

There is no mni clutter whose core is the projective plane of order 3

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Abstract: In 1998, Lütolf and Margot [18] claimed, in a sidenote, that the projective plane of order 3, also called the ternary plane, or $PG(2, 3)$, cannot be the core of any minimally non-ideal clutter. In this paper, we show that, if C is a minimally non-ideal clutter whose core is the $PG(2, 3)$ then C must be the blocker of the $PG(2, 3)$. Since no projective plane of order $n \geq 3$ is minimally non-ideal, the nonexistence of C is a consequence of the fact that a clutter is minimally non-ideal if and only its blocker is minimally non-ideal.

Keywords: Clutters, Blockers, Ideal and minimally nonideal clutters, Projective planes.

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1. Introduction

The Set Covering problem, which is to find to a vector x of zeros and ones, that maximizes a linear function cx subject to linear constraints $Ax \geq 1$, where A stands for a matrix of zeros and ones and 1 stands for a column-vector of zeros and ones, is a pro-eminent problem in the operations research community. Clutters and their blockers provide a way to understand and characterize the structure of set families, which can lead to more efficient solution procedures for the set covering problem. Long standing questions are: under which conditions do integrality properties hold? Can we devise “efficient” cutting plane algorithms for the set covering problem that would not plague the solution process with additional extreme points?

A clutter is a pair $C \equiv (V, E)$, where E is a family of subsets of a finite and nonempty set V such that no element in E contains another. A clutter C is ideal if the polyhedron $Q(A) \equiv \{x: Ax \geq 1, x \geq 0\}$ is integral, where $A \equiv M(C)$ is a matrix whose rows are the characteristic vectors of the elements of E , i.e., all the extreme points of $Q(A)$ have integer components. The matrix A is usually referred to as the (element-member) incidence matrix of the clutter C . A minor of C is a clutter that stems from the clutter C after one or more operations of *deletion* or *contraction* of an element of V . A clutter is minimally non-ideal, or mni for short, if it is not ideal but all its proper minors are. Lehman [17] proved that an mni clutter C (that is not a specific *Delta* clutter) has a precise structure, it is the extension of a *core* clutter, identified as $\text{core}(C)$. This core clutter is defined by a Lehman matrix. A Lehman matrix is an r -regular matrix Y of zeros and ones such that $YZ^T = dI + 11^T$ holds for some positive integer d and some matrix Z of zeros and ones. Point-line incidence matrices of projective planes are natural candidates.

In 1998, Lütolf and Margot [18] provided a computational study of mni clutters and constructed a partial catalog with clutters up to 20 vertices. In a sidenote, they claimed, maybe as an observation from the computations, that the projective plane of order 3 - also called the ternary plane, or $PG(2, 3)$ -, though not mni, cannot be the core of any mni clutter. There, the incidence matrix of the $PG(2, 3)$ is identified as $L_{13}^4(1)$. In 2019, Mayhew et al [19] provided a noncomputational proof, but we believe there is a flaw in their proof of Lemma 26 and also some fundamental details are missing in the proof of Theorem 22.

In this paper, we will show that if C is an mni clutter such that the core of C is the $PG(2, 3)$ (the only nonisomorphic projective plane of order 3) then C must be the blocker of the $PG(2, 3)$. But, this would imply that the $PG(2, 3)$, being the blocker of the blocker of the $PG(2, 3)$, is also mni, because a clutter is mni

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if and only its blocker is mni, a contradiction, because the only projective plane that is also an mni clutter is the Fano plane, also called $PG(2, 2)$ [21]. Thus, the $PG(2, 3)$ cannot be the core of an mni clutter.

Our main result, Theorem 5.4, stems basically from putting three results together. The first two hold more generally when $\text{core}(C)$ is a projective plane of any order. So, the first is [19, Lemma 23]: every edge of C that is not an edge of $\text{core}(C)$ must be an edge of the blocker of $\text{core}(C)$. The second is [19, Lemma 25]: certain edges, called 0-corners, whose characterization only depends on $\text{core}(C)$ must be edges of C . The third result (our Corollary 5.2 and Proposition 5.3) that use some results from [9] enables us to conclude that the nontrivial edges of the blocker of the $PG(2, 3)$ are precisely the 0-corners. Hence, C must be the blocker of the $PG(2, 3)$.

The paper is structured as follows. In Section 2, we review the definitions of clutter, blocker and minor. In Section 3, we recall the definition of mni clutters and Lehman's characterization of mni clutters, which involves the concept of Lehman matrix and core clutter. Then, in Section 4, we review the definition and some basic properties of projective planes of any order. We also review the connection with mni clutters and recall some results from [19], whose proofs we choose to include using our own words for completeness and for coherency with the notation used (for clarity purposes). In some cases our proofs are shorter. Finally, in Section 5, we look specifically into the $PG(2, 3)$. Building upon the results of [9], we characterize the blocker of $PG(2, 3)$ in terms of projective triangles. We also show that, in the $PG(2, 3)$ context, projective triangles and 0-corners [19] are the same sets. We conclude this section with the main result. In Section 6, we explain why we believe there is a flaw in the proof of [19, Lemma 26] and address some research questions.

2. Clutters

A clutter is a pair $C \equiv (V, E)$, where E is a family of subsets of a finite and nonempty set V such that no element in E contains another [11, 7]. The set V is often referred to as the *ground set* and the set E as the *edge (or member) set* (of the clutter C). When E is the empty set or contains the empty set we say that the clutter is trivial. In combinatorics, a clutter is a *Sperner family* [10]; in graph theory, it is an hypergraph where no edge properly contains another [27, 1]; in other areas of mathematics it is an *independence system* or an *irredundant system*.

The (vertex-edge) incidence matrix, or clutter matrix, of C is the matrix, denoted $M(C)$, whose rows (resp., columns) are indexed by $E(C)$ (resp., $V(C)$) and contain the characteristic vectors of the edges in terms of its elements. From the definition of clutter, the relation " $\geq$ (componentwise)" does not hold for any pair of rows of $M(C)$. Obviously, the clutter matrix of C is not unique because depending on the ordering imposed on vertices and edges we may have a different matrix. If $A \equiv M(C_1)$ and $B \equiv M(C_2)$ we will say that clutters C_1, C_2 are the same if A, B are isomorphic, in the sense that there are permutation matrices P, Q such that $B = PAQ^T$.

A few examples are as follows. If G is a simple graph (i.e., a graph without loops and multiple edges) then $C \equiv (V(G), E(G))$ is a clutter; if $C(G)$ is the collection of all minimal vertex covers of G (also called minimal transversals) then $C \equiv (V(G), C(G))$ is also a clutter; if $P_{s,t}(G)$ is the collection of all edge-sets of (s, t) -paths, where $s, t \in V(G)$ are distinct, then $C \equiv (E(G), P_{s,t}(G))$ is a clutter; if $C_{s,t}(G)$ is the collection of all minimal (s, t) -cuts of G , where $s, t \in V(G)$ are distinct, then $C \equiv (E(G), C_{s,t}(G))$ is a clutter. If, additionally, G is connected graph then, if $T(G)$ is the collection of the edge sets of spanning trees of G then $C \equiv (E(G), T(G))$ is a clutter; if $C(G)$ is the collection of the edge sets of minimal edge cutsets in G then $C \equiv (E(G), C(G))$ is a clutter. A particular example, that plays a specific role, is the *Delta* clutter. For a given integer $t \geq 2$, the clutter $\Delta_t \equiv (V, E)$ is such that $V \equiv \{0, 1, \dots, t\}$ and the edge set composed of the t -element set $\{1, 2, \dots, t\}$ and the two-element sets $\{0, i\}$, for every $i = 1, 2, \dots, t$.

