \left( \frac{H_i(y)}{n_i} - \frac{H_j(y)}{n_j} \right) - \frac{\lambda_i}{n_i}, \quad (7a)$$

$$= -\mu_{jk} - \varepsilon \left( \frac{H_i(y)}{n_i} - \frac{H_j(y)}{n_j} \right) - \frac{\lambda_i}{n_i}, \quad (7b)$$

$$\leq \varepsilon \cdot \frac{4L(|I| - 1)}{\underline{n}}. \quad (7c)$$

where inequality (7c) follows from the non-negativity of the dual multipliers and equation (6).

From Proposition 2.5 there exists a  $\Delta > 0$  such that for any VP configuration of a relevant pair  $w_i F'(y_i) - w_j F'(y_j) \geq \Delta$ .

For  $\varepsilon \in (0, \frac{\Delta \underline{n}}{4L(|I| - 1)})$ , this lower bound contradicts equation (7c). Hence, for sufficiently small  $\varepsilon$ , no VP outcome can occur in any optimal solution of (4), and every such solution is fair in the sense of Definition 1.1.  $\square$

For the theoretical development above we penalize all pairs  $(i, j)$  with  $j < i$ . In practice, one may restrict the penalty to a subset  $I'$  of relevant pairs, e.g., those that tie in an optimal solution of model (2). This simply reduces the number of terms in  $H_i(y)$ , and the bounds in the proof of Theorem 3.2 remain

valid (with  $|I| - 1$  replaced by  $|I'|$ ). In particular, the fairness guarantee (i.e., absence of VP outcomes for sufficiently small  $\varepsilon$ ) persists. However, the degree of distinctness may change; i.e., although restricting the penalty to  $I'$  ensures that the original ties are discouraged, it no longer guarantees that the entire coverage vector is distinct.

Intuitively, the penalty  $\Psi_\varepsilon(y)$  plays a role similar to “sharpening” of the optimum by discouraging equal coverage levels. This property is reminiscent of the notion of sharp minima and the associated superstability property in the sense of Polyak [20, Chapter 5]. Here, a minimizer,  $y^\star$ , of a function  $g(\cdot)$  satisfies an error bound of the form  $g(y) \geq g(y^\star) + \alpha\|y - y^\star\|$  for some  $\alpha > 0$ .

Our setting differs from this in two respects:

First, by Assumption 3.1, the perturbed objective remains continuously differentiable in a neighborhood of the optimal solution, and the leading term  $\sum_i w_i n_i F(y_i)$  is strictly convex in  $y$ ; thus, in the original problem we obtain a unique minimizer and the typical continuity of an optimal solution with respect to small parameter changes. However, the perturbed problem is not guaranteed to produce a unique minimum in coverages without convexity assumptions on  $h$ , and even in that case a sharp minimum in Polyak’s technical sense is not achieved.

Second, sharp minima in Polyak’s framework are typically associated with non-smoothness (e.g., kinks or norms in the objective), while our construction uses a smooth penalty that assigns strictly larger objective values to tied solutions without introducing an infinite barrier or a non-smooth kink at the optimum. In our case, distinctness of the coverage profile arises from this smooth but concentrated penalization of ties, rather than from a sharp-minimum or superstability mechanism.

We now return to the case of benign ties, where the original model admits a continuum of optimal solutions with  $y_1 = y_2 = 1$ . These solutions correspond to different ways of splitting the shared resource between the two users. In this setting, the perturbation acts only within the set of full-coverage-optimal solutions: since any reduction of coverage strictly increases the primary loss, the perturbed problem does not sacrifice primary optimality for sufficiently small  $\varepsilon$ . Hence, in the small- $\varepsilon$  regime characterized in Theorem 3.2, such benign ties may remain intact. To summarize, the perturbation serves only to select among primary-optimal allocations when distinct refinements exist that do not worsen the primary loss.

In the next section, we provide numerical experiments showing how the perturbation serves as a consistent and interpretable tie-breaking mechanism that aligns with the underlying fairness structure.

#### 4. Case study

As a numerical case study, we analyze the vaccine distribution model presented in [7]. This model considers the 254 counties in Texas with five priority groups:

(i) 0–3 year olds, (ii) 4–24 year olds, (iii) 25–64 year olds at high risk, (iv) pregnant women, and (v) infant caregivers – competing for four vaccine types – (i) pre-filled syringe for infants (PFS baby), (ii) standard pre-filled syringe (PFS), (iii) multi-dose vial (MDV), and (iv) live attenuated influenza vaccine (LAIV). In our notation, the five priority groups correspond to five users  $i = 1, \dots, 5$ , while the four vaccine types correspond to four resource types  $k = 1, \dots, 4$ . Each user is eligible to receive only a subset of the available vaccines. The suitability matrix,  $K_i \subseteq K$ , defining these eligibility sets is given in Table 1. Huang et al. [7] consider individual county level data, however for the purposes of this work we aggregate the data to the state-level to define the input parameters of model (2). Table 2 presents the data we use. We use the same function  $F(z) = (1 - z)^2$  as in [7].

