

Variational Formulations for Fractional Integral and Differential Equations

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Abstract: Given a fractional-order linear equation $L^\alpha u = f$, we define an appropriate symmetric bilinear form so that the fractional operator L^α is symmetric with respect to that bilinear form. Using the bilinear form, we then define a functional of the fractional calculus of variations proving that the solutions of the given fractional-order equation are critical points of the fractional variational functional. In the case of fractional integral equations, the provided bilinear form is non-degenerate, and all critical points are solutions of the given equation. In the case of fractional differential equations, a relation with the least-squares method is obtained.

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1. Introduction

In recent years, fractional (non-integer order) differential and integral equations have attracted much attention, see, e.g., [10, 14] and references therein.

The fractional Calculus of Variations was introduced in 1996–1997, with the work of Riewe in Physics, to obtain a Lagrangian for a dissipative system with a damping force proportional to the velocity [17, 18]. Recently, Torres introduced a fractional-order calculus of variations by providing new formulas of integration by parts that involve only left or only right fractional operators [23]. This allowed him to obtain, in contrast with all previously available results of the literature, Euler–Lagrange equations that involve only left or only right fractional operators, thus providing a long-sought answer to some non-causality concerns on the fractional calculus of variations [11]. Moreover, and as a consequence of the new perspective to the subject, the first example of a Lagrangian involving only left fractional derivatives was obtained for which the respective Euler–Lagrange equation coincides with the equation of motion of a dissipative system [23].

Here, we interpret the fractional integration by parts formulas of [23] as the symmetry of the fractional-order operator with respect to suitable bilinear forms. This new interpretation allows us to associate fractional integral and differential equations to fractional-order problems of the calculus of variations, in line with the classical (integer-order) calculus of variations [3, 7, 8, 12, 20].

To the best of our knowledge, the interpretation of the one-sided fractional integration-by-parts formulas of [23], as symmetry relations associated with bilinear forms, together with the resulting variational formulations, has not been previously reported. In fact, while the (direct) fractional calculus of variations is well established, see, e.g., the books [1, 2, 15, 16], the inverse problem is less studied. Precisely, the inverse problem of the fractional calculus of variations has only been solved for specific fractional partial differential equations – the fractional wave equation, the fractional diffusion equation, and the incompressible Stokes’ equations [6]; and for fractional ordinary differential equations involving simultaneously left and right fractional operators [9, 21, 22]. Here, using the recent approach of [23], we are able to deal with more standard fractional integral and fractional ordinary differential equations involving only left (or only right)

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operators.

The paper is organized as follows. In Section 2, we solve the inverse problem of the fractional calculus of variations for fractional integral equations $\mathcal{I}^\alpha u = f$ (see Theorem 2.3), obtaining also new insights to the well-studied direct problem of the fractional calculus of variations (see Corollary 2.4). A fractional variational formulation for fractional differential equations $\mathcal{D}^\alpha u = f$ is then obtained in Section 3 (see Theorem 3.3). It should be observed that the situations studied in Sections 2 and 3 are intrinsically different and require the use of different bilinear forms so that the considered fractional operators become symmetric with respect to the bilinear form. Moreover, our variational formulation for fractional integral operators $\mathcal{I}^\alpha$ is stronger than the one obtained for fractional differential operators $\mathcal{D}^\alpha$, in the sense Theorem 2.3 provides a necessary and sufficient condition while, in contrast, Theorem 3.3 is only an implication. We end our paper with Section 4, establishing a link with the least-squares method.

Along the text, α 's are always real numbers between zero and one: $\alpha \in (0, 1)$. Our results are formal and the reader is assumed to be familiar with the classical fractional calculus of Riemann–Liouville [19]. In particular, the reader is referred to [19] for the domains and ranges of the considered operators.