The *blocker* of a clutter $C \equiv (V, E)$, denoted $\text{bl}(C) \equiv (V, F)$, is a clutter with edge set consisting of all minimal sets $B \subseteq V$ such that $|B \cap A| \geq 1$, for every $A \in E$. Of course, the blocker of a trivial clutter is also trivial. It can be shown that $\text{bl}(\text{bl}(C)) = C$ [14, 11]. The elements of F are often called *minimal covers* or *minimal transversals* (of the clutter C). It may be shown that, when G is a simple graph and s, t are two distinct vertices then the blocker of $C \equiv G$ is the clutter of all minimal vertex covers (of G); the blocker of the clutter of all (s, t) -paths of G is the clutter of all minimal (s, t) -cuts of G . If G is a connected graph then the blocker of $C \equiv (E(G), T(G))$, the clutter of all spanning trees of G , is $C \equiv (E(G), C(G))$, the clutter of all minimal edge cutsets in G . Also $\text{bl}(\Delta_t) = \Delta_t$.

If $C \equiv (V, E)$ is a nontrivial clutter with n vertices and $m \geq 1$ edges then the *cover polyhedron* of C is $Q(A) \equiv \{ \mathbf{x} : A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0} \} \subseteq \mathbb{R}^n$, where $A \equiv M(C)$ and $\mathbf{1}$ (resp., $\mathbf{0}$) denotes a column-vector of all-ones (resp., all-zeros). The set $Q(A)$ is a nonempty pointed rational polyhedron, with recession cone $Q(A)_\infty = \mathbb{R}_+^n$. Every extreme point of $Q(A)$ belongs to the hypercube $[0, 1]^n$ and the set of integral extreme

points (i.e., with integer components) of $Q(A)$ coincide with the set of characteristic vectors of edges of the blocker $\text{bl}(C)$. Therefore, the rows of $B \equiv M(\text{bl}(C))$ are the integral extreme points of $Q(A)$. Reciprocally, since $C = \text{bl}(\text{bl}(C))$, the rows of $A \equiv M(C)$ are the integral extreme points of $Q(B)$.

There is a minor relation on clutters that is similar to the minor relation on graphs. If $C \equiv (V, E)$ is a clutter and $j \in V$, then we may delete j to get the clutter $C \setminus j$, the *deletion of C by j* , with vertex set $V \setminus \{j\}$ and edge set consisting of all inclusion-minimal $A \in E$ that do not contain j . One may observe that if $Q(A) \subseteq \mathbb{R}^n$ denotes the cover polyhedron of C then the cover polyhedron of $C \setminus j$ is $\{\mathbf{x}_{-j} : \mathbf{x} \in Q(A), x_j = 1\} \subseteq \mathbb{R}^{n-1}$, where $\mathbf{x}_{-j}$ denotes the subvector of $\mathbf{x}$ without the j -th component. Alternatively, we may *contract j* to get the clutter C/j , the *contraction of C by j* , with vertex set $V \setminus \{j\}$ and edge set consisting of all inclusion-minimal $A \setminus \{j\}$, for all $A \in E$. One may observe that the cover polyhedron of C/j is $\{\mathbf{x}_{-j} : \mathbf{x} \in Q(A), x_j = 0\} \subseteq \mathbb{R}^{n-1}$. Any two of these operations commute, so that we may say C' is a minor of C if C' may be obtained from C by a sequence of deletions and contractions using vertices from some subset $J \subseteq V$. Consequently, if (J_0, J_1) denotes a partition of J , the ground set of $C' \equiv C/J_0 \setminus J_1$ is $V \setminus (J_0 \cup J_1)$ and the edge set is composed by the inclusion minimal sets in $\{A \setminus J_0 : A \in E, A \subseteq V \setminus J_1\}$. We say that the minor is proper when J is nonempty. Furthermore, $\text{bl}(C') \equiv \text{bl}(C) \setminus J_0 / J_1$ [25], [7, page 13].

3. Ideal Clutters

A nontrivial clutter C is *ideal* if its cover polyhedron $Q(A)$ is integral, i.e., every extreme point of $Q(A)$ is integral [13, 21, 22]. By definition, a trivial clutter is ideal. Lehman [16] showed that, a clutter is ideal if and only if its blocker is ideal. Seymour [26] showed that, if a clutter is ideal then all its minors are ideal. The clutters defined by each the following matrices are ideal,

$$\underbrace{\begin{array}{c} 124 \\ 135 \\ 236 \\ 456 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}}_{M(Q_6)}, \quad \underbrace{\begin{array}{c} 13 \\ 124 \\ 235 \\ 346 \\ 45 \\ 156 \\ 26 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M(\text{bl}(Q_6))},$$

where Q_6 stands for the clutter whose vertices are the edges of the K_4 and whose edges are the three-edge sets that define triangles. In fact, the matrix A on the right, not the one on the left, defines a *Totally Dual Integral* linear inequality system " $A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ " [24, pp 310-311], [20, p155]. A clutter C is *minimally nonideal*, or *mni* for short, if C is not ideal but all its proper minors are. The clutters defined by each the following matrices are mni,

$$\underbrace{\begin{array}{c} 124 \\ 235 \\ 346 \\ 457 \\ 561 \\ 672 \\ 713 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M(\mathcal{F}_7)}, \quad \underbrace{\begin{array}{c} 123456 \\ 01 \\ 02 \\ 03 \\ 04 \\ 05 \\ 06 \end{array} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M(\Delta_6)}$$

where, $\mathcal{F}_7$ stands for the Fano plane, the unique projective plane of order 2, and Δ_6 stands for the Delta clutter with $t = 6$. In fact, for the first matrix A , the incidence matrix of the Fano plane, $\hat{\mathbf{x}} \equiv \mathbf{1}/3$ is a fractional extreme point of $Q(A)$. The incidence matrix of any of its minors is totally unimodular. Hence, $C \equiv \mathcal{F}_7$ is mni. For the second matrix A , $\hat{\mathbf{x}} \equiv (5, 1, 1, 1, 1, 1, 1)/6$ is a fractional extreme point of $Q(A)$. The incidence matrix of any of its minors is totally unimodular. Hence, $C \equiv \Delta_6$ is mni. In fact, every Delta clutter Δ_t , $t \geq 2$, is mni.

From Seymour's and Lehman's results cited in the first paragraph of this section, a clutter is mni if and only if its blocker is mni. One may easily prove that, if C is mni then its cover polyhedron $Q(A)$ has a unique

extreme point $\hat{x}$ with some fractional component. In fact, not only all of the components are fractional but also $\hat{x}$ is a nondegenerate basic feasible solution of linear inequality system " $Ax \geq 1, x \geq 0$ " [17]. Therefore, there exists a maximal square matrix $\hat{A}$ of A that is nonsingular and such that $\hat{x}$ is the unique solution of $\hat{A}x = 1$. In this setting, we will call $\hat{A}$ the *core* of A at $\hat{x}$.

Lehman [17] took this analysis further and characterized an mni clutter in a result that we will address from now on as *Lehman's Theorem*, see below. But, first, we recall the definition of Lehman matrix. We say that Y , a $n \times n$ matrix of zeros and ones, is a *Lehman matrix* if, for some positive integer d , there exists some matrix Z , also square and of zeros and ones, such that $YZ^T = dI + 11^T$ [17] - see also [8]. Since $dI + 11^T$ is nonsingular, only nonsingular matrices may be Lehman. Consequently, d defines uniquely the pair (Y, Z) , if it exists. Besides, since $dI + 11^T$ is symmetric, if Y is a Lehman matrix, so is Z . Then, we will say (Y, Z) form a *Lehman pair* (for a given d). One may show that (Y, Z) form a *Lehman pair* for at most one value of d . Bridges and Ryser [4] showed that, if (Y, Z) form a Lehman pair of $n \times n$ matrices, for a given d , then there is a divisor pair (r, s) of $n + d$ such that Y is r -regular (every row and column sum is r) and Z is s -regular. Besides, $YZ^T = Z^TY$.

