

Open problems on Steiner trees and maximal distance minimizers

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Abstract: In this work, I collect and discuss a series of open questions in one-dimensional geometric optimization in Euclidean spaces. The focus is on two classes of problems: maximal distance minimizers and Steiner trees. Maximal distance minimizers concern finding a connected set of minimal length whose closed r -neighborhood covers a given compact set, whereas Steiner trees aim to find a minimum-length set connecting a prescribed set of points. For both problems, I briefly summarize known results and highlight the remaining open questions. While some questions can be approached with elementary methods, others remain highly challenging.

Keywords: Steiner trees, maximal distance minimizers.

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1. Preliminary remarks

The aim of this paper is to collect several open questions concerning two problems in one-dimensional geometric optimization: maximal distance minimizers and Steiner trees. These problems have different formulations, but they have an important common feature: in both cases one looks for connected one-dimensional sets of minimal length. The Steiner problem has a long history and has been studied extensively from geometric, variational, and algorithmic points of view; see, for example, [16, 24, 27, 15] and references therein. The problem of maximal distance minimizers appeared relatively recently and remains much less studied. One of the motivations for the present paper is the feeling that questions of this kind receive less attention than they deserve. The novelty of the paper lies in bringing together open problems from these two related areas, comparing a classical and well-developed theory with a less familiar one. Some of the questions below are classical or well known to specialists, while others come from recent work or informal discussions. I also hope that the paper may be useful not only for specialists, but also for researchers who are new to the field: it contains genuine open problems which can be understood, and in some cases approached, without a large amount of preliminary theory.

2. Maximal distance minimizers

For a given compact set $M \subset \mathbb{R}^d$ and a given positive number $r > 0$, the aim is to find a connected compact set Σ of minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) such that $M \subset \overline{B_r(\Sigma)}$, where $\overline{B_r(\Sigma)}$ denotes the closed r -neighborhood of the sought set Σ . Such a solution Σ is called a *maximal distance minimizer* (MDM or r -minimizer).

A solution always exists and has no loops (homeomorphic image of a circle), see [26].

For detailed information on MDMs, see the review [9].

2.1. Explicit examples

Question 2.1. What is a maximal distance minimizer for a circle of radius R ($M = \partial B_R$) with $R > r$? This is open for $4.98r > R > r$.

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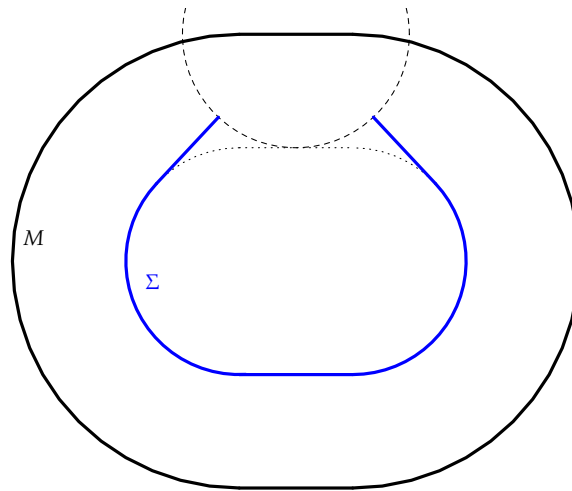


Figure 1. A horseshoe for a closed curve with a large radius of curvature

Miranda, Paolini, and Stepanov conjectured in [23] that any minimizer for a circle of radius $R > r$ is a horseshoe, i.e., a union of an arc of a circle of radius $R - r$ and two segments of its tangent lines.

The conjecture is confirmed in [7] for $R > 4.98r$. For $4.98r > R > r$, the conjecture remains open.

Question 2.2. What is the set of maximal distance minimizers when the input set M is an R -stadium (which is the boundary of the R -neighborhood of a segment)? This is open for $r < R < 5r$.