2. Variational Formulation of Fractional Integral Equations

In this section, we use the fractional calculus of variations of [23] to obtain a variational formulation for a given fractional integral equation or a given system of fractional integral equations of the form

$$\mathcal{I}^\alpha u = f, \quad (1)$$

where u denotes a scalar or vector integrable function on the interval $[a, b]$ and $\mathcal{I}^\alpha$ denotes the left $\mathcal{I}_{a+}^\alpha$ or the right $\mathcal{I}_{b-}^\alpha$ fractional integral of order α . Our aim here is to obtain a functional $\mathcal{J}_I^\alpha[u]$ such that its critical points coincide with the solutions of the given fractional integral equation (1). In other words, while in [23] Torres has introduced a fractional calculus of variations and studied its direct problem, here we are interested in its inverse problem: to begin with a fractional equation and obtain the associated fractional variational functional.

2.1. The bilinear form associated with the fractional integral equation

Let us define the functional $\langle \cdot, \cdot \rangle$ by

$$\langle v, u \rangle := \int_a^b v(t)u^*(t)dt, \quad (2)$$

where u^* is the dual function of u ($u^*(t) = u(b - t + a)$, see Definition 3.1 of [23]).

We use here the symbol $\langle \cdot, \cdot \rangle$ to stress that the functional (2) satisfies the same properties of the scalar product of two elements of a Hilbert space. The proof of Proposition 2.1 is a simple exercise and is left to the reader.

Proposition 2.1. *The functional $\langle v, u \rangle$ is a bilinear form (linear in both arguments v and u), and symmetric ($\langle v, u \rangle = \langle u, v \rangle$). Moreover, $\langle v, u \rangle$ is non-degenerate (if $\langle v, u \rangle = 0$ for all v , then $u = 0$; if $\langle v, u \rangle = 0$ for all u , then $v = 0$). In the space of functions satisfying the symmetry condition $u(t) = u^*(t)$, the functional $\langle v, u \rangle$ is also positive definite ($\langle u, u \rangle \geq 0$ for all u , with $\langle u, u \rangle = 0$ if, and only if, $u = 0$).*

2.2. Integration by parts for fractional integrals

We now provide a new look to the formulas of integration by parts for fractional integrals proved in [23].

Proposition 2.2. *Let $\mathcal{I}^\alpha$ be a given linear fractional integral operator. Then $\mathcal{I}^\alpha$ is symmetric with respect to the bilinear form (2), that is,*

$$\langle \mathcal{I}^\alpha u_1, u_2 \rangle = \langle \mathcal{I}^\alpha u_2, u_1 \rangle. \quad (3)$$

Proof. Relation (3) follows directly from our definition (2) and Theorems 3.12 and 3.13 of [23]. $\square$

The importance of Proposition 2.2 is that it allows one to reinterpret fractional integration-by-parts formulas as symmetry properties of the fractional operator. Indeed, in the classical theory, the identity

$$\int_a^b (Lu)v dt = \int_a^b u(Lv)dt$$

is often viewed as a manifestation of self-adjointness. Similarly, relation (3) shows that the fractional integral operator $\mathcal{I}^\alpha$ becomes symmetric when viewed through the bilinear form $\langle \cdot, \cdot \rangle$.

This observation provides a bridge between fractional calculus and the classical inverse problem of the calculus of variations. Rather than constructing a Lagrangian by ad hoc methods, one starts from a symmetry property induced by the fractional integration-by-parts formula and obtains the variational functional in a systematic way.

2.3. Variational formulation for the fractional integral equation

Our first main result follows from Propositions 2.1 and 2.2. The bilinear forms introduced in this work play a role analogous to the inner products that appear in the classical variational formulation of differential equations. In the standard Hilbert-space setting, a linear operator can often be represented, variationally, whenever it is symmetric with respect to a suitable bilinear form. This observation lies at the heart of many variational methods in mathematical physics and numerical analysis [5, 12].

For fractional operators involving only left or only right Riemann–Liouville operators, the standard L^2 inner product does not provide the appropriate framework. The duality theory developed in [23] suggests replacing the classical scalar product by bilinear forms involving the dual transformation

$$u \mapsto u^*, \quad u^*(t) = u(b - t + a).$$

The resulting bilinear forms naturally encode the duality between left and right fractional operators and transform the integration-by-parts formulas of fractional calculus into symmetry relations.