Lehman's Theorem [17] is as follows. Let $C \neq \Delta_t$ be a clutter with $n \geq 3$ vertices. The clutter C is mni if and only if, for $A = M(C)$ and $B = M(\text{bl } C)$:

1. $Q(A)$ (resp. $Q(B)$) has a unique extreme point $\hat{x}$ (resp. $\hat{y}$) with some fractional component;
2. $0 < \hat{x}$ (resp. $\hat{y}$) < 1 and $\hat{x}$ (resp. $\hat{y}$) is defined by a unique core $\hat{A}$ (resp. $\hat{B}$).
3. the pair $(\hat{A}, \hat{B})$ is Lehman, where $\hat{A}$ is r -regular and $\hat{B}$ is s -regular, for two positive integers r, s such that, for some positive integer d , $rs = n + d$.
4. every row of A (resp. B) that is not a row of $\hat{A}$ (resp. $\hat{B}$) has more than r (resp. s) ones.

Thus, in particular, if C is mni then the rows of $A \equiv M(C)$ can be permuted so that the first $n \equiv V(C)$ define, for some positive integer d , a nonsingular r -regular matrix, for some divisor r of $n + d$, and the remaining rows have at least $r + 1$ ones. When C is mni, we will refer as $\text{core}(C)$, the clutter C' such that $\hat{A} \equiv M(C')$. The fact that $\hat{A}$ is a Lehman matrix will naturally leads us to investigate the link with projective planes.

A corollary of Lehman's Theorem applies to the *set covering* problem, see [17], e.g.. Let A be a $m \times n$ matrix of zeros and ones, with $n \geq 3$. The set covering polyhedron $P \equiv \{x: Ax \geq 1, x \geq 0\}$ is integral if and only if, for every w with components in $\{0, 1, +\infty\}$, the linear optimization problem $\min \{wx: Ax \geq 1, x \geq 0\}$ has, at least, one optimal integral solution, whenever the optimal value is finite.

4. Projective Planes

Every (finite) projective plane defines a clutter [3, 17]. A *projective plane* (finite and nondegenerate) C is an ordered pair (V, E) , where V denotes a finite nonempty set - whose elements can be associated with points on the plane - and E denotes a family of subsets of V - whose elements can be associated with lines on the plane that pass through some of the points in V - satisfying the following three axioms - see, e.g., [12, Section 8.2]:

- (AX1) Any two distinct points are in one and only one common line;
- (AX2) Any two distinct lines contain one and only one common point;
- (AX3) There exist four points, no three of which are in a common line.

The third axiom excludes a few cases that we would call *degenerate* - see [2, Theorem 1, page 7] - that include the *Delta* clutter that we have seen before. The most well-known projective plane is the Fano plane with 7 points and 7 lines. Projective planes are the basis of popular game DOBBLE.

If $C \equiv (V, E)$ is a projective plane then there is an integer $n \geq 2$ such that $|V| = |E| = n^2 + n + 1$, and every line contains exactly $n + 1$ points, and every point belongs to exactly $n + 1$ lines. We will, then, say that C is a projective plane of order n , in which case its (point-line) incidence matrix is $(n^2 + n + 1) \times (n^2 + n + 1)$ and $(n + 1)$ -regular. There is only one (nonisomorphic) projective plane of order 2, the Fano plane. When n is an integer power of a prime, there is a constructive process of a projective plane of order n that uses vectorial spaces of dimension 3 and finite (Galois) fields of order n [2, 3]. These projective planes are identified uniquely by $PG(2, n)$. Thus, the Fano plane is the $PG(2, 2)$. When n is not an integer power of a prime, there is a negative result due to Bruck and Reiser [5]: when $n \pmod{4} \in \{1, 2\}$, existence of a projective plane of order n implies that n is the sum of two squares, each one the square of an integer. Therefore, there is

no projective plane of order $n = 6, 14, 21, 22$, etc. The following scheme identifies the number of (nonisomorphic) projective planes of a given order known so far. The proof of the nonexistence of a projective plane of order 10 is computational[15],

$$\begin{array}{cccccccccccccccccccc} \overset{1}{\textcircled{2}}, \overset{1}{\textcircled{3}}, \overset{1}{\textcircled{4}}, \overset{1}{\textcircled{5}}, \overset{0}{6}, \overset{1}{\textcircled{7}}, \overset{1}{\textcircled{8}}, \overset{4}{\textcircled{9}}, \overset{0}{10}, \textcircled{11}, 12, \textcircled{13}, \overset{0}{14}, 15, \overset{\geq 22}{\textcircled{16}}, \\ \textcircled{17}, 18, \textcircled{19}, 20, \overset{0}{21}, \overset{0}{22}, \textcircled{23}, 24, \overset{\geq 193}{\textcircled{25}}, 26, \overset{\geq 13}{\textcircled{27}}, 28, \textcircled{29}, \overset{0}{30}, \textcircled{31}, \textcircled{32}, \dots \end{array}$$

If A is the incidence matrix of a projective plane of order n then A is a Lehman matrix relative to $d \equiv n$ and (A, A) is a Lehman pair because

$$A^T A = A A^T = nI + \mathbf{1}\mathbf{1}^T, \quad \det(A) = \pm(n+1)n^{n(n+1)/2}.$$

We need the following lemma in order to prove Theorem 4.2 below.

Lemma 4.1. *If $C \equiv (V, E)$ is a projective plane of order $n \geq 3$ then there is no other line contained in the union of three distinct lines.*

Proof. By contradiction, suppose that for some line L , distinct of three other lines L_x, L_y, L_z , we have that $L \subseteq L_x \cup L_y \cup L_z$. Since L has $n+1$ points and $n+1 \geq 4$ then, at least, one of the sets $L \cap L_x$, $L \cap L_y$ or $L \cap L_z$, has more than one point, a contradiction. $\square$

Lemma 4.1 does not hold for the Fano plane, the only projective plane that defines an mni clutter.

Theorem 4.2. [19, 21] *If C is a projective plane of order $n \geq 3$ then C is not mni.*

Proof. Suppose C is mni and let L_x, L_y, L_z be three distinct lines such that $L_x \cap L_y \cap L_z$ is the empty set - we call such set of lines non copunctual - the existence of such a triplet follows from the definition of projective plane, which implies that there is no point that belongs to every line. Let $\{x\} \equiv L_y \cap L_z$, $\{y\} \equiv L_x \cap L_z$ and $\{z\} \equiv L_x \cap L_y$. By Lemma 4.1, the only lines contained in $T \equiv L_x \cup L_y \cup L_z$ are the three lines L_x, L_y, L_z , themselves. Now, let $\mathcal{H}$ the minor of C that results from deleting $V \setminus T$ and contracting $T \setminus \{x, y, z\}$. After deletion, we get a clutter that has only three edges. The subsequent contraction leads us to $\mathcal{H} \equiv \Delta_2$, which is nonideal, a contradiction. Therefore, C is not mni. $\square$

A *blocking set* of a projective plane C is a set of points that intersect every line [6]. By definition, every line intersects every other line in exactly one point. Therefore, we will say that a blocking set is *nontrivial* if it does not contain a line. Hence, a nontrivial blocking set is a set S of points that, for every line L , intersects both L and $V \setminus L$. In other words, S is a nontrivial blocking set if and only if $V \setminus S$ is a nontrivial blocking set. In clutter language, the minimal blocking sets are the edges of the blocker. The projective plane $PG(2, 2)$ has no nontrivial blocking sets [6].