In [7], it is proved that for a closed curve with minimal radius of curvature $R > 5r$, any MDM is a horseshoe, i.e., a union of an arc of a curve parallel to an arc of M (and hence having a radius of curvature at least $R - r$) and two segments of its tangent lines (see Figure 1). Note that if a curve has a strictly positive radius of curvature, then it is at least $C^{1,1}$ -smooth. For such curves, the parallel (offset) curve is well-defined by the map $s \mapsto \gamma(s) - r\vec{n}(s)$, where $\vec{n}(s)$ is the unit normal vector to a given curve $M = \gamma(s)$. Thus, for $R > 5r$, every solution for an R -stadium is a horseshoe. It is also known (see [9] and Figure 2) that for $R < 1.75r$ the solution for an R -stadium is not a horseshoe.

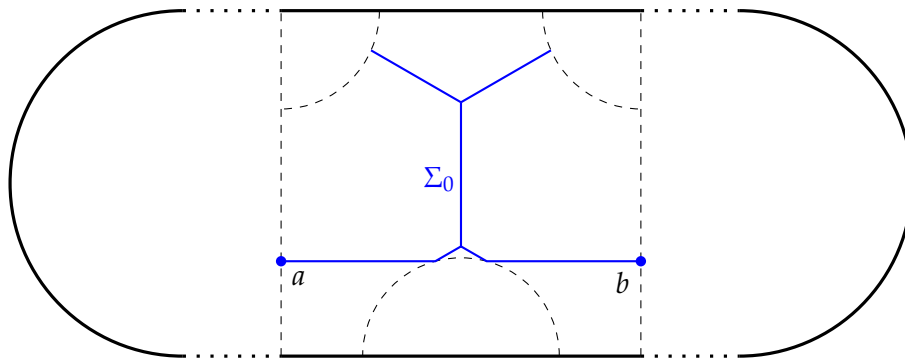


Figure 2. A competitor shorter than the horseshoe for a stadium

Thus, the following subquestions remain open:

- Determine the supremum of C such that there exists an MDM for a Cr -stadium that is not a horseshoe.
- Determine the infimum of c such that, for any closed curve with minimal radius of curvature cr , every MDM is a horseshoe.

It might also be helpful to know that any minimizer for an R -stadium consists of line segments and of curves parallel to the curves contained in an R -stadium (these curves coincide with arcs of circles of radius $R - r$).

Question 2.3. Characterize the set of MDMs for explicitly given sets.

For example, one may consider:

- A disc or ball $\overline{B_R}$ in $\mathbb{R}^d$, $d \geq 2$. This is a very challenging question; apart from degenerate cases, only the usual regularity properties, such as the absence of loops, are known.
- A rectangle (the boundary). For sufficiently small $r > 0$, the question is answered, see [4].
- (Regular) n -gons.
 - Partial conjecture. For $n \geq 6$, an MDM for a regular n -gon is expected to be a polyline.
 - Some remarks. Although this question appears to be less difficult than determining minimizers for more general sets, a complete and rigorous proof is still lacking. The known regularity properties imply that every solution is a finite union of line segments meeting at angles of at least $2\pi/3$. The problem is already nontrivial in the case $n = 3$. The methods developed for the rectangle case may prove useful; see [4].

2.2. Regularity and uniqueness

Question 2.4. *Is any MDM in $\mathbb{R}^d$ a union of a finite number of injective curves? Open for $d \geq 3$.*

In $\mathbb{R}^2$, the answer is positive, see [17]. In higher dimensions some information about local behavior is proved in the same paper: there are at most three tangent rays at any point of a minimizer and the angles between the tangent rays are at least $2\pi/3$. Recall that a solution always exists and never contains loops, see [26].

Question 2.5. *Let Σ be an r -minimizer for some M . Is Σ the unique r -minimizer for $\overline{B_r(\Sigma)}$?*

It is clear that if Σ is an r -minimizer for some M , then it is an r -minimizer for $\overline{B_r(\Sigma)}$; see [2]. A weaker form of this question is obtained by replacing r in the hypothesis with some positive $r_0 < r$.