**Table 1.** Eligibility matrix,  $K_i$ , from [7]. An entry  $(i, k)$  of 1 denotes that user  $i$  is eligible for resource  $k$ .

| User/Resource |                   | $k = 1$  | $k = 2$ | $k = 3$ | $k = 4$ |
|---------------|-------------------|----------|---------|---------|---------|
|               |                   | PFS baby | PFS     | MDV     | LAIV    |
| $i = 1$       | 0–3 years         | 1        | 0       | 0       | 0       |
| $i = 2$       | 4–24 years        | 0        | 1       | 1       | 1       |
| $i = 3$       | 25–64 (high-risk) | 0        | 1       | 1       | 0       |
| $i = 4$       | Pregnant women    | 0        | 1       | 1       | 0       |
| $i = 5$       | Infant caregivers | 0        | 1       | 1       | 1       |

**Table 2.** Aggregated user and resource data for the baseline instance**(a)** User data

| User $i$ | $n_i$     | $f_i$  |
|----------|-----------|--------|
| 1        | 1,568,427 | 0.1549 |
| 2        | 7,632,499 | 0.7158 |
| 3        | 3,276,939 | 0.5219 |
| 4        | 342,432   | 0.5378 |
| 5        | 681,930   | 0.7266 |

**(b)** Resource data

| Resource $k$ | $b_k$   |
|--------------|---------|
| 1            | 17,711  |
| 2            | 100,365 |
| 3            | 354,228 |
| 4            | 118,076 |

The aim of this case study is to illustrate: (i) how IP outcomes arise in this real-world instance, and (ii) how the perturbation of Section 3 can restore distinctness while preserving fairness, and (iii) how the behavior depends on the choice of  $h$  and  $\varepsilon$ .

In the sections below, the final coverage vectors are displayed until the third or fourth decimal place, however the numerical solutions are computed to certified global optimality with optimality tolerances on the order of  $10^{-9}$ .

#### 4.1. Equal weights

We first consider an instance with uniform weights:  $w_i = 1$  for all  $i \in I$ , as assumed in the original setting of [7]. Solving model (2) yields the final coverage levels:

$$y^{F,\text{eq}} = [0.166, 0.731, 0.649, 0.649, 0.731]. \quad (8)$$

There are two ties: between users 2 and 5, and between users 3 and 4. The corresponding allocation shares on the shared vaccines (PFS, MDV, LAIV) are non-zero for both users in each tied pair; thus, these ties arise from splitting shared resources between groups with identical priority. This instance lies outside Assumption 2.1(a), since the weights are not strictly ordered; thus, the taxonomy in Proposition 2.6 does not apply. We report it here to illustrate that IP outcomes appear naturally in real-world policy-making, and to motivate the perturbation as a tie-breaking device.

Solving the perturbed model (4) with  $h(d) = 1/(d^2 + 10^{-6})$  and  $\varepsilon = 10^{-5}$  removes these ties yielding:

$$y^{P,\text{eq}} = [0.166, 0.731, 0.649, 0.647, 0.732]. \quad (9)$$

The changes in coverage between equations (8) and (9) are small, however the groups whose coverages change are those for which the magnitude of the marginal impact on the loss is smallest. This behavior foreshadows the more systematic results we present next.

#### 4.2. Unequal weights

Next, we present a setting that falls under the taxonomy of Section 2. Consider the strictly distinct weight vector

$$\bar{w} = [0.767, 0.702, 0.806, 1.079, 0.677], \quad (10)$$

which induces a strict priority ordering (after relabeling users by weight). This vector was constructed via a simple randomized search. Solving model (2) with these weights and the same inputs yields

$$y^{F,\bar{w}} = [0.1662, 0.7313, 0.6404, 0.7313, 0.7266]. \quad (11)$$

We now have a single tie between users 2 and 4 both of whom have coverages strictly below 1. In an optimal solution, user 2 receives no doses of shared resource types PFS and MDV (i.e.,  $x_{22} = x_{23} = 0$ ), while user 4 receives positive allocations of both (i.e.,  $x_{42}, x_{43} > 0$ ). Thus, for this relevant pair, the shared resources are allocated exclusively to the higher-priority user with  $\bar{w}_4 > \bar{w}_2$ . This is exactly Proposition 2.6 (B), and, indeed, pattern (P3) occurs; i.e., for every shared resource, only the higher-priority user receives a positive allocation. The solution is fair in the sense of Definition 1.1 even though  $y_2^{F,\bar{w}} = y_4^{F,\bar{w}}$ .