Since, from Theorem 4.2, projective planes C' of order $n \geq 3$ are not mni, the following question is addressed. Is it possible to define an mni clutter C by simply adding a sufficient large number of edges to C' in such a way that $\text{core}(C) = C'$? Theorem 4.3 below states a necessary condition: each one of those additional edges must be a nontrivial blocking set.

Theorem 4.3. [19, Lemma 23] *Let C be a mni clutter such that $C' \equiv \text{core}(C)$ is a projective plane. Then: (i) $C' = \text{core}(\text{bl } C)$; (ii) every edge $R \in E(C) \setminus E(C')$ is a nontrivial blocking set of C' .*

Proof. Assume that $C \equiv (V, E)$ has n vertices. Since C is mni then, from Lehman's Theorem [17], there exists some submatrix $\hat{A} \equiv M(C')$, $n \times n$, of $A \equiv M(C)$, and some submatrix $\hat{B} = M(\text{core}(\text{bl } C))$ of $B \equiv M(C)$, such that $\hat{A}(\hat{B})^T = \mathbf{1}\mathbf{1}^T + dI$, for some positive integer d . Since C' is a projective plane of some order $k \geq 2$ then $\hat{A}\hat{A}^T = \mathbf{1}\mathbf{1}^T + kI$. Since $\hat{A}$ is Lehman relatively to at most one value of d , $\hat{B} = \hat{A}$ and $d = k$. Thus, $\hat{A} = M(\text{core}(\text{bl } C))$ and, therefore, $C' \equiv \text{core}(\text{bl } C)$.

Since $C \equiv \text{bl } \text{bl}(C)$, if R is an edge C but not of C' then R is a minimal cover of $\text{bl}(C)$. In particular, R is a blocking set of $C' \equiv \text{core}(\text{bl } C)$. From the definition of clutter, no edge of C that is not an edge of C' may contain an edge of C' . Therefore, R is a nontrivial blocking set of C' . $\square$

Let $C \equiv (V, E)$ be an mni clutter such that $C' \equiv \text{core}(C)$ is a projective plane of order $n \geq 3$, and let L_x, L_y, L_z be three distinct non copunctual lines C' satisfying (1) - see below -, defining a *triangle* $T \equiv L_x \cup L_y \cup L_z$.

Lemma 4.4. [19, Lemma 24] Let C' be a projective plane of order $n \geq 3$, and let L_x, L_y, L_z be three distinct non copunctual lines such that

$$\{x\} = L_y \cap L_z, \quad \{y\} = L_x \cap L_z, \quad \{z\} = L_x \cap L_y, \quad (1a)$$

$$X = L_x \setminus \{y, z\}, \quad Y = L_y \setminus \{x, z\}, \quad Z = L_z \setminus \{x, y\}. \quad (1b)$$

If C is an mni clutter such that $C' = \text{core}(C)$ and $R \subseteq T \equiv L_x \cup L_y \cup L_z$ an edge of C but not of C' then $|R \cap \{x, y, z\}| \neq 2$ and

$$|R \cap \{x, y, z\}| = 0 \Rightarrow R = X \cup Y \cup Z \equiv T \setminus \{x, y, z\}, \quad (2a)$$

$$|R \cap \{x, y, z\}| = 3 \Rightarrow R = X' \cup Y' \cup Z' \cup \{x, y, z\}, \quad (2b)$$

where X', Y', Z' denote, respectively, proper subsets of X, Y, Z , and

$$R \cap \{x, y, z\} = \{x\} \Rightarrow R = X' \cup Y \cup Z \cup \{x\}, \quad X' \subseteq X, \text{ nonempty}, \quad (2c)$$

$$R \cap \{x, y, z\} = \{y\} \Rightarrow R = X \cup Y' \cup Z \cup \{y\}, \quad Y' \subseteq Y, \text{ nonempty}, \quad (2d)$$

$$R \cap \{x, y, z\} = \{z\} \Rightarrow R = X \cup Y \cup Z' \cup \{z\}, \quad Z' \subseteq Z, \text{ nonempty}. \quad (2e)$$

Proof. By definition of projective plane, the three sets X, Y, Z are disjoint and each has cardinality $n - 1$. The set $R \setminus \{x, y, z\}$ is a subset of $X \cup Y \cup Z$, so that there are subsets $X', Y', Z' \subseteq X, Y, Z$ such that $R \setminus \{x, y, z\} = X' \cup Y' \cup Z'$. If $\{x, y, z\} \subseteq R$ then $R = X' \cup Y' \cup Z' \cup \{x, y, z\}$ so that, in this case, all such subsets must be proper or, otherwise, R would contain some line of C' , a contradiction to the definition of clutter.

Now, suppose that $x \notin R$. By hypothesis, $x \notin L_x$ or, otherwise, $x \in L_x \cap L_y \cap L_z$, a contradiction. Suppose, by contraction, that it exists $x' \in X \setminus R$, and let $L_{x'}$ be the line that contains both x, x' . Since $\{x'\} = L_x \cap L_{x'}$, $\{x\} = L_y \cap L_{x'} = L_z \cap L_{x'}$ then $T \cap L_{x'} = \{x, x'\}$. Since $x, x' \notin R$, it would mean that $R \cap L_{x'} = \emptyset$, a contradiction because, from Theorem 4.3, R is a nontrivial blocking set of C' . Therefore, $X \subseteq R$. In particular, if $x, y, z \notin R$ then $R = X \cup Y \cup Z$.

If $|R \cap \{x, y, z\}| = 2$ then, without loss of generality, $R \cap \{x, y, z\} = \{x, y\}$. Since $z \notin R$ then $Z \subseteq R$. Therefore, $L_z \equiv Z \cup \{x, y\} \subseteq R$, a contradiction because L_z, R are both edges of C . Finally, if $|R \cap \{x, y, z\}| = 1$ then, without loss of generality, $R \cap \{x, y, z\} = \{x\}$. Since $y, z \notin R$ then $R = \{x\} \cup X' \cup Y \cup Z$. If X' were the empty set then $L_x \cap R = X \cap R = \emptyset$, a contradiction because, from Theorem 4.3, R is a nontrivial blocking set of C' . $\square$

From Lemma 4.4, we may classify the edges of C , which are not edges of C' , relatively to a fixed triangle T in the following way [19]. Each one of the edges $R \subseteq T$ of C that is not an edge of C' can be classified as 0-corner, a 1-corner or a 3-corner, depending on R being characterized as in (2a), (2c)-(2e), or (2b), respectively. The classification is unique relatively to a given triangle T . If it exists, the 0-corner is uniquely determined by the triangle T . See Figure 1b.

Lemma 4.5. [19, Lemma 25] Let $C \equiv (V, E)$ be an mni clutter such that $C' = \text{core}(C)$ is a projective plane of order $n \geq 3$. If L_x, L_y, L_z are three distinct non copunctual lines of C' then the respective 0-corner is an edge of C and not an edge of C' .

Proof. Let L_x, L_y, L_z be three distinct non copunctual lines of C' defining (1) and let $T \equiv L_x \cup L_y \cup L_z$. We start by supposing, by contradiction, that, for every edge $R \subseteq T$ of C that is not an edge of C' , R is a 3-corner of T , i.e., $\{x, y, z\} \subseteq R$. Let $\mathcal{H}$ be the minor of C that results from eliminating $V \setminus T$, the vertices of C that are not in T , and contracting $X \cup Y \cup Z \equiv T \setminus \{x, y, z\}$. The set of vertices of $\mathcal{H}$ is $\{x, y, z\}$ and every edge A of $\mathcal{H}$ is such that $A = R \setminus (X \cup Y \cup Z)$, for some edge $R \subseteq T$ of C . (i) If R is an edge of C' then, from Lemma 4.1, R is one of the edges L_x, L_y, L_z . If, for example, $R = L_x \equiv X \cup \{y, z\}$ then $A = R \setminus (X \cup Y \cup Z) = \{y, z\}$. Therefore, the set of edges of $\mathcal{H}$ includes those that are not dominated among $\{x, y\}, \{x, z\}, \{y, z\}$. (ii) If R is not an edge of C' then, from Lemma 4.4, $R = \{x, y, z\} \cup X' \cup Y' \cup Z'$, for some proper subsets X', Y', Z' of X, Y, Z , respectively. The edge $A \equiv R \setminus (X \cup Y \cup Z) \equiv \{x, y, z\}$ is dominated by any of the three previous edges, so that $\mathcal{H} \equiv \Delta_2$, which is not ideal, a contradiction. Therefore, C must contain, at least, a 0-corner or a 1-corner (relative to the triangle T).