Some points of an MDM are more important than others: a point $x \in \Sigma$ is called *energetic* if there exists a point $y \in M$ such that $|xy| = r$ and $B_r(y) \cap \Sigma = \emptyset$. The set of energetic points of Σ is denoted by G_Σ .

Question 2.6. *Fix any $M \subset \mathbb{R}^d$ and $r > 0$. Assume that M is not contained in any closed ball of radius r . Does the set G_Σ of energetic points determine an MDM Σ ? In other words, let Σ_1 and Σ_2 be r -minimizers for M . Does*

$$G_{\Sigma_1} = G_{\Sigma_2}$$

imply that $\Sigma_1 = \Sigma_2$?

The relationship between Questions 2.5 and 2.6 is itself a nice question.

2.3. An algebraic point of view

Reflections from a spherical mirror play an important role in the geometry of MDMs. In particular, the MDM for a 3-point set M often coincides with a solution to Alhazen's billiard problem, which reduces to solving a certain fourth-degree equation and thus cannot be solved with a straightedge (ruler) and a compass, see [14]. The following question is already open for $n = 4$.

Question 2.7. *Let $M \subset \mathbb{R}^2$ be a finite set of cardinality n . How does the explicit description of a minimizer Σ depend on M in terms of the degrees of the resulting equations?*

For simplicity, one may assume that the points of M have rational coordinates. Then it is enough to find the largest degree of the extension of $\mathbb{Q}$ generated by the set of energetic points G_Σ .

3. Steiner trees

For a given compact set $\mathcal{A} \subset \mathbb{R}^d$, the aim is to find a closed set St with minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) such that $St \cup \mathcal{A}$ is connected. Such a solution St is called a *Steiner set*. If St is connected (which is the case when $\mathcal{A}$ is finite or totally disconnected), it is called a *Steiner tree*. The problem of finding such a set is called the Steiner problem. Traditionally, the Steiner problem is formulated for a finite set of terminals in the Euclidean plane. Here the more general formulation is used. In 2013, Paolini and Stepanov showed the existence of a solution, see [27]. Note that the length of the solution may be infinite if no rectifiable set connects $\mathcal{A}$. If a solution St has finite length, then it is compact, has no loops, and consists of countably many connected components. The edges of the locally finite graph $St \setminus \mathcal{A}$ are straight line segments. The maximal degree (in the graph-theoretic sense) of a vertex is at most 3. Moreover,

only terminals can have degree 1 or 2; all the other vertices have degree 3 and are called *Steiner* (branching) points. The angle between any two adjacent edges of $\mathcal{S}t$ is at least $2\pi/3$; in particular, in a neighborhood of a branching point, $\mathcal{S}t$ is a regular tripod (all three angles are equal to $2\pi/3$). If every terminal has degree 1, then $\mathcal{S}t$ is called *full*.

3.1. History of the problem

A detailed historical study [3] showed that the statement and basic results about the Steiner problem were rediscovered (at least) three times: it was first stated by Gergonne in 1811, by Gauss in 1836, and by Jarník and Kössler [21] in 1934. The problem has become well-known as the “Steiner problem” after the great success of the book “What is Mathematics?” by Courant and Robbins [11].

A new wave of interest arose in the 1960s in connection with research at the Bell Telephone Company. Melzak [22] provided an explicit algorithm for finding a Steiner tree for any finite set of points in the plane. Then Gilbert and Pollak [16] published their examination of the properties of minimum Steiner trees. Garey, Graham, and Johnson [15] proved that the Steiner problem is NP-hard. The GeoSteiner software [34] for solving the planar Steiner problem was released in 1985 and has since been improved; it is still maintained by Winter, Zachariasen and Warme.

The present paper does not address the open questions concerning the approximation of solutions to the Steiner problem and its computational complexity from an algorithmic point of view.

Interested readers are referred to the recent paper by Fleischmann, Gavva, and Karthik [12] for the introduction to the contemporary open questions and the references therein.