Next we apply the perturbation. Solving model (4) with the same data, the function  $h(d) = 1/(d^2 + 10^{-6})$ , and  $\varepsilon = 10^{-5}$ , yields

$$y^{P, \bar{w}} = [0.1662, 0.7313, 0.6402, 0.7333, 0.7266], \quad (12)$$

in which all coverages are distinct.

The fairness structure is preserved: whenever a lower-priority user attains a larger final coverage than a higher-priority one, the lower-priority user receives no shared resources (as required by Definition 1.1). The perturbation acts as a secondary criterion on top of the original loss  $L(y) = \sum_i w_i n_i (1 - y_i)^2$ . To minimize its impact on  $L$ , the optimizer perturbs users with the smallest magnitude of the marginal cost  $|\partial L / \partial y_i| = 2w_i n_i (1 - y_i^*)$ . For the tie between users 2 and 4, the magnitude  $|\partial L / \partial y_4|$  is about an order of magnitude smaller than  $|\partial L / \partial y_2|$ ; hence, the optimizer slightly increases the coverage of the higher-priority group 4 while keeping the other coverages essentially unchanged. The penalty  $h(\cdot)$  determines how far the two tied values separate; the direction of movement is governed by the marginal loss and the fairness structure.

For this instance, Assumption 3.1 holds and Theorem 3.2 guarantees the existence of a threshold  $\bar{\varepsilon} > 0$  such that every optimal solution of the perturbed problem is fair for all  $\varepsilon \in (0, \bar{\varepsilon})$ . Although the proof shows that  $\bar{\varepsilon}$  depends on an instance-specific gap parameter  $\Delta > 0$  (which measures the minimal difference  $w_i F'(y_i) - w_j F'(y_j)$  over relevant pairs at optimality), the theorem does not yield a simple value of  $\bar{\varepsilon}$  that is known prior to the optimization. In practice, as we demonstrated above, we can gauge the scale of this small perturbation. For  $h(d) = 1/(d^2 + 10^{-6})$ , a direct bound on  $|h'(d)|$  over  $[-1, 1]$  yields  $L \approx 6.5 \times 10^8$ . With  $\underline{n} = \min_{i \in I} n_i = 342,432$  and  $|I| = 5$ , the quantity

$$\frac{\underline{n}}{4L(|I| - 1)} = \frac{342,432}{16L} \approx 3.3 \times 10^{-5}$$

provides a natural scale for  $\varepsilon$ , up to the unknown factor  $\Delta$ . Our choice  $\varepsilon = 10^{-5}$  is of the same small order of magnitude and strictly below this crude scale. Consistently with Theorem 3.2, the resulting solution (12) is empirically fair for all relevant pairs and restores distinctness.

### 4.3. Effect of penalty sharpness

To highlight the role of the penalty shape, we next consider a more sharply peaked function,  $h(d) = 1/(d^4 + 10^{-6})$ , with the same weights as equation (10) and the baseline data from Table 1 and Table 2. For this penalty, the derivative is  $h'(d) = -4d^3/((d^4 + 10^{-6})^2)$ , which is bounded on  $[-1, 1]$ , and a direct calculation yields an upper bounding constant  $L \approx 3.4 \times 10^7$  for  $|h'(d)|$  on this interval. With  $\underline{n} = \min_i n_i = 342,432$  and  $|I| = 5$ , the quantity

$$\frac{\underline{n}}{4L(|I| - 1)} = \frac{342,432}{16L} \approx 6.3 \times 10^{-4}$$

provides an estimate of the scale of  $\varepsilon$ , up to the unknown instance-specific factor  $\Delta$  that appears in the proof of Theorem 3.2. In our experiments with this sharper penalty we choose  $\varepsilon = 6 \times 10^{-4}$ , which is of the same small order of magnitude as this scale.

Solving the perturbed model (4) for this instance yields the final coverage vector

$$y_{\text{sharp}}^{P, \bar{w}} = [0.1662, 0.7313, 0.6404, 0.7313, 0.7266], \quad (13)$$

which coincides (up to numerical precision) with the unperturbed solution in equation (11). Relative to the milder penalty solution (12), the effect of a sharper penalty depends critically on the scale of  $\varepsilon$ . For very small  $\varepsilon$  (e.g.,  $\varepsilon \in \{10^{-6}, 10^{-7}\}$  in this instance), the perturbation acts only within the primary-optimal set and leaves the IP tie between users 2 and 4 unchanged (since separating the tied coverages requires a non-negligible increase in the primary loss  $L(y)$ ). Thus, distinctness emerges only when  $\varepsilon$  is chosen large enough for the secondary criterion to become active while still remaining within the fairness-preserving range of Theorem 3.2.