From this perspective, the bilinear forms are not auxiliary objects but rather the geometric structures that make a variational interpretation possible.

In some sense, our next result can be interpreted as a fractional analogue of the classical principle that symmetric operators admit natural quadratic variational formulations [3, 8]. The strategy adopted here follows a classical principle of variational analysis: rather than searching directly for a Lagrangian, one first identifies a bilinear form with respect to which the governing operator is symmetric. Once such a structure is available, the associated quadratic functional arises naturally. This philosophy goes back to the classical theory of operator equations and weak formulations: see, for instance, [12].

Theorem 2.3. *A function u is a solution of the fractional integral equation (1) if, and only if, u is a critical point of the fractional variational functional*

$$\mathcal{J}_I^\alpha[u] := \frac{1}{2} \langle \mathcal{I}^\alpha u, u \rangle - \langle f, u \rangle. \quad (4)$$

Proof. We perform the first variation of $\mathcal{J}_I^\alpha[u]$. Let $\Phi(\varepsilon) = \mathcal{J}_I^\alpha[u + \varepsilon v]$. By definition, function u is a critical point of $\mathcal{J}_I^\alpha[u]$ if, and only if, $\varepsilon = 0$ is a critical point of the real function $\Phi(\varepsilon)$, that is, $\Phi'(0) = 0$. By (4) and the bilinearity of $\langle \cdot, \cdot \rangle$, we have

$$\Phi(\varepsilon) = \frac{1}{2} \langle \mathcal{I}^\alpha u, u \rangle + \frac{1}{2} \varepsilon \langle \mathcal{I}^\alpha u, v \rangle + \frac{1}{2} \varepsilon \langle \mathcal{I}^\alpha v, u \rangle + \frac{1}{2} \varepsilon^2 \langle \mathcal{I}^\alpha v, v \rangle - \langle f, u \rangle - \varepsilon \langle f, v \rangle,$$

and, by Proposition 2.2,

$$\Phi'(0) = 0 \Leftrightarrow \frac{1}{2} \langle \mathcal{I}^\alpha u, v \rangle + \frac{1}{2} \langle \mathcal{I}^\alpha v, u \rangle - \langle f, v \rangle = 0$$

is equivalent to

$$\langle \mathcal{I}^\alpha u - f, v \rangle = 0. \quad (5)$$

Since v is arbitrary, it follows from (5) and the non-degeneracy of the bilinear form $\langle \cdot, \cdot \rangle$ (see Proposition 2.1) that (5) is equivalent to $\mathcal{I}^\alpha u - f = 0$. The proof is complete. $\square$

Theorem 2.3 not only solves the inverse fractional variational problem for (1), but also gives new insights to the direct problem of the fractional calculus of variations [1, 2, 15, 16, 23]. Indeed, we are not aware of any analogous result in the existing literature.

Corollary 2.4. *Let $\mathcal{I}^\alpha$ be the left $\mathcal{I}_{a+}^\alpha$ or the right $\mathcal{I}_{b-}^\alpha$ Riemann–Liouville fractional integral of order $\alpha \in (0, 1)$. If u is a minimizer of the fractional variational functional*

$$\mathcal{J}_I^\alpha[u] = \int_a^b \left(\frac{1}{2} (\mathcal{I}^\alpha u)(t) - f(t) \right) u(b - t + a) dt, \quad (6)$$

then u satisfies the Euler–Lagrange equation $(\mathcal{I}^\alpha u)(t) = f(t)$ for $t \in [a, b]$.

We illustrate Theorem 2.3 with a simple fractional integral equation.