3.2. Uniqueness

Question 3.1. *Does there exist a finite set of terminals in $\mathbb{R}^d$, $d \geq 3$, with two different Steiner trees $\mathcal{S}t_1$ and $\mathcal{S}t_2$ such that the directions of all segments incident to terminals are the same (in other words, the trees coincide in a neighborhood of the terminals)?*

In the plane, K. Oblakov proved that it is impossible [25].

Question 3.2. *Is a solution to the Steiner tree problem unique for a general input?*

A positive answer for a finite planar input was obtained in [1]. Moreover, it was proved there that in $\mathbb{R}^2$ the set of n -point ambiguous configurations has the Hausdorff dimension at most $2n - 1$ (where a configuration is called *ambiguous* if there are several Steiner trees for it). In particular, this means that the $2n$ -dimensional Hausdorff measure of the set of all ambiguous n -point configurations is 0, i.e., a general planar n -input has a unique Steiner tree.

3.3. Gilbert–Pollak conjecture on the Steiner ratio

A *minimum spanning tree* (MST) is a minimal connection of a given set A of points by segments with endpoints in A . The *Steiner ratio* is the infimum of the ratio of the length of a Steiner tree to the length of a minimum spanning tree for a finite set of points.

Question 3.3. *What is the Steiner ratio in $\mathbb{R}^d$, $d \geq 2$?*

The most well-known open problem in this area is the determination of the Steiner ratio, namely the proof or disproof of the Gilbert–Pollak conjecture, which states that the planar Steiner ratio is attained for an equilateral triangle (and thus equals $\frac{\sqrt{3}}{2} = 0.866\dots$). As of 2026, this conjecture remains open; see [20]. Graham [18] offered \$1000 for a proof (his paper also contains an overview of the problem and previous attempts to solve it). The classical best published lower bound is approximately 0.824, obtained by Chung and Graham in 1985 [10]. A recent preprint reports a computer-assisted improvement to 0.8559 [33].

As Graham and Hwang showed in 1976, in any Euclidean space the Steiner ratio is at least $1/\sqrt{3}$, see [19]. It is also known that for any metric space the Steiner ratio is at least $1/2$, see [31] (Steiner problem can be naturally posed there, see [27]). The ratio for the vertices of a regular simplex, including the tetrahedron, is better than the planar value, but it is not optimal. Smith and Smith constructed in [32] so-called d -sausage configurations which provide the best known and conjecturally optimal value of the Steiner ratio in $\mathbb{R}^d$, $d \geq 2$. For $d = 3$ this value is approximately 0.784.

3.4. Density-type questions

The following questions, suggested by Guy David in a personal communication, concern the local complexity of Steiner trees in terminal-free regions.

Let $St \subset \mathbb{R}^d$, $d \geq 2$, be a Steiner tree with terminal set $\mathcal{A}$. Throughout this subsection, assume that, for some $x \in \mathbb{R}^d$ and $r > 0$,

$$\mathcal{A} \cap B_r(x) = \emptyset, \quad \mathcal{H}^1(St) < \infty.$$

One may additionally require that $St \cap \overline{B_r(x)}$ is connected or that St is full.

The planar case with $\mathcal{A} \subset \partial B_r(x)$ is a natural special case. Indeed, if $St \cap \overline{B_r(x)}$ is connected, then it is a Steiner tree for

$$\mathcal{A}_r := St \cap \partial B_r(x);$$

otherwise, a shorter tree connecting $\mathcal{A}_r$ could replace the part of St inside the ball. Moreover, if $\mathcal{A} \subset \partial B_r(x)$, then the finite-length assumption is automatic, since $\partial B_r(x)$ is a competitor and

$$\mathcal{H}^1(St) \leq 2\pi r.$$

Question 3.4. For fixed $d \geq 2$ and $t \in (0, 1)$, determine

$$N_d(t) := \sup \#(\partial B_{tr}(x) \cap St),$$

where the supremum is taken over all configurations satisfying the assumptions above. In particular, what is $N_2(1/2)$?

For $t \in (0, 1)$, the ball $B_{tr}(x)$ is separated from the complement of $B_r(x)$ by a terminal-free region of width $(1 - t)r$. Thus, $t = 1/2$ is only a representative value, while the behavior as $t \rightarrow 1^-$ is essential.