Suppose, by contradiction, that C does not contain any 0-corner. Therefore, from Lemma 4.4, for every edge $R \subseteq T$ of C that is not an edge of C' , R is a 3-corner or a 1-corner. By hypothesis, there is, at least, one

1-corner. Let $R_1, R_2, \dots, R_d$, with $d \geq 1$, denote all 1-corners, namely,

$$\begin{aligned} R_i \cap \{x, y, z\} &= \{x\}, & i &= 1, 2, \dots, p, \\ R_i \cap \{x, y, z\} &= \{y\} \text{ ou } \{z\}, & i &= p+1, \dots, d. \end{aligned}$$

Without loss of generality, $p \geq 1$. From Lemma 4.4, $R_i \equiv \{x\} \cup X'_i \cup Y \cup Z$, for $i = 1, 2, \dots, p$, where X'_i denotes a nonempty subset of X . Let $W \subseteq X$ be such that $W \cap X'_i \neq \emptyset$, for every $i = 1, 2, \dots, p$. Among all the possible choices for W (e.g., $W = X$) choose one that is minimal, so that, for each $w \in W$, there is some $i \in \{1, 2, \dots, p\}$ such that $X'_i \cap W = \{w\}$. Note that $x, y, z \notin W$ because $W \subseteq X$. Now, consider the following two distinct possibilities: (i) $p = d$ and (ii) $p < d$.

If (i) holds, $R_i \cap \{x, y, z\} = \{x\}$, $i = 1, 2, \dots, p = d$. Let $\mathcal{H}$ be the minor of C that results from eliminating $V \setminus T$, the vertices of C that are not in T , and contracting $(X \setminus W) \cup Y \cup Z \equiv T \setminus (\{x, y, z\} \cup W)$. Therefore, $\{x, y, z\} \cup W$ is the set of vertices of $\mathcal{H}$ and every edge A of $\mathcal{H}$ is such that $A = R \setminus ((X \setminus W) \cup Y \cup Z)$, for some edge $R \subseteq T$ of C .

(i.1) If R is an edge of C' then, from Lemma 4.1, R is one of the edges L_x, L_y, L_z . If $R = L_x \equiv X \cup \{y, z\}$ then $A = R \setminus ((X \setminus W) \cup Y \cup Z) = \{y, z\} \cup W$. If $R = L_y \equiv Y \cup \{x, z\}$ then $A = R \setminus ((X \setminus W) \cup Y \cup Z) = \{x, z\}$. If $R = L_z \equiv Z \cup \{x, y\}$ then $A = R \setminus ((X \setminus W) \cup Y \cup Z) = \{x, y\}$.

(i.2) If R is an edge of C but not an edge of C' then, from Lemma 4.4, R is a 3-corner or a 1-corner. If R is a 3-corner then, from Lemma 4.4, $R = \{x, y, z\} \cup X' \cup Y' \cup Z'$, for some proper subsets X', Y', Z' of X, Y, Z , respectively, so that $A = \{x, y, z\} \cup W'$, for $W' \equiv X' \cap W \subseteq W$. If R is a 1-corner then, for some i , $R \equiv R_i \equiv \{x\} \cup X'_i \cup Y \cup Z$, for some $X'_i \subseteq X$, nonempty. Thus, $A = \{x\} \cup W'$, for $W' = X'_i \cap W \subseteq W$. By the definition of W , W' is nonempty. The minimality of W implies that, for each $w \in W$, $A = \{x, w\}$ for some R_i .

Summarising, by dominance, the only edges of $\mathcal{H}$ are $\{y, z\} \cup W, \{x, z\}, \{x, y\}$ and $\{x, w\}$, for each $w \in W$. Thus, $\mathcal{H} \equiv \Delta_{|W|+2}$, which is not ideal, a contradiction to the fact that C is mni.

If (ii) holds, $R_i \cap \{x, y, z\} = \{x\}$, $i = 1, 2, \dots, p$ and there is, without loss of generality, some $j > p$ such that $R_j \cap \{x, y, z\} = \{y\}$. Let $\mathcal{H}$ be the minor of C that results from eliminating $V \setminus T \cup \{z\}$, the vertices of C that are not in $T \setminus \{z\}$, and contracting $(X \setminus W) \cup Y \cup Z \equiv T \setminus (\{x, y, z\} \cup W)$. Therefore, $\{x, y\} \cup W$ is the set of vertices of $\mathcal{H}$ and every edge A of $\mathcal{H}$ is such that $A = R \setminus ((X \setminus W) \cup Y \cup Z)$, for some edge $R \subseteq T \setminus \{z\}$ of C .

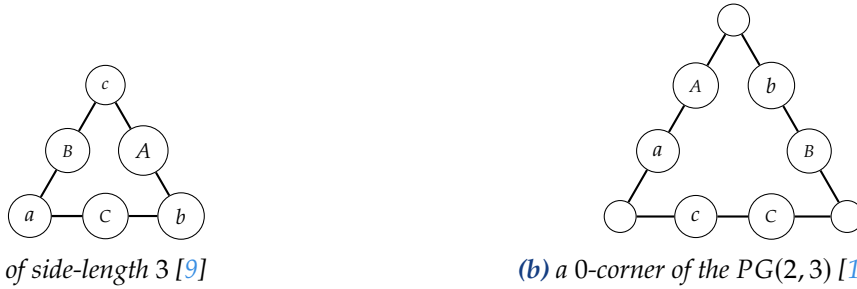
(ii.1) If R is an edge of C' then, from Lemma 4.1, R is one of the edges L_x, L_y, L_z . Since $z \notin R$, $R = L_z \equiv Z \cup \{x, y\}$, so that $A = R \setminus ((X \setminus W) \cup Y \cup Z) = \{x, y\}$ is an edge of $\mathcal{H}$, if it is not dominated.

(ii.2) If R is an edge of C but not an edge of C' then, from Lemma 4.4, R is a 1-corner (R cannot be a 3-corner because $z \notin R$), so that $R \equiv R_i$, for some $i \leq d$. If $i \leq p$ then $R \equiv \{x\} \cup X'_i \cup Y \cup Z$, for some $X'_i \subseteq X$, nonempty, so that $A = \{x\} \cup W'$, for $W' = X'_i \cap W \subseteq W$. From the definition of W , W' is nonempty. If $i > p$ then, from Lemma 4.4, $A = \{y\} \cup W$. The minimality of W implies that, for each $w \in W$, $A = \{x, w\}$, for some R_i , with $i \leq p$.