Example 2.5. Let $a = 0$, $b = 1$, and let $\mathcal{I}^\alpha = \mathcal{I}_{0+}^\alpha$ be the left Riemann–Liouville fractional integral of order $\alpha \in (0, 1)$. Consider

$$\mathcal{I}_{0+}^\alpha u(t) = 1, \quad t \in [0, 1]. \quad (7)$$

It is well known that

$$\mathcal{I}_{0+}^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} t^{\alpha+\beta-1}, \quad \beta > 0.$$

Taking $\beta = 1 - \alpha$, we obtain

$$\mathcal{I}_{0+}^\alpha \left(\frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \right) = 1.$$

Hence,

$$u(t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \quad (8)$$

is a solution of (7). Note that since $0 < \alpha < 1$, the function (8) belongs to $L^1(0, 1)$. According to Theorem 2.3, equation (7) is associated with the variational functional

$$\mathcal{J}_I^\alpha[u] = \frac{1}{2} \langle \mathcal{I}_{0+}^\alpha u, u \rangle - \langle 1, u \rangle, \quad (9)$$

where

$$\langle v, u \rangle = \int_0^1 v(t) u^*(t) dt, \quad u^*(t) = u(1 - t).$$

Thus,

$$\mathcal{J}_I^\alpha[u] = \frac{1}{2} \int_0^1 (\mathcal{I}_{0+}^\alpha u)(t) u(1 - t) dt - \int_0^1 u(1 - t) dt. \quad (10)$$

Let v be an admissible variation and set

$$\Phi(\varepsilon) = \mathcal{J}_I^\alpha[u + \varepsilon v].$$

Using the bilinearity of $\langle \cdot, \cdot \rangle$ and the symmetry property

$$\langle \mathcal{I}_{0+}^\alpha u, v \rangle = \langle \mathcal{I}_{0+}^\alpha v, u \rangle,$$

we obtain

$$\Phi'(0) = \langle \mathcal{I}_{0+}^\alpha u - 1, v \rangle.$$

Therefore, u is a critical point of $\mathcal{J}_I^\alpha$ if and only if

$$\langle \mathcal{I}_{0+}^\alpha u - 1, v \rangle = 0$$

for all admissible variations v . Since the bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate, this is equivalent to

$$\mathcal{I}_{0+}^\alpha u(t) = 1,$$

that is, to the original fractional integral equation (7). Consequently, the function (8) is precisely a critical point of the functional (9). This example shows explicitly how the proposed bilinear form transforms a fractional integral equation into a variational problem whose critical points coincide with the solutions of the original equation.

3. Variational Formulation of Fractional Differential Equations

We now address the problem of obtaining a variational formulation for a fractional differential equation

$$\mathcal{D}^\alpha u = f, \quad (11)$$

where u denotes a scalar or vector integrable function on the interval $[a, b]$ and $\mathcal{D}^\alpha$ denotes the left $\mathcal{D}_{a+}^\alpha$ or the right $\mathcal{D}_{b-}^\alpha$ Riemann–Liouville fractional derivative of order α .

The situations considered in Section 2 and now here in Section 3 are fundamentally different since $\mathcal{D}^\alpha$ is not symmetric with respect to the bilinear form (2). Indeed, it follows from the integration by parts formulas involving only left or only right fractional derivatives (Theorems 3.14 and 3.15 of [23]) that $\langle \mathcal{D}^\alpha u_1, u_2 \rangle = -\langle \mathcal{D}^\alpha u_2, u_1 \rangle$, that is, $\mathcal{D}^\alpha$ is anti-symmetric (or skew-symmetric) with respect to the bilinear form (2).

To solve this difficulty, and find a variational formulation for the fractional differential equation (11), we change the bilinear form (2) so that the fractional derivative $\mathcal{D}^\alpha$ becomes symmetric with respect to the new bilinear form. For that, we propose the bilinear form $(\cdot, \cdot)$ defined by

$$(v, u) := \langle v, \mathcal{D}^\alpha u \rangle = \int_a^b v(t) (\mathcal{D}^\alpha u)^*(t) dt. \quad (12)$$