### 4.4. Effect of increased resource availability

As discussed in Section 2, IP outcomes become more pronounced and less benign as users approach full coverage. To illustrate the effect of increasing resource availability, we consider a version of the baseline instance (with weights given by equation (10)) in which each resource supply is scaled by a factor of seven. The unperturbed model then yields

$$y^{F,\text{scaled}} = [0.234, 0.994, 0.994, 0.996, 0.994], \quad (14)$$

assigning nearly complete coverage to four user groups. User 1 is only eligible for a non-shared resource (PFS baby) and receives a strictly smaller, distinct coverage. With  $\bar{w}_3 > \bar{w}_2$  and  $y_2^{F,\text{scaled}} = y_3^{F,\text{scaled}} < 1$ , this again corresponds to an IP outcome of case (B) in Proposition 2.6: a non-saturated meaningful tie in which the higher-priority user receives the shared allocations (pattern (P3)) with a fair but non-distinct solution.

Solving the perturbed model (4) for this scaled instance, with  $h(d) = 1/(d^2 + 10^{-6})$  and  $\varepsilon = 10^{-5}$  (as in the previous experiments) yields

$$y^{P,\text{scaled}} = [0.234, 0.994, 0.995, 0.998, 0.991], \quad (15)$$

restoring distinctness while keeping values close to the unperturbed optimal solution. The optimal allocation of shared resources (PFS, MDV, LAIV) corresponding to equation (15) is fair in the sense of Definition 1.1 and distinct for all relevant pairs. This illustrates that IP outcomes become more pronounced as resources become abundant, and that the perturbation mechanism can correct these outcomes even when the total supply is still insufficient to cover all users fully.

#### 4.5. Numerical performance

All numerical instances were solved on a Dell laptop with 16 GB RAM, using GAMS Studio 1.21.5 and BARON version 25.10.15 [27]. All instances reported here were solved essentially instantaneously to global optimality and without numerical warnings. This applies both to the baseline instances and to the scaled-supply variants, for the milder penalty  $h(d) = 1/(d^2 + \delta)$  and for the sharper penalty  $h(d) = 1/(d^4 + \delta)$  with the small values of  $\varepsilon$  used in our experiments. However, sharper penalties combined with larger values of  $\varepsilon$  may lead to ill-conditioned optimization problems in principle (a behavior we observed in our experiments not reported here). In practice, our findings suggest that milder penalties of the form  $1/(d^2 + \delta)$ , together with appropriate  $\varepsilon$ , are sufficient to remove non-benign ties while having no practically relevant adverse effect on convergence speed or numerical stability.

### 5. Conclusions

Systematic tie-breaking mechanisms in optimization-based allocation are relevant in high-stakes settings where transparency, interpretability, and adherence to prioritization principles are essential. In this work, we studied the occurrence of non-distinct outcomes in convex resource allocation problems under a fairness framework in the sense of [23]. We showed that such outcomes – characterized by ties in final coverages across users – can naturally arise even in realistic, policy-relevant instances under both uniform and non-uniform weighting schemes.

Our structural analysis, based on relevant priority pairs and the priority taxonomy, clarified when these indeterminate outcomes are benign and when they give rise to configurations that obscure the intended prioritization. We showed that indeterminate outcomes may persist not only in regimes of relative resource abundance, but also in moderately scarce regimes where the total supply is insufficient to satisfy all users fully. In these cases, ties emerge not from slack constraints but from overlapping eligibility sets and the nature of the convex loss function. Thus, distinctness cannot be inferred from resource capacity alone.

To resolve these ambiguities, we revisited a classical idea: small perturbations to the objective function. We proposed a smooth, separable penalty term that discourages equal coverage levels and proved that, for sufficiently small perturbation parameters, every optimal solution to the perturbed problem remains fair in the original sense. The perturbation acts as a principled secondary criterion: it operates only within the set of optimal solutions of the base model and selects those that break ties in a manner consistent with the existing priority structure. We also discussed how the perturbation can be restricted to tied relevant pairs, preserving the original allocation for users who already have unique coverage. While such perturbations have a long history in the optimization literature (see, also [17]), we are unaware of their application to *fairness-constrained* convex allocation models.

Finally, our numerical study – based on a vaccine distribution model for the 2009 H1N1 pandemic – demonstrated that, despite the simplicity of our approach, it yields distinct and interpretable solutions without impacting fairness when parameters are chosen to satisfy the analytical bounds. Such allocations are easy to publicly explain for a policymaker. Thus, our method is suitable as a lightweight, principled

enhancement to existing allocation models when interpretability and distinctness are critical.

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The relevant datasets and code for this work are available at:

<https://github.com/bissil/FairAllocation>.

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