Summarising, by dominance, the only edges of $\mathcal{H}$ are $\{y\} \cup W, \{x, y\}$ and $\{x, w\}$, for each $w \in W$. Thus, $\mathcal{H} \equiv J_{|W|+1}$, which is not ideal, a new contradiction to the fact that C is mni. $\square$

5. The particular case of $PG(2, 3)$

We will now focus on the $PG(2, 3)$, the projective plane of order 3. The (point-line) incidence matrix of the $PG(2, 3)$ can be seen in any book on Projective Geometry or on WIKIPEDIA, [3, 28, 29]. The matrix below is PAQ^T , where A denotes the $PG(2, 3)$ incidence matrix exhibited in [29], for the permutation matrix P defined by permutation $\sigma \equiv (1, 2, 3, 9, 5, 13, 4, 12, 8, 7, 6, 10, 11)$ and the the permutation matrix Q defined by permutation $\mu \equiv (1, 2, 7, 8, 3, 6, 10, 4, 5, 9, 11, 12, 13)$,



(a) a triangle of side-length 3 [9]

(b) a 0-corner of the $PG(2, 3)$ [19]

Figure 1. Two distinct definitions of a triangle

$$\begin{array}{c}
 \begin{array}{c}
 L_1 \\
 L_2 \\
 L_3 \\
 L_4 \\
 L_5 \\
 L_6 \\
 L_7 \\
 L_8 \\
 L_9 \\
 L_{10} \\
 L_{11} \\
 L_{12} \\
 L_{13}
 \end{array}
 \left[\begin{array}{cccccccccccccc}
 x & a_1 & a_2 & a_3 & y_1 & y_2 & y_3 & b_1 & b_2 & b_3 & y_4 & u & v \\
 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0
 \end{array} \right].
 \end{array} \tag{3}$$

Di Paola had introduced a similar definition of triangle in the projective plane context [9]. A *triangle* T in a projective plane C is the union of three sets of points S_a, S_b, S_c - the *sides* of T -, having the same cardinality - the *side-length* of T -, such that

$$a = S_b \cap S_c, \quad b = S_a \cap S_c, \quad c = S_a \cap S_b,$$

are three distinct points - the *corners* of T -, and for which there are three distinct non copunctual lines L_a, L_b, L_c , that include the sides. *i.e.*, $S_a \subseteq L_a, S_b \subseteq L_b, S_c \subseteq L_c$. Figure 1a illustrates a triangle with side-length 3. The sets $S_a \setminus \{b, c\}, S_b \setminus \{a, c\}$ and $S_c \setminus \{a, b\}$ are the *in-sides* of T . We say that a triangle T is a *projective triangle* if every line L of C that has a nonempty intersection with two of the in-sides (the two intersections must be distinct because C is a projective plane) then L also has a nonempty intersection with the third in-side.

In a projective plane $C \equiv (V, E)$ of order n , any three distinct non copunctual lines define a projective triangle of side-length $n + 1$. In fact, if $S_a \equiv L_a, S_b \equiv L_b, S_c \equiv L_c$ denote three distinct non copunctual lines then they have the same cardinality - $n + 1$ - and, because of the projective plane definition, the three points $a \equiv L_b \cap L_c, b \equiv L_a \cap L_c, c \equiv L_a \cap L_b$ are distinct. Therefore, L_a, L_b, L_c define a triangle. Such a triangle is a projective triangle because if L has a nonempty intersection with $L_a \setminus \{b, c\}$ and $L_b \setminus \{a, c\}$ then none of the corners a, b, c belong to L , so that, from the projective plane definition, $L \cap (L_c \setminus \{a, b\})$ is nonempty.

Di Paola [9] refers a nontrivial blocking set as *blocking coalition*, and a nontrivial blocking set of minimum cardinality as *minimum blocking coalition*. We remark that 'of minimum cardinality' and 'minimal' is not necessarily the same. Di Paola [9], and Richardson [23], proved that the size of a minimum blocking coalition is, respectively, 6, 7, 9, 12, 13, 13 for projective planes of order 3, 4, 5, 7, 8, 9 - a fact that can be verified computationally, we believe. Di Paola also showed several properties of projective triangles. For example, in a projective plane of order n , if T is a projective triangle of side-length $(n + 3)/2$ then T is a blocking coalition [9, Theorem 13]. If $n \in \{3, 5, 7\}$ then there is, at least, one projective triangle of side-length $(n + 3)/2$. Such a projective triangle is also a minimum blocking coalition [9, Theorem 14]. If $n = 3$ (resp. $n = 5$) the reciprocal is also true, *i.e.*, a minimum blocking coalition is also a projective triangle of side-length 3 (resp. 4) [9, Theorems 15, 16].

Our next result establishes that, when $n = 3$, a nontrivial blocking set is minimal if and only if it is of minimum cardinality - a result that, via [9, Theorem 13], can be verified computationally. In fact, the

blocker of the clutter $C \equiv PG(2, 3)$ has 247 edges, the 13 edges of C and 234 minimal nontrivial blocking sets. The number of projective triangles of the $PG(2, 3)$ is also 234.

Proposition 5.1. *Every minimal nontrivial blocking set of the $PG(2, 3)$ has cardinality 6.*

Proof. Let R be a minimal nontrivial blocking set of the $PG(2, 3)$, i.e., R is an edge of the blocker of the $PG(2, 3)$ that is not an edge of the $PG(2, 3)$. If $|R| \geq 8$ then $V \setminus R$ is also a nontrivial blocking set of the $PG(2, 3)$, which is a contradiction because the minimum nontrivial blocking set of the $PG(2, 3)$ has cardinality 6 and $|V \setminus R| \leq 5$. Therefore, R has cardinality 6 or 7. Let x, y, z be three non collinear points of R , i.e., there is no line that contains the three points. Let L_x be the (unique) line that contains y, z and, analogously, define lines L_y, L_z . The sets of vertices in each one of these three lines define a triangle of side-length 4, which is projective because each one these three sets of vertices fills completely the corresponding line. Denote corresponding in-sides by

$$X \equiv \{x_1, x_2\} \subseteq L_x, \quad Y \equiv \{y_1, y_2\} \subseteq L_y, \quad Z \equiv \{z_1, z_2\} \subseteq L_z,$$

- see Figure 2. Let L_x^i be the (unique) line that contains x, x_i , for $i = 1, 2$. Analogously, identify L_y^i, L_z^i , for $i = 1, 2$.

Each one of the two lines L_x^1, L_x^2 intersects each one of the two lines L_z^1, L_z^2 originating four points - see Figure 2. None of these four points belong to any of the lines L_x, L_y, L_z . - in fact, let $\{w\} \equiv L_x^a \cap L_z^b$, where $a, b \in \{1, 2\}$. Then: $w \notin L_x$ because $x_a \in L_x \setminus L_z$; $w \notin L_y$ because $z \in L_x \setminus L_y$; $w \notin L_z$ because $z_b \in L_z \setminus L_x$ - and the four are distinct - in fact, if $\{u\} \equiv L_x^1 \cap L_z^1$ and $\{v\} \equiv L_x^1 \cap L_z^2$ then $u \neq v$, for $x \in L_x^1 \setminus L_z^1, L_x^1 \setminus L_z^2$. Similar arguments hold for any other pair of intersections. Denote the set made of these four points by

$$D \equiv V \setminus (L_x \cup L_y \cup L_z) \equiv \{d_1, d_2, d_3, d_4\}.$$

By symmetry, each one of the two lines L_x^1, L_x^2 intersects each one of the two lines L_y^1, L_y^2 in those same four points, and also each one of the two lines L_y^1, L_y^2 intersects each one of the two lines L_z^1, L_z^2 in those same four points.

Since R is a nontrivial blocking set, R intersects each one of the in-sides in, at most, one point (for if R intersect one of the in-sides in two points then R would contain all the points of the corresponding line, a contradiction). Therefore, if $|R \cap D| = 0$ then $|R| = 6$. If $|R \cap D| \geq 2$, say $d_1, d_2 \in R$, there is $v \in \{x, y, z\}$ such that $d_1 \in L_v^1, d_2 \in L_v^2$. Therefore, the four lines incident in v are still covered by $R \setminus \{v\}$, while the remaining lines of the $PG(2, 3)$, not containing v , remain covered by $R \setminus \{v\}$ too, a contradiction because R is minimal and $R \setminus \{v\}$ is still a nontrivial blocking set. Thus, we conclude that if $|R| > 6$ then $|R \cap D| = 1$. For the remainder of the proof we assume $|R| > 6$, so that R intersects each one of the three in-sides.