Proposition 3.1. *The fractional derivative operator $\mathcal{D}^\alpha$ is symmetric with respect to the bilinear form (12), that is,*

$$(\mathcal{D}^\alpha u_1, u_2) = (\mathcal{D}^\alpha u_2, u_1). \quad (13)$$

Proof. From definition (12), and because $\langle \cdot, \cdot \rangle$ is symmetric, we have

$$(\mathcal{D}^\alpha u_1, u_2) = \langle \mathcal{D}^\alpha u_1, \mathcal{D}^\alpha u_2 \rangle = \langle \mathcal{D}^\alpha u_2, \mathcal{D}^\alpha u_1 \rangle = (\mathcal{D}^\alpha u_2, u_1),$$

which proves the result. $\square$

Proposition 3.2. *The bilinear form $(\cdot, \cdot)$ is degenerate. Indeed, every function u satisfying $\mathcal{D}^\alpha u = 0$ belongs to its kernel.*

Proof. If $\mathcal{D}^\alpha u = 0$, then

$$(v, u) = \langle v, \mathcal{D}^\alpha u \rangle = 0$$

for every v . Hence, u belongs to the kernel of $(\cdot, \cdot)$. $\square$

Using the bilinear form $(\cdot, \cdot)$, we are now in conditions to give a variational formulation for the fractional differential equation (11) (our next result, Theorem 3.3). It is important to note, however, that while Theorem 2.3 gives a necessary and sufficient condition for u to be a solution of the fractional integral equation (1), Theorem 3.3 only gives a necessary condition for u to be a solution of the fractional differential equation (11). Indeed, as one can understand from the proof of Theorem 3.3, the bilinear form $(\cdot, \cdot)$ is degenerate and functional $\mathcal{J}_D^\alpha[u]$ may admit more critical points than the solutions of the given fractional differential equation.

Theorem 3.3. *If function u is a solution of the fractional differential equation (11), then u is a critical point of the fractional variational functional*

$$\mathcal{J}_D^\alpha[u] := \frac{1}{2} (\mathcal{D}^\alpha u, u) - (f, u). \quad (14)$$

Proof. Let $\Phi(\varepsilon) := \mathcal{J}_D^\alpha[u + \varepsilon v]$. Using Proposition 3.1 and (12), we obtain that

$$\Phi(\varepsilon) = \frac{1}{2} \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha u \rangle + \varepsilon \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha v \rangle + \frac{1}{2} \varepsilon^2 \langle \mathcal{D}^\alpha v, \mathcal{D}^\alpha v \rangle - \langle f, \mathcal{D}^\alpha u \rangle - \varepsilon \langle f, \mathcal{D}^\alpha v \rangle.$$

By definition, function u is a critical point of (14) if $\varepsilon = 0$ is a critical point of $\Phi(\varepsilon)$. Differentiating $\Phi(\varepsilon)$ with respect to ε and putting $\varepsilon = 0$ gives, by Fermat's theorem, that

$$\begin{aligned} \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha v \rangle - \langle f, \mathcal{D}^\alpha v \rangle &= 0 \Leftrightarrow \langle \mathcal{D}^\alpha u - f, \mathcal{D}^\alpha v \rangle = 0 \\ &\Leftrightarrow (\mathcal{D}^\alpha u - f, v) = 0. \end{aligned} \quad (15)$$

If u is solution of (11), then $\mathcal{D}^\alpha u - f = 0$ and (15) holds, that is, u is a critical point of $\mathcal{J}_D^\alpha[u]$. $\square$

Theorem 3.3 can be interpreted as a fractional analogue of the classical fact that symmetric operators generate quadratic variational functionals.

Remark 3.4. The degeneracy of $(\cdot, \cdot)$ is the fundamental reason why Theorem 3.3 only provides an implication and not an equivalence. In contrast with the integral case, vanishing of $(\mathcal{D}^\alpha u - f, v)$ for all admissible variations does not imply uniqueness of the residual $\mathcal{D}^\alpha u - f$.

The degeneracy of the bilinear form $(\cdot, \cdot)$ deserves further discussion. In classical variational theory, non-degeneracy is essential to obtain an equivalence between the Euler–Lagrange equation and the original operator equation. In the present setting, the kernel of the operator $\mathcal{D}^\alpha$ induces a kernel in the bilinear form

$$(v, u) = \langle v, \mathcal{D}^\alpha u \rangle,$$

so that critical points of the functional may exist that do not necessarily correspond to solutions of the differential equation. For this reason, Theorem 3.3 provides only a necessary condition.