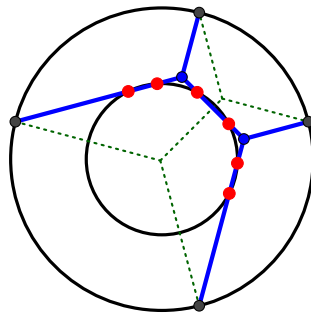


Figure 3. A planar Steiner tree with six intersection points on $\partial B_{r/2}(x)$.

The construction shown in Figure 3, due to M. Khristoforov (personal communication), shows that

$$N_2(1/2) \geq 6.$$

It remains open whether equality holds.

Question 3.5. For fixed $d \geq 2$ and $t \in (0, 1)$, determine

$$M_d(t) := \sup \#\{y \in St \cap B_{tr}(x) : y \text{ is a branching point of } St\}.$$

In particular, what is $M_d(1/2)$?

In joint work with D. Cherkashin and P. Prozorov [6], a planar full Steiner tree for $\mathcal{A}$ with four accumulation points on $\partial B_r(x)$ was constructed. Consequently,

$$M_2(t), N_2(t) \longrightarrow \infty \quad \text{as } t \rightarrow 1^-.$$

Question 3.6. For $d > 2$, assuming that St is full, determine

$$L_d(t) := \sup \frac{\mathcal{H}^1(St \cap B_{tr}(x))}{r}, \quad t \in (0, 1).$$

In [6], the following estimate was proved:

$$L_d(t) \leq \frac{(144d)^{d-2}}{(1-t)^{d-2}}.$$

3.5. Connection with maximal distance minimizers

A natural way to investigate the connection is to consider the case of a finite set M in the maximal distance minimizers problem. A minimizer for a finite set $M \subset \mathbb{R}^d$ is a finite union of at most $2\sharp M - 3$ segments. In this case the maximal distance minimizers problem comes down to connecting r -neighborhoods of all the points from M .

A topology T of a labeled Steiner tree St is the corresponding abstract graph with labeled terminals and unlabeled Steiner points. It was shown in [2] that if a Steiner tree St for n terminals in $\mathbb{R}^d$ preserves its topology for any sufficiently small perturbation of the terminals, then St is an r -minimizer for some n -tuple M and sufficiently small r . Recall that in the plane a Steiner tree for a random input is unique with probability one. Also in the plane a somewhat opposite statement holds: if St is a full planar Steiner tree for n terminals and is not unique, then St cannot be a minimizer for any M being an n -tuple of points.

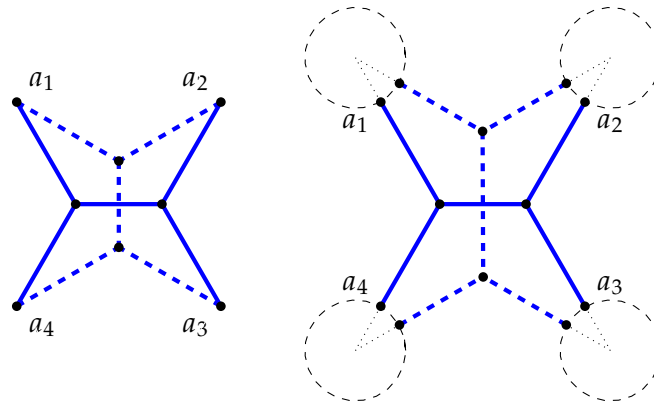


Figure 4. A non-unique Steiner tree that is not a maximal distance minimizer

To illustrate this statement, consider a square $a_1a_2a_3a_4$. There are two Steiner trees for a_1, a_2, a_3, a_4 (see the left-hand side of Figure 4); let us choose the solid one. If the solid tree were a maximal distance minimizer for some $r > 0$, then it would be a maximal distance minimizer for the four points obtained by extending the solid segments incident to a_1, a_2, a_3, a_4 by distance r . However, as can be seen on the right-hand side of Figure 4, for these four points the dotted tree is shorter (of the two relevant topologies, the one with the longer central segment has smaller total length). Thus, the solid Steiner tree in the left-hand picture cannot be a maximal distance minimizer.