Suppose $R \cap D = \{d_1\}$. Let $v \in \{x, y, z\}$, arbitrary. If each one of the four lines that is incident in v contain some point in $R \setminus \{v\}$ then $R \setminus \{v\}$ would remain a nontrivial blocking set of the $PG(2, 3)$. In fact, the four lines incident in v are still covered by $R \setminus \{v\}$, while the remaining lines of the $PG(2, 3)$, not containing v , remain covered by $R \setminus \{v\}$ too, a contradiction because R is minimal and $R \setminus \{v\}$ is still a nontrivial blocking set. Therefore, for each $v \in \{x, y, z\}$, there exists a line containing v that intersects R uniquely at v .

Without loss of generality, suppose those three lines are L_x^1, L_y^1, L_z^1 , and denote by x_2, y_2, z_2 the three points of in-sides that belong to R , i.e., $R = \{x, y, z, x_2, y_2, z_2, d_1\}$. Since $d_1 \notin L_x^1 \cup L_y^1 \cup L_z^1$ then $d_1 \in L_x^2 \cap L_y^2 \cap L_z^2$, so that, without loss of generality,

$$L_x^2 \equiv \{x, x_2, d_1, d_2\}, \quad L_y^2 \equiv \{y, y_2, d_1, d_3\}, \quad L_z^2 \equiv \{z, z_2, d_1, d_4\},$$

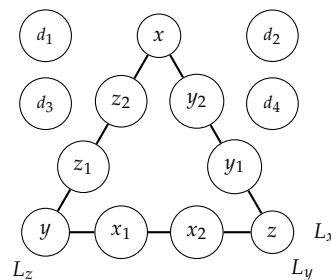


Figure 2. The 13 points of $PG(2, 3)$ and the projective triangle

o que implica que

$$L_x^1 \equiv \{x, x_1, d_3, d_4\}, \quad L_y^1 \equiv \{y, y_1, d_2, d_4\}, \quad L_z^1 \equiv \{z, z_1, d_2, d_3\}.$$

Let L_{xz} be the (unique) line that contains x_1, z_1 . This line does not contain the points x, y, z (because $x, z_1 \in L_z, y, x_1 \in L_x, z, x_1 \in L_x$), it cannot contain either x_2, z_2 (because $x_2, x_1 \in L_x, z_2, z_1 \in L_z$), it cannot contain either d_2, d_3, d_4 (because $d_2, z_1 \in L_z^1, d_3, x_1 \in L_x^1, d_4, x_1 \in L_x^1$). Therefore, $y_1 \in L_{xz}$ (because $L_y^1 = \{y, y_1, d_2, d_4\}$) and $d_1 \in L_{xz}$ (because $L_y^2 = \{y, y_2, d_1, d_3\}$). Therefore, $L_{xz} = \{x_1, y_1, z_1, d_1\}$. Then, the remaining three lines of the $PG(2, 3)$ are $\{x_2, y_2, z_1, d_4\}$, $\{x_1, y_2, z_2, d_2\}$ and $\{x_2, y_1, z_2, d_3\}$. Therefore, for example, $R \setminus \{x_2\}$ remains a nontrivial blocking set, a contradiction because R is minimal. $\square$

Let T_0 be a minimal nontrivial blocking set of the $PG(2, 3)$. From Proposition 5.1, T_0 is a minimum blocking coalition. Then, from [9, Teorema 15], T_0 is a projective triangle of side-length 3. Reciprocally, let T_0 be a projective triangle of side-length 3. From [9, Teorema 14], T_0 is a minimum blocking coalition, so that T_0 is also a minimal nontrivial blocking set of the $PG(2, 3)$. Thus, we have proved the following.

Corollary 5.2. *The set of edges of the blocker of the $PG(2, 3)$ is the set of edges of the $PG(2, 3)$ gathered with the projective triangles of side-length 3.*

Our next result shows that the concepts of 0-corner and projective triangle coincide.

Proposition 5.3. *In the $PG(2, 3)$, T_0 is a projective triangle of side-length 3 if and only if T_0 is a 0-corner.*

Proof. Let $T_0 \equiv \{a, b, c, A, B, C\}$ denote a projective triangle of side-length 3 - see Figure 1a. Then, T_0 is the union of three sets of points S_a, S_b, S_c , all of cardinality 3, for which there are three distinct and non copunctual lines L_a, L_b, L_c , that satisfy

$$S_a \equiv \{b, A, c\} \subseteq L_a, \quad S_b \equiv \{a, B, c\} \subseteq L_b, \quad S_c \equiv \{a, C, b\} \subseteq L_c.$$

Let L_x be the the (unique line of the $PG(2, 3)$ that contains the points a, A ; let L_y be the the (unique line that contains the points b, B ; let L_z be the the (unique line that contains the points c, C . By construction, these are three distinct lines, any of which is distinct from any of the lines L_a, L_b, L_c .

Let $x \equiv L_y \cap L_z$; let $y \equiv L_x \cap L_z$; and let $z \equiv L_x \cap L_y$. Next, we prove that $x \notin T_0$. If $x = a$ then $a, b \in L_y$, a contradiction because both $a, b \in L_c$; If $x = b$ then $a, b \in L_x$, a contradiction because both $a, b \in L_c$; If $x = c$ then $a, c \in L_x$, a contradiction because both $a, c \in L_b$; If $x = A$ then $b, A \in L_y$, a contradiction because both $b, A \in L_a$; If $x = B$ then $a, B \in L_x$, a contradiction because both $a, B \in L_b$; If $x = C$ then $a, C \in L_x$, a contradiction because both $a, C \in L_c$. Analogously, we conclude that $y, z \notin T_0$.

Suppose, by contradiction, that $x = y = z$. Then, there are three points a', b', c' , all distinct of a, b, c, A, B, C , such that

$$L_x \equiv \{a, A, x, a'\}, \quad L_y \equiv \{b, B, x, b'\}, \quad L_z \equiv \{c, C, x, c'\}.$$

Let $L \equiv \{d, D, x, d'\}$ the fourth line of the $PG(2, 3)$ that contains x , where d, D, d' are three points distinct of $x, a, b, c, A, B, C, a', b', c'$ (therefore, at this point, all 13 points of the $PG(2, 3)$ are identified). If L_1 is the unique line that passes through a, d then: (i) $x, D, d' \notin L_1$ because $L_1 \cap L = \{d\}$; (ii) $A, a' \notin L_1$ because $L_1 \cap L_x = \{a\}$; (iii) $B, c \notin L_1$ because $L_1 \cap L_b = \{a\}$; (iv) $b, C \notin L_1$ because $L_1 \cap L_c = \{a\}$. Thus, $L_1 \equiv \{a, d, b', c'\}$. If L_2 is the unique line that passes through a, D then: (i) $x, d, d' \notin L_2$ because $L_2 \cap L = \{D\}$; (ii) $A, a' \notin L_2$ because $L_2 \cap L_x = \{a\}$; (iii) $b', c' \notin L_2$ because $L_2 \cap L_1 = \{a\}$; (iv) $B, c \notin L_2$ because $L_2 \cap L_b = \{a\}$; (v) $b, C \notin L_2$ because $L_2 \cap L_c = \{a\}$. We have reached a contradiction, because, then, it is not possible to define L_2 . We conclude that L_x, L_y, L_z are not copunctual. More, x, y, z are all distinct for if $x = L_y \cap L_z = L_x \cap L_z = y$ then $x \in L_x \cap L_y \cap L_z$ which would imply $x = y = z$, a contradiction. Therefore, L_x, L_y, L_z are three distinct, non copunctual lines, so that $L_x \cup L_y \cup L_z \setminus \{x, y, z\} = \{a, A, b, B, c, C\} \equiv T_0$ is a 0-corner.