For the least-squares interpretations below, assume that f and the residuals $\mathcal{D}^\alpha u - f$ of all admissible functions u are square-integrable and reflection-symmetric on $[a, b]$. All minimizations in this interpretation are over this restricted admissible class. On the reflection-symmetric subspace, the bilinear form (2) induces the usual L^2 norm. Under these assumptions, Section 4 shows that the functional (14) admits a natural least-squares interpretation:

$$\mathcal{J}_D^\alpha[u] = \frac{1}{2} \|\mathcal{D}^\alpha u - f\|^2 - \frac{1}{2} \|f\|^2,$$

which differs from the least-squares functional only by a constant. Therefore, minimizing $\mathcal{J}_D^\alpha$ corresponds to minimizing the residual of the fractional differential equation. Section 4 establishes a direct connection between the proposed fractional variational formulation and the classical least-squares approach for operator equations. It also suggests possible numerical applications, since least-squares formulations are known to provide stable and flexible approximation schemes for differential and integral equations.

4. Relation with least-squares methods

Variational formulations play a central role in modern numerical analysis because they provide the natural framework for Galerkin, mixed, hybrid and least-squares approximation methods: see, e.g., [4, 5, 12].

From definitions (14) and (12),

$$\begin{aligned} \mathcal{J}_D^\alpha[u] &= \frac{1}{2} \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha u \rangle - \langle f, \mathcal{D}^\alpha u \rangle = \left\langle \frac{1}{2} \mathcal{D}^\alpha u - f, \mathcal{D}^\alpha u \right\rangle \\ &= \int_a^b \left[\frac{1}{2} (\mathcal{D}^\alpha u)(t) - f(t) \right] (\mathcal{D}^\alpha u)^*(t) dt \\ &= \int_a^b \left[\frac{1}{2} (\mathcal{D}^\alpha u)(t) - f(t) \right] (\mathcal{D}^\alpha u)(b - t + a) dt. \end{aligned}$$

Variational formulations can be handled by several numerical methods [13]. More generally, variational formulations provide the natural starting point for weak formulations, Galerkin approximations, finite element methods, and least-squares methods: see, e.g., [4, 5, 12]. Here we interpret Theorem 3.3 in terms of the least-squares method. Indeed,

$$\begin{aligned} \mathcal{J}_D^\alpha[u] &= \frac{1}{2} \langle \mathcal{D}^\alpha u, \mathcal{D}^\alpha u \rangle - \langle f, \mathcal{D}^\alpha u \rangle \\ &= \frac{1}{2} \langle \mathcal{D}^\alpha u - f, \mathcal{D}^\alpha u \rangle - \frac{1}{2} \langle f, \mathcal{D}^\alpha u \rangle \\ &= \frac{1}{2} \langle \mathcal{D}^\alpha u - f, \mathcal{D}^\alpha u - f \rangle + \frac{1}{2} \langle \mathcal{D}^\alpha u - f, f \rangle - \frac{1}{2} \langle f, \mathcal{D}^\alpha u \rangle \\ &= \frac{1}{2} \langle \mathcal{D}^\alpha u - f, \mathcal{D}^\alpha u - f \rangle - \frac{1}{2} \langle f, f \rangle \\ &= \frac{1}{2} \left(\|\mathcal{D}^\alpha u - f\|^2 - \|f\|^2 \right), \end{aligned}$$

where $\|v\| := \sqrt{\langle v, v \rangle}$ is the L^2 norm on the reflection-symmetric subspace, on which the bilinear form (2) is positive definite. The relation with the least-squares method is clear since the functionals $\mathcal{J}_D^\alpha[u]$ and $\frac{1}{2} \|\mathcal{D}^\alpha u - f\|^2$ differ from a constant term and thus have the same critical points.

Follows a simple illustrative example.