This is not a coincidence: if two different Steiner trees for the same finite input have non-collinear segments incident to the same terminal (which is always true in the plane by Oblakov's theorem, see [25]), then, due to the strictness of the triangle inequality, the topology of the second Steiner tree yields a shorter competitor for the problem derived from the first one. Thus, the situation is clear for full Steiner trees with finite planar input. All possible generalizations remain open.

Question 3.7. Let $\mathcal{A} \subset \mathbb{R}^d$ be a compact totally disconnected set with at least two Steiner trees, say S and T . Can S be a solution to the maximal distance minimizers problem?

Recall the partial results known in the case when $d = 2$: if S or T is full then the answer is negative. For infinite $\mathcal{A}$, the negative answer follows from the fact that every planar maximal distance minimizer is a finite union of simple curves; see [17]. A finite $\mathcal{A}$ is covered in [2].

3.6. Nonplanar explicit examples

Recall that Steiner trees in a Euclidean d -dimensional space with $d \geq 3$ have the same local structure as in the plane, i.e., every branching point has degree 3, and the three adjacent segments have pairwise angles $2\pi/3$ and are therefore coplanar. However, the global structure is much more complicated. In particular, there is no explicit algorithm which may be caused by the fact that Steiner points are no longer expressed in radicals [30].

To the best of my knowledge, all known series of explicit solutions are planar. Even for the vertices of regular simplices [13] in high dimensions the Steiner trees remain unknown.

Thus, almost all questions in high dimensions remain open. In particular, I would like to generalize the examples from the paper [5]:

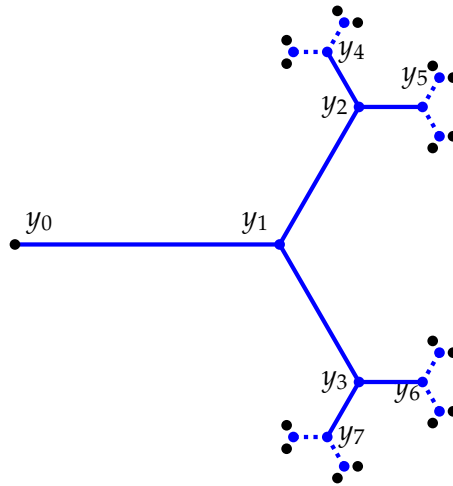


Figure 5. Planar universal Steiner tree

Question 3.8. For a regular n -gon lying in the plane $x = 1$ and centered at $(1, 0, 0)$, let Q be the set of its vertices and consider the union of the origin and all $f^k(Q)$, $k \in \mathbb{N}$, where f is a homothety with the center at the origin and a sufficiently small scale factor λ . What is the structure of the corresponding Steiner tree?

Generalizations of the universal Steiner tree problem are also of interest. For the planar case, see [29], [8], and Figure 5.

Question 3.9. Can one construct a non-planar example of a unique full Steiner tree with infinitely many branching points?

Work on the self-similar construction based on homothetic copies of a regular polygon, as well as on the non-planar universal Steiner tree problem, is ongoing, and some progress has already been made.

4. Conclusion

This paper brings together open problems concerning maximal distance minimizers and Steiner trees, two classes of one-dimensional geometric optimization problems linked by the search for connected sets of minimal length. Its contribution is expository and problem-oriented: it collects questions that are scattered across the literature or arise from recent discussions, records known partial results and explicit examples, and highlights structural parallels between the two theories.

The questions discussed above concern explicit minimizers, regularity and uniqueness, algebraic aspects, the Steiner ratio, local density, and higher-dimensional constructions. Several of them appear accessible with existing geometric and variational techniques, while others expose substantial gaps in the current understanding, especially in dimensions greater than two. By placing these problems in a common framework, the paper is intended to provide a reference point for further work and to indicate directions in which methods developed for one problem may be useful for the other.

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