Reciprocally, let T_0 be a 0-corner. Therefore, there are three distinct, non copunctual lines L_x, L_y, L_z such that $T_0 = (L_x \cup L_y \cup L_z) \setminus \{x, y, z\}$, where $x \equiv L_y \cap L_z, y \equiv L_x \cap L_z$ and $z \equiv L_x \cap L_y$. Without loss of generality,

$$L_x \equiv \{y, z, a, A\}, \quad L_y \equiv \{x, z, b, B\}, \quad L_z \equiv \{x, y, c, C\}.$$

Let L_a be the (unique) line of the $PG(2, 3)$ that contains b, c . Therefore, L_a is distinct from L_x, L_y, L_z . Besides, since: (i) $x, B, z \notin L_a$ because $b = L_y \cap L_a$; (ii) $y, C, x \notin L_a$ because $c = L_z \cap L_a$, we conclude that $L_x \cap L_a$ is a ou A . Without loss of generality, $L_x \cap L_a = A$. Therefore, $L_a \equiv \{b, c, A, d\}$, for some d distinct of $a, b, c, A, B, C, x, y, z$.

Let L_b be the (unique) line that contains a, c . Therefore, L_b is distinct from L_x, L_y, L_z, L_a . Besides, since: (i) $y, A, z \notin L_b$ because $a = L_x \cap L_b$; (ii) $y, C, x \notin L_b$ because $c = L_z \cap L_b$; (iii) $b, A, d \notin L_b$ because $c = L_a \cap L_b$; we conclude that $L_y \cap L_b = B$. Therefore, $L_b \equiv \{a, c, B, e\}$, for some e distinct from $a, b, c, d, A, B, C, x, y, z$.

Let L_c be the (unique) line that contains a, b . Therefore, L_c is distinct from L_x, L_y, L_z, L_a, L_b . Besides, since: (i) $y, A, z \notin L_c$ because $a = L_x \cap L_c$; (ii) $x, B, z \notin L_c$ because $b = L_y \cap L_c$; (iii) $c, A, d \notin L_c$ because $b = L_a \cap L_c$; (iv) $c, B, e \notin L_c$ because $a = L_b \cap L_c$; we conclude that $L_z \cap L_c = C$. Therefore, $L_c \equiv \{a, b, C, f\}$, for some f distinct from $a, b, c, d, e, A, B, C, x, y, z$.

By construction, L_a, L_b, L_c are three distinct, non copunctual lines with

$$S_a \equiv \{b, A, c\} \subseteq L_a, \quad S_b \equiv \{a, B, c\} \subseteq L_b, \quad S_c \equiv \{a, C, b\} \subseteq L_c,$$

where a, b, c, A, B, C all distinct. Therefore, $S_a \cup S_b \cup S_c \equiv \{a, b, c, A, B, C\} = T_0$ is a triangle of side-length 3. It remains to be proved that this triangle is a projective triangle. Let L be any line such that $A, B \in L$. Since: (i) $a, y, z \notin L$ because $A = L_x \cap L$; (ii) $x, b, z \notin L$ because $B = L_y \cap L$; (iii) $c, b, d \notin L$ because $A = L_a \cap L$; (iv) $a, c, e \notin L$ because $B = L_b \cap L$; we conclude that $L_z \cap L = C$. Analogously, we conclude that, if $A, C \in L$ then $L_y \cap L = B$, and if $B, C \in L$ then $L_x \cap L = A$. Therefore, T_0 is a projective triangle. $\square$

This development proves our main theorem.

Theorem 5.4. *There is no mni clutter C such that $\text{core}(C)$ is a projective plane of order 3.*

Proof. Let C be an mni clutter such that $\text{core}(C) = PG(2, 3)$. From Theorem 4.3, every edge of C that is not an edge of the $PG(2, 3)$ must be an edge of the blocker of the $PG(2, 3)$. Hence, from Corollary 5.2, it must be a projective triangle of side-length 3 of the $PG(2, 3)$. Or, from Proposition 5.3, it must be a 0-corner of the $PG(2, 3)$. On the other hand, from Lemma 4.5, every 0-corner of the $PG(2, 3)$ is an edge of C . Hence, C must be the blocker of the $PG(2, 3)$, a contradiction to the fact that, from Theorem 4.2, the $PG(2, 3)$ is not mni. $\square$

Take for example, the three non copunctual lines L_1, L_2, L_4 of the $PG(2, 3)$ identifiable in (3). For $T \equiv L_1 \cup L_2 \cup L_4 \equiv \{x, a_1, a_2, y_1, y_2, y_3, b_1, b_2, y_4\}$, the minor $\mathcal{H}$ of the $PG(2, 3)$ that results from deleting all points in $I \equiv V \setminus T \equiv \{a_3, b_3, u, v\}$, the vertices that are not in T , and contracting all points in $J \equiv T \setminus \{x, y_1, y_2\} \equiv \{a_1, b_1, a_2, b_2, y_3, y_4\}$, the vertices in the 0-corner of T , is defined by the following incidence matrix

$$\begin{array}{c} \{y_1, y_2\} \\ \{x, y_1\} \\ \{x, y_2\} \end{array} \begin{array}{c} x \quad y_1 \quad y_2 \\ \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right] \end{array}.$$

Thus, for $C \equiv PG(2, 3)$, we have $\mathcal{H} \equiv C \setminus I/J \equiv \Delta_2$, which is not ideal. Moreover, the fact that

$$\Delta_2 = \text{bl}(\Delta_2) \equiv \text{bl}(C \setminus I/J) = \text{bl}(C)/I \setminus J$$

gives a hint of where to find a verification, in terms of extreme points of the cover polyhedron, that $\text{bl}(C)$ is not mni.

6. Further comments and future research

In the proof of Lemma 26 in [19], one may use the first four lines of the matrix in (3) to define the lines L_1, L_2, L_3, L_4 so that $X \equiv \{x, a_1, a_2, a_3, y_1, y_2, y_3, b_1, b_2, b_3, y_4\}$ - we are using the same notation to identify the points. The clutter $\mathcal{H}$ obtained from $C \equiv (V, E)$ - the clutter whose set of edges is defined by the lines of the $PG(2, 3)$ and the 0-corners - by deleting all points in $V \setminus X \equiv \{u, v\}$ and contracting all points in $X \setminus \{x, a_1, a_2, a_3, y_1, y_2, y_3\} \equiv \{b_1, b_2, b_3, y_4\}$ is defined by the following incidence matrix

$$\begin{array}{c} \{x, a_1, y_1\} \\ \{a_1, a_3\} \\ \{a_2\} \\ \{a_1, y_1, y_2\} \\ \{y_1, y_3\} \end{array} \begin{array}{c} x \quad a_1 \quad a_2 \quad a_3 \quad y_1 \quad y_2 \quad y_3 \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array} \right] \end{array}$$

which does not characterize the Fano plane - contrary to what is stated in the proof of Lemma 26 - and, in fact, such $\mathcal{H}$ is ideal. Furthermore, in the proof of Theorem 22 in [19], it is claimed that the only nontrivial blocking sets of the $PG(2, 3)$ are the 0-corners, addressing the proof of this fact to [9]. But, while this is certainly true, Di Paola's result applies to minimum (cardinality) nontrivial blocking sets of the $PG(2, 3)$ (not to minimal, though our Proposition 5.1 establishes that they are the same). Furthermore, Di Paola uses a different definition of triangles and, it was not clear how the two definitions of triangles are related. So, we felt that this was a point that needed to be clarified (through our Proposition 5.3).

As future research, we would like to address the general question of whether there is some mini clutter whose core is a projective plane of order $n \geq 4$.

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