Example 4.1. We illustrate the least-squares interpretation with a simple fractional differential equation. Let $a = 0$, $b = 1$, and let $\mathcal{D}^\alpha = \mathcal{D}_{0+}^\alpha$ be the left Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$. Consider

$$\mathcal{D}_{0+}^\alpha u(t) = 1, \quad t \in (0, 1). \quad (16)$$

Since

$$\mathcal{D}_{0+}^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \quad \beta > \alpha,$$

taking $\beta = \alpha + 1$ gives

$$\mathcal{D}_{0+}^\alpha \left(\frac{t^\alpha}{\Gamma(\alpha+1)} \right) = 1.$$

Thus, the exact solution is

$$u(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}. \quad (17)$$

The least-squares formulation consists in minimizing the residual

$$R(u) = \mathcal{D}_{0+}^\alpha u - 1$$

through the functional

$$\mathcal{L}[u] = \frac{1}{2} \|\mathcal{D}_{0+}^\alpha u - 1\|_{L^2(0,1)}^2. \quad (18)$$

Equivalently, in our notation, this differs from $\mathcal{J}_D^\alpha[u]$ only by the constant $\frac{1}{2} \|1\|_{L^2(0,1)}^2$. To obtain a finite-dimensional approximation, let us look for u in the one-dimensional trial space

$$V_1 = \text{span } t^\alpha.$$

Thus,

$$u_c(t) = ct^\alpha.$$

Using

$$\mathcal{D}_{0+}^\alpha t^\alpha = \Gamma(\alpha+1),$$

we obtain

$$R(u_c)(t) = c\Gamma(\alpha+1) - 1.$$

Therefore,

$$\mathcal{L}[c] = \frac{1}{2} \int_0^1 (c\Gamma(\alpha+1) - 1)^2 dt = \frac{1}{2} (c\Gamma(\alpha+1) - 1)^2.$$

The minimizer satisfies

$$\frac{d\mathcal{L}}{dc} = \Gamma(\alpha+1)(c\Gamma(\alpha+1) - 1) = 0,$$

and hence

$$c = \frac{1}{\Gamma(\alpha+1)}.$$

Consequently,

$$u_c(t) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

which coincides with the exact solution (17).

This example shows explicitly how the least-squares formulation minimizes the residual of the fractional differential equation. In this simple case, the chosen trial space already contains the exact solution, and therefore the least-squares approximation recovers it exactly.

From a computational perspective, the variational formulations obtained here may provide an alternative route for the numerical treatment of fractional equations. Indeed, once a variational principle is available, one may exploit the large body of methods developed for variational problems, including Galerkin, mixed finite element, hybrid finite element, and least-squares approaches [4, 5].

Besides its theoretical interest, a variational formulation is often the first step towards the construction of numerical approximation schemes. This is one of the reasons why variational principles play such an important role in applied mathematics and scientific computing [5, 12].

The representation

$$\mathcal{J}_D^\alpha[u] = \frac{1}{2}\|\mathcal{D}^\alpha u - f\|^2 - \frac{1}{2}\|f\|^2$$

shows that the proposed variational formulation is closely related to residual minimization methods. Indeed, minimizing $\mathcal{J}_D^\alpha$ is equivalent to minimizing the norm of the residual

$$R(u) = \mathcal{D}^\alpha u - f.$$

This interpretation places our approach within the broad family of least-squares variational principles that have been successfully used for ordinary differential equations, partial differential equations, inverse problems, and numerical approximation: see, e.g., [4, 13].

From this perspective, Theorem 3.3 provides a bridge between fractional differential equations and variational residual minimization methods. This connection may be useful for the development of finite element, spectral, and collocation schemes for fractional equations.

The investigation of such numerical schemes for fractional equations is an interesting topic for future research.

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References

- [1] R. Almeida, S. Pooseh and D. F. M. Torres, *Computational methods in the fractional calculus of variations*, Imperial College Press, London, 2015. <https://doi.org/10.1142/p991>
- [2] R. Almeida, D. Tavares and D. F. M. Torres, *The variable-order fractional calculus of variations*, Springer, Cham, 2019. <https://doi.org/10.1007/978-3-319-94006-9>
- [3] G. A. Bliss, On the inverse problem of the calculus of variations, *Ann. of Math.* **9** (1908), no. 3, 127–140. <https://doi.org/10.2307/1967456>
- [4] P. B. Bochev and M. D. Gunzburger, *Least-squares finite element methods*, Springer, New York, 2009. <https://doi.org/10.1007/b13382>
- [5] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer, New York, 1991. <https://doi.org/10.1007/978-1-4612-3172-1>
- [6] J. Cresson, Inverse problem of fractional calculus of variations for partial differential equations, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010), no. 4, 987–996. <https://doi.org/10.1016/j.cnsns.2009.05.036>
- [7] D. R. Davis, The inverse problem of the calculus of variations in higher space, *Trans. Amer. Math. Soc.* **30** (1928), no. 4, 710–736. <https://doi.org/10.2307/1989443>
- [8] J. Douglas, Solution of the inverse problem of the calculus of variations, *Trans. Amer. Math. Soc.* **50** (1941), 71–128. <https://doi.org/10.2307/1989912>
- [9] F. Jiao and Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.* **62** (2011), 1181–1199. <https://doi.org/10.1016/j.camwa.2011.03.086>

- [10] K. Lan, Equivalence of higher order linear Riemann-Liouville fractional differential and integral equations, *Proc. Amer. Math. Soc.* **148** (2020), no. 12, 5225–5234. <https://doi.org/10.1090/proc/15169>
- [11] M. J. Lazo and C. E. Krumreich, The action principle for dissipative systems, *J. Math. Phys.* **55** (2014), no. 12, 122902, 11 pp. <https://doi.org/10.1063/1.4903991>
- [12] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*. Vol. I, Springer, New York-Heidelberg, 1972.
- [13] T. Liu and Z. Cai, *Inverse problems in global flow diagnostics*, Springer, Cham, 2024. <https://doi.org/10.1007/978-3-031-42474-8>
- [14] V. T. Luong, N. D. Huy, N. V. Minh and N. N. Vien, On asymptotic periodic solutions of fractional differential equations and applications, *Proc. Amer. Math. Soc.* **151** (2023), no. 12, 5299–5312. <https://doi.org/10.1090/proc/16484>
- [15] A. B. Malinowska, T. Odziejewicz and D. F. M. Torres, *Advanced methods in the fractional calculus of variations*, Springer, Cham, 2015. <https://doi.org/10.1007/978-3-319-14756-7>
- [16] A. B. Malinowska and D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imperial College Press, London, 2012. <https://doi.org/10.1142/p871>
- [17] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, *Phys. Rev. E* (3) **53** (2) (1996) 1890–1899. <https://doi.org/10.1103/PhysRevE.53.1890>
- [18] F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E* (3) **55** (3) (1997) 3581–3592. <https://doi.org/10.1103/PhysRevE.55.3581>
- [19] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993.
- [20] C. E. Stromquist, A second inverse problem of the calculus of variations, *Ann. of Math.* **9** (1908), no. 2, 57–68. <https://doi.org/10.2307/1967242>
- [21] K. M. Teng, H. E. Jia and H. F. Zhang, Existence and multiplicity results for fractional differential inclusions with Dirichlet boundary conditions, *Appl. Math. Comput.* **220** (2013), 792–801. <https://doi.org/10.1016/j.amc.2012.08.050>
- [22] Y. Tian and J. J. Nieto, The applications of critical-point theory to discontinuous fractional-order differential equations, *Proc. Edinb. Math. Soc.* **60** (2017), no. 4, 1021–1051. <https://doi.org/10.1017/S001309151600050X>
- [23] D. F. M. Torres, The duality theory of fractional calculus and a new fractional calculus of variations involving left operators only, *Mediterr. J. Math.* **21** (2024), no. 3, Paper No. 106, 16 pp. <https://doi.org/10.1007/s00009-024-02652